Yuan and Yang *Boundary Value Problems* (2016) 2016:201
DOI 10.1186/s13661-016-0708-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A class of compressible non-Newtonian fluids with external force and vacuum under no compatibility conditions

Hongjun Yuan and Zan Yang*

*Correspondence:
yangzan953@163.com
Institute of Mathematics, Jilin University, Changchun, 130012, P.R. China

Abstract
We are concerned with the Cauchy problem for a class of compressible non-Newtonian fluids on the whole one-dimensional space with external force and vacuum. It is proved that the Cauchy problem for a class of compressible non-Newtonian fluids with external force and vacuum admits a unique local strong solution under no compatibility conditions.

MSC: 76N10; 76A05

Keywords: non-Newtonian fluid; vacuum; strong solution

1 Introduction and main results

We consider the one-dimensional equations of compressible non-Newtonian fluids which read as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - [(u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x]_x + \pi_x = f, \end{cases} \tag{1.1}$$

where $t \geq 0$, $x \in \mathbb{R}$, $\mu_0 > 0$, the unknown functions $\rho = \rho(x,t)$, $u = u(x,t)$ and $\pi(\rho) = A\rho^\gamma$ ($A > 0$, $\gamma > 1$) denote the density, the velocity and the pressure, separately. Without loss of generality, we set $A = 1$. We consider the Cauchy problem with (ρ, u) vanishing at infinity. For given initial functions, we require that

$$\rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{1.2}$$

The motion of the fluid is driven by an external force $f(t,x,u)$. In this paper, for any $(t,x,u) \in (0,T_0] \times (-\infty,+\infty) \times (-\infty,+\infty)$, we assume $f(t,x,u)$ to satisfy

$$\begin{cases} fu \leq -u^2, \\ fu_t \leq -uu_t, \\ f_t u_t \leq -u_t^2, \end{cases} \tag{1.3}$$

and to satisfy the structural conditions

$$\begin{cases} f \in L^\infty(0,T,L^2(\mathbb{R})), \\ f_x \in L^\infty(0,T,L^2(\mathbb{R})), \\ f_u \le -B, \end{cases} \tag{1.4}$$

where B is a positive constant.

In recent years, there has been an increasing recognition of the importance of non-Newtonian fluids. Non-Newtonian fluids arise in a large number of problems such as in the field of biomechanics, chemistry, hemorheology, glaciology, and geology. For the study of the non-Newtonian fluids, sparking the increasing interest, see [1–3].

When the initial vacuum is allowed, Lions [4] obtained the global weak solutions for the isentropic fluids for large initial data. Li and Xin [5] obtained that the Cauchy problem of the Navier-Stokes equations for viscous compressible barotropic flows in two or three spatial dimensions with vacuum as far field density admits global well-posedness and large time asymptotic behavior of strong and classical solutions. Huang and Li [6] establish the global existence and uniqueness of strong and classical solutions to the Cauchy problem for the barotropic compressible Navier-Stokes equations in two spatial dimensions with smooth initial data with vacuum. Liang and Lu [7] obtained the global-in-time existence of a unique classical solution with large initial data for the Cauchy problem for a compressible viscous fluid in one-dimensional space. Recently, Li and Liang [8] found that the two-dimensional Cauchy problem of the compressible Navier-Stokes equations admits a unique local classical solution provided the initial density decays not too slow at infinity.

Up to now, the results about non-Newtonian fluids are quite few. Recently, Yuan and Xu [9] obtained an existence result on local solutions. They obtained local existence and uniqueness of solution by using a classical energy method. For related results we refer the reader to [9–16] and the references therein.

For the Cauchy problem (1.1)-(1.2), it is still open even for the local existence of strong solutions under no compatibility conditions when the far field density is vacuum. Moreover, the system (1.1) is with strong nonlinearity, so we are facing another difficulty. In fact, this is the aim of this paper. Reference [8] motivated our study. Compared with [9, 15], the advantage of this paper is there is no need for compatibility conditions, and compared with [8], our problem is nonlinear. In this paper, we will obtain a unique strong solutions for (1.1) under $4 < p < +\infty$.

The authors in [8] bounded the $L^p(\mathbb{R}^2)$-norm of u just in the terms of $\|\rho^{1/2}u\|_{L^2(\mathbb{R}^2)}$ and $\|u_x\|_{L^2(\mathbb{R}^2)}$ by Hardy type and Poincaré type inequalities. In a similar way, we bounded the $L^k(\mathbb{R})$ $(k > p)$-norm of u just in the terms of $\|\rho^{1/2}u\|_{L^2(\mathbb{R})}$ and $\|u_x\|_{L^p(\mathbb{R})}$. However, the application of a Sobolev embedding inequality in $\mathbb{R}$ is very different from $\mathbb{R}^2$. For this, we use truncation techniques which are needed to obtain the local existence of strong solutions.

The rest of the paper is organized as follows: Firstly, we shall give some elementary facts and inequalities which will be needed in later analysis in Section 2. Sections 3 is devoted to the *a priori* estimates which are needed to obtain the local existence and uniqueness of strong solution. Finally, Theorem 1.2 is proved in Section 4.

Definition 1.1 If all derivatives involved in (1.1) for (ρ, u) are regular distributions, and equations. (1.1) hold almost everywhere in $\mathbb{R} \times (0, T)$, then (ρ, u) is called a strong solution to (1.1).

Theorem 1.2 *For constant* $4 < p < +\infty$, *assume that the initial data* (ρ_0, u_0) *satisfies*

$$\rho_0 \geq 0, \qquad u_0 \in L^2(\mathbb{R}), \qquad u_{0x} \in L^2(\mathbb{R}), \qquad \rho_0^{\frac{1}{2}} u_0 \in L^2(\mathbb{R}), \tag{1.5}$$

where $(t, x, u) \in (0, T_0] \times \mathbb{R} \times \mathbb{R}$. *Further, for constant* $p \leq q < +\infty$, *assume that* ρ_0 *also satisfies*

$$\Phi\rho_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap W^{1,q}(\mathbb{R}),$$

where

$$\Phi \triangleq \left(e + x^2\right)^{1+\zeta_0} \tag{1.6}$$

and ζ_0 *is a positive constant.*

Then there exists a positive time T_0 *such that the problem* (1.1)-(1.2) *has a unique strong solution* (ρ, u) *on* $\mathbb{R} \times (0, T_0]$ *satisfying*

$$\begin{cases} \rho \in C([0, T_0]; L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap W^{1,q}(\mathbb{R})), \\ \Phi\rho \in L^\infty(0, T_0; L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap W^{1,q}(\mathbb{R})), \\ u, \sqrt{\rho}u, \sqrt{t}\sqrt{\rho}u_t \in L^\infty(0, T_0; L^2(\mathbb{R})), \\ \sqrt{\rho}u_t, \sqrt{t}u_{tx} \in L^2(\mathbb{R} \times (0, T_0)), \\ u_x \in L^\infty(0, T_0; L^2(\mathbb{R})) \cap L^\infty(0, T_0; L^p(\mathbb{R})), \\ u_x \in L^2(0, T_0; H^1(\mathbb{R})) \cap L^{(q+1)/q}(0, T_0; W^{1,q}(\mathbb{R})), \\ \sqrt{t}u_x \in L^2(0, T_0; W^{1,q}(\mathbb{R})), \\ \sqrt{t}u_{xx} \in L^\infty(0, T_0; L^2(\mathbb{R})). \end{cases} \tag{1.7}$$

Moreover,

$$\inf_{0 \leq t \leq T_0} \int_{\Omega_R} \rho(x, t)\, \mathrm{d}x \geq \frac{1}{4} \int_{\mathbb{R}} \rho_0(x)\, \mathrm{d}x. \tag{1.8}$$

2 Preliminaries

We have the following results concerning local existence theory on bounded intervals whose existence can be found in [11, 17–19].

Lemma 2.1 *For* $R > 0$ *and* $\Omega_R \triangleq \{x \in \mathbb{R} | |x| < R\}$, *assume that* (ρ_0, u_0) *satisfies*

$$\rho_0 \in H^1(\Omega_R), \qquad \inf_{x \in \Omega_R} \rho_0(x) > 0, \qquad u_0 \in H_0^1(\Omega_R) \cap H^2(\Omega_R). \tag{2.1}$$

Let f as in (1.3) *and* (1.4). *Then there exist a small time $T_R > 0$ and a unique classical solution (ρ, u) to the following initial-boundary value problem*:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - [(u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x]_x + \pi_x = f, \\ \pi(\rho) = \rho^\gamma, \quad \gamma > 1, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \quad x \in \Omega_R, \\ u(x) = 0, \quad x \in \partial\Omega_R, t > 0, \end{cases} \tag{2.2}$$

on $\Omega_R \times (0, T_R]$ such that

$$\begin{cases} \rho \in C([0, T_R]; H^1(\Omega_R)), \qquad u \in C([0, T_R]; H_0^1(\Omega_R)) \cap L^\infty(0, T_R; H^2(\Omega_R)), \\ \rho_t \in C([0, T_R]; L^2(\Omega_R)), \qquad u_t \in L^2(0, T_R; H_0^1(\Omega_R)), \\ \sqrt{\rho} u_t \in L^\infty(0, T_R; L^2(\Omega_R)), \\ [(u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x]_x \in C([0, T_R]; L^2(\Omega_R)). \end{cases} \tag{2.3}$$

Remark 2.2 Assume $v \in H_0^1(\Omega_R) \cap H^2(\Omega_R)$, then $v \in W_0^{1,p}(\Omega_R)$ with $p > 4$.

Lemma 2.3 *For either $s = 2$ or $s = p$, $m \in [s, \infty)$ and $\vartheta \in (m + 1/2, \infty)$, there exists a positive constant C such that, for either $\Omega = \mathbb{R}$ or $\Omega = \Omega_R$ with $R \geq 1$ and for any $v \in W_{\mathrm{loc}}^{1,s}(\Omega)$,*

$$\left(\int_\Omega |v|^m \left(e + x^2\right)^{-\vartheta} \mathrm{d}x \right)^{1/m} \leq C\|v\|_{L^s(\Omega_1)} + C\|v_x\|_{L^s(\Omega)}. \tag{2.4}$$

A consequence of Lemma 2.3 will play a crucial role in our analysis.

Proof First, we begin with the case $v \in W_{\mathrm{loc}}^{1,2}(\Omega)$. For all $R \geq 1$, since

$$\int_{-R}^{R} \left|v(x)\right|^2 \mathrm{d}x = R \int_{-1}^{1} \left|v(R\tau)\right|^2 \mathrm{d}\tau,$$

we observe there exists $R_0 \in [\frac{1}{2}, 1]$ such that

$$\left(\int_{-1}^{1} \left|v(R_0\tau)\right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \leq 2 \left(\int_{-1}^{1} \left|v(x)\right|^2 \mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.5}$$

By calculation, we get

$$\int_{R_0}^{R} \frac{\mathrm{d}}{\mathrm{d}s} \left(\int_{-1}^{1} \left|v(s\tau)\right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \mathrm{d}s = \left(\int_{-1}^{1} \left|v(R\tau)\right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} - \left(\int_{-1}^{1} \left|v(R_0\tau)\right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}}. \tag{2.6}$$

By virtue of

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s} \left(\int_{-1}^{1} \left|v(s\tau)\right|^2 \mathrm{d}\tau \right) \\ &\quad = \int_{-1}^{1} 2\left|v(s\tau)\right| \left| \frac{\partial v(s\tau)}{\partial s} \right| \mathrm{d}\tau \end{aligned}$$

$$\leq 2\left(\int_{-1}^{1}\left|v(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left|\frac{\partial v(s\tau)}{\partial s}\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}$$
$$\leq 2\left(\int_{-1}^{1}\left|v(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left|v'(s\tau)\tau\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}$$
$$\leq 2\left(\int_{-1}^{1}\left|v(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left|v'(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{-1}^{1}\left|v(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \leq \left(\int_{-1}^{1}\left|v'(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}. \tag{2.7}$$

By some directly calculations, we get

$$\begin{aligned}
&\int_{R_0}^{R}\left(\int_{-1}^{1}\left|v'(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \mathrm{d}s \\
&\quad = \int_{R_0}^{R}\left(s\int_{-1}^{1}\left|v'(s\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\left(\frac{1}{s}\right)^{\frac{1}{2}} \mathrm{d}s \\
&\quad \leq \left(\int_{R_0}^{R} s\int_{-1}^{1}\left|v'(s\tau)\right|^2 \mathrm{d}\tau\, \mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{R_0}^{R}\frac{1}{s}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad \leq \left(\int_{R_0}^{R}\int_{-s}^{s}\left|v_x(x)\right|^2 \mathrm{d}x\, \mathrm{d}s\right)^{\frac{1}{2}}\left(\ln\frac{R}{R_0}\right)^{\frac{1}{2}}
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\int_{R_0}^{R}\int_{-s}^{s}\left|v_x(x)\right|^2 \mathrm{d}x\, \mathrm{d}s \\
&\quad \leq \int_{0}^{R}\int_{x}^{R}\left|v_x(x)\right|^2 \mathrm{d}s\, \mathrm{d}x + \int_{-R}^{0}\int_{-x}^{R}\left|v_x(x)\right|^2 \mathrm{d}s\, \mathrm{d}x \\
&\quad \leq \int_{0}^{R}\left|v_x(x)\right|^2 (R-x)\, \mathrm{d}x + \int_{-R}^{0}\left|v_x(x)\right|^2 (R+x)\, \mathrm{d}x \\
&\quad = R\int_{-R}^{R}\left|v_x(x)\right|^2 \mathrm{d}x.
\end{aligned} \tag{2.9}$$

Combining (2.5)-(2.9), we get

$$\begin{aligned}
&\left(\int_{-1}^{1}\left|v(R\tau)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad \leq CR^{\frac{1}{2}}\left(\ln(2R)\right)^{\frac{1}{2}}\left[\left(\int_{-1}^{1}\left|v(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{-R}^{R}\left|v_x(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{2.10}$$

We denote

$$|||v|||_2 \triangleq \left(\int_{-1}^{1}\left|v(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{-R}^{R}\left|v_x(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{2.11}$$

and rewrite (2.10) as

$$\int_{-1}^{1}\left|v(R\tau)\right|^2\,\mathrm{d}\tau \le CR\ln(2R)|||v|||_2^2. \tag{2.12}$$

Multiplying (2.12) by R, we obtain

$$\begin{aligned}\int_{-R}^{R}\left|v(x)\right|^2\,\mathrm{d}x &\le CR^2\ln(2R)|||v|||_2^2\\ &\le CR^2\ln\left(e+R^2\right)|||v|||_2^2\\ &\le CR^2\left(e+R^2\right)|||v|||_2^2.\end{aligned} \tag{2.13}$$

The Gagliardo-Nirenberg inequality implies that, for all $m_1 \in (2,+\infty)$,

$$\left(\int_{-R}^{R}\left|v(x)\right|^{m_1}\,\mathrm{d}x\right)^{\frac{1}{m_1}} \le C\|v\|_{L^2(\Omega)}^{\frac{m_1+2}{2m_1}}\|v_x\|_{L^2(\Omega)}^{\frac{m_1-2}{2m_1}}. \tag{2.14}$$

Consequently, we obtain after using (2.11), (2.13), and (2.14)

$$\begin{aligned}&\int_{-R}^{R}\left|v(x)\right|^{m_1}\,\mathrm{d}x\\ &\quad\le C\|v\|_{L^2(\Omega)}^{\frac{m_1+2}{2}}\|v_x\|_{L^2(\Omega)}^{\frac{m_1-2}{2}}\\ &\quad\le C\left[R^2\left(e+R^2\right)|||v|||_2^2\right]^{\frac{m_1+2}{4}}\left[R^2\left(e+R^2\right)|||v|||_2^2\right]^{\frac{m_1-2}{4}}\\ &\quad\le CR^{m_1}\left(e+R^2\right)^{\frac{m_1}{2}}|||v|||_2^{m_1}.\end{aligned} \tag{2.15}$$

Next, we discuss the case $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$. Since

$$\int_{-R}^{R}\left|v(x)\right|^p\,\mathrm{d}x = R\int_{-1}^{1}\left|v(R\tau)\right|^p\,\mathrm{d}\tau,$$

we observe there exists $R_0 \in [\frac{1}{2},1]$ such that

$$\left(\int_{-1}^{1}\left|v(R_0\tau)\right|^p\,\mathrm{d}\tau\right)^{\frac{1}{p}} \le 2\left(\int_{-1}^{1}\left|v(x)\right|^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.16}$$

By some direct calculations, we get

$$\int_{R_0}^{R}\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{-1}^{1}\left|v(s\tau)\right|^p\,\mathrm{d}\tau\right)^{\frac{1}{2}}\mathrm{d}s = \left(\int_{-1}^{1}\left|v(R\tau)\right|^2\,\mathrm{d}\tau\right)^{\frac{1}{p}} - \left(\int_{-1}^{1}\left|v(R_0\tau)\right|^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}. \tag{2.17}$$

By virtue of

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{-1}^{1}\left|v(s\tau)\right|^p\,\mathrm{d}\tau\right)\\ &\quad= \int_{-1}^{1}p\left|v(s\tau)\right|^{p-1}\left|\frac{\partial v(s\tau)}{\partial s}\right|\mathrm{d}\tau\end{aligned}$$

$$\leq p\left(\int_{-1}^{1}\left|v(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{p-1}{p}}\left(\int_{-1}^{1}\left|\frac{\partial v(s\tau)}{\partial s}\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}$$
$$\leq p\left(\int_{-1}^{1}\left|v(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{p-1}{p}}\left(\int_{-1}^{1}\left|v'(s\tau)\tau\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}$$
$$\leq p\left(\int_{-1}^{1}\left|v(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{p-1}{p}}\left(\int_{-1}^{1}\left|v'(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{-1}^{1}\left|v(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}\leq\left(\int_{-1}^{1}\left|v'(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}. \tag{2.18}$$

For all $R\geq 1$, we get

$$\begin{aligned}
&\int_{R_0}^{R}\left(\int_{-1}^{1}\left|v'(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s\\
&\quad=\int_{R_0}^{R}\left(s\int_{-1}^{1}\left|v'(s\tau)\right|^{p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\frac{1}{s}\right)^{\frac{1}{p}}\mathrm{d}s\\
&\quad\leq\left(\int_{R_0}^{R}\int_{-1}^{1}\left|v_x(x)\right|^{p}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_{R_0}^{R}\left(\frac{1}{s}\right)^{\frac{1}{p-1}}\mathrm{d}s\right)^{\frac{p-1}{p}}\\
&\quad\leq\left(\int_{R_0}^{R}\int_{-1}^{1}\left|v_x(x)\right|^{p}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}\left[\left(\int_{R_0}^{R}\left(\frac{1}{s}\right)^{\frac{3-p}{p-1}}\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{R_0}^{R}\frac{1}{s}\,\mathrm{d}s\right)^{\frac{1}{2}}\right]^{\frac{p-1}{p}}\\
&\quad\leq\left(\int_{R_0}^{R}\int_{-1}^{1}\left|v_x(x)\right|^{p}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\ln\frac{R}{R_0}\right)^{\frac{p-1}{2p}}R^{\frac{p-2}{p}}\\
&\quad\leq\left(\int_{R_0}^{R}\int_{-1}^{1}\left|v_x(x)\right|^{p}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\ln(2R)\right)^{\frac{p-1}{2p}}R^{\frac{p-2}{p}}\\
&\quad\leq R^{\frac{p-1}{p}}\left(e+R^{2}\right)^{\frac{1}{2}}\left(\int_{R_0}^{R}\int_{-1}^{1}\left|v_x(x)\right|^{p}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}\tag{2.19}$$

We denote

$$|||v|||_{p}\triangleq\left(\int_{-1}^{1}\left|v(x)\right|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}+\left(\int_{-R}^{R}\left|v_x(x)\right|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.20}$$

Combining (2.17)-(2.20), we get

$$\int_{-1}^{1}\left|v(R\tau)\right|^{p}\,\mathrm{d}\tau\leq CR^{p-1}\left(e+R^{2}\right)^{\frac{p}{2}}|||v|||_{p}^{p}. \tag{2.21}$$

Multiplying (2.21) by R, we obtain

$$\int_{-R}^{R}\left|v(x)\right|^{p}\,\mathrm{d}x\leq CR^{p}\left(e+R^{2}\right)^{\frac{p}{2}}|||v|||_{p}^{p}. \tag{2.22}$$

The Gagliardo-Nirenberg inequality implies that, for all $m_2 \in (p, +\infty)$,

$$\begin{aligned}
&\int_{-R}^{R} \left|v(x)\right|^{m_2} \mathrm{d}x \\
&\quad \le C\|v\|_{L^p(\Omega)}^{\frac{pm_2-m_2+p}{p}} \|v_x\|_{L^p(\Omega)}^{\frac{m_2-p}{p}} \\
&\quad \le C\left(R^p\left(e+R^2\right)^{\frac{p}{2}} |||v|||_p^p\right)^{\frac{m_2}{p}} \\
&\quad \le CR^{m_2}\left(e+R^2\right)^{\frac{m_2}{2}} |||v|||_p^{m_2}. \qquad (2.23)
\end{aligned}$$

By a simple integration by parts, (2.13), (2.15), (2.22), and (2.23), we obtain for all $R \ge 1$, either $s = 2$ or $s = p$, $m \in [s, \infty)$, and, for any $v \in W^{1,s}_{\mathrm{loc}}(\Omega)$,

$$\begin{aligned}
&\int_{-R}^{R} \left|v(x)\right|^{m} \left(e+x^2\right)^{-\vartheta} \mathrm{d}x \\
&\quad \le \int_{-R}^{R} \left|v(x)\right|^{m} \mathrm{d}x \cdot \left(e+R^2\right)^{-\vartheta} + \int_{-R}^{R} \left(\int_0^x \left|v(\tau)\right|^m \mathrm{d}\tau\right) \frac{2\vartheta x}{(e+x^2)^{\vartheta+1}} \mathrm{d}x \\
&\quad \le |||v|||_s^m \frac{(R^2)^{\frac{m}{2}}(e+R^2)^{\frac{m}{2}}}{(e+R^2)^{\vartheta}} + C|||v|||_s^m \int_{-R}^{R} \frac{2x \cdot x^{m-1}(e+x^2)^{\frac{m}{2}}}{(e+x^2)^{\vartheta}} \mathrm{d}x \\
&\quad \le |||v|||_s^m \left(e+R^2\right)^{m-\vartheta} + C|||v|||_s^m \int_{-R}^{R} \frac{2x(e+x^2)^{m-\frac{1}{2}}}{(e+x^2)^{\vartheta}} \mathrm{d}x \\
&\quad \le |||v|||_s^m + C|||v|||_s^m \int_{-R}^{R} \left(e+x^2\right)^{m-\frac{1}{2}-\vartheta} \mathrm{d}\left(e+x^2\right) \\
&\quad \le C|||v|||_s^m \left(1 + \int_{e+R^2}^{e+R^2} y^{m-\frac{1}{2}-\vartheta} \mathrm{d}y\right) \\
&\quad = C|||v|||_s^m,
\end{aligned}$$

if $\vartheta > m + \frac{1}{2}$. □

Lemma 2.4 *Let* Φ *and* ζ_0 *be as in* (1.5) *and* Ω *as in Lemma* 2.3. *For* $\gamma > 1$, *assume that* $\rho \in L^1(\Omega) \cap L^\gamma(\Omega)$ *is a non-negative function such that*

$$\int_{\Omega_{N_1}} \rho \,\mathrm{d}x \ge Q_1, \qquad \int_{\Omega} \rho^\gamma \,\mathrm{d}x \le Q_2, \qquad (2.24)$$

for positive constants Q_1, Q_2, *and* $N_1 \ge 1$ *with* $\Omega_{N_1} \subset \Omega$. *Then there is a positive constant* C *depending only on* Q_1, Q_2, N_1, γ, *and* ζ_0 *such that*

$$\left\|v\Phi^{-1}\right\|_{L^s(\Omega)} \le C\left\|\rho^{1/2}v\right\|_{L^2(\Omega)} + C\|v_x\|_{L^s(\Omega)}, \qquad (2.25)$$

for every $v \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$. *Moreover, for* $\varepsilon > 0$ *and* $\zeta > 0$ *there is a positive constant* C *depending only on* ε, ζ, Q_1, Q_2, N_1, γ, *and* ζ_0 *such that*

$$\left\|v\Phi^{-\zeta}\right\|_{L^{(s+\varepsilon)/\tilde{\zeta}}(\Omega)} \le C\left\|\rho^{1/2}v\right\|_{L^2(\Omega)} + C\|v_x\|_{L^s(\Omega)}, \qquad (2.26)$$

with $\tilde{\zeta} = \min\{1, \zeta\}$.

Proof It follows from (2.24) and similar arguments to Lemma 3.2 of [20] that there exists a positive constant C depending only on Q_1, Q_2, N_1, and γ, such that

$$\|v\|^2_{L^s(\Omega_{N_1})} \le C\int_{\Omega_{N_1}} \rho v^2 \,\mathrm{d}x + C\|v_x\|^2_{L^s(\Omega_{N_1})}. \tag{2.27}$$

There exists a positive constant ζ_0 such that

$$\zeta\frac{s+\varepsilon}{\tilde{\zeta}}(1+\zeta_0) > \frac{s+\varepsilon}{\tilde{\zeta}} + \frac{1}{2},$$

which together with (2.4) and (2.27) gives (2.25) and (2.26). The proof of Lemma 2.4 is finished. □

3 *A priori* estimates

For $R > 4R_0 \ge 4$, assume that the smooth (ρ_0, u_0) satisfies, in addition to (2.1),

$$1/2 \le \int_{\Omega_{R_0}} \rho_0(x)\,\mathrm{d}x \le \int_{\Omega_R} \rho_0(x)\,\mathrm{d}x \le 3/2. \tag{3.1}$$

Lemma 2.1 thus shows that there exists some $T_R > 0$ such that the initial-boundary value problem (2.2) has a unique classical solution (ρ, u) on $\Omega_R \times [0, T_R]$ satisfying (2.3).

For Φ, ζ_0, and q as in Theorem 1.2, the main aim of this section is to derive the key *a priori* estimate on ψ defined by

$$\begin{aligned}\psi(t) \triangleq{}& 1 + \left\|\rho^{1/2}u\right\|_{L^2(\Omega_R)} + \|u\|_{L^2(\Omega_R)} + \|u_x\|_{L^2(\Omega_R)} + \|u_x\|_{L^p(\Omega_R)} \\ &+ \|\Phi\rho\|_{L^1(\Omega_R)\cap H^1(\Omega_R)\cap W^{1,q}(\Omega_R)}.\end{aligned} \tag{3.2}$$

Proposition 3.1 *Assume that* (ρ_0, u_0) *satisfies* (2.1) *and* (3.1). *Let* (ρ, u) *be the solution to the initial-boundary value problem* (2.2) *on* $\Omega_R \times (0, T_R]$ *obtained by Lemma* 2.1. *Then there exist positive constants* T_0 *and* M *both depending only on* γ, q, μ_0, ζ_0, N_0, *and* K_0 *such that*

$$\begin{aligned}&\sup_{0\le t\le T_0} \psi(t) + \int_0^{T_0}\left(\|u_{xx}\|^{(q+1)/q}_{L^q(\Omega_R)} + t\|u_{xx}\|^2_{L^q(\Omega_R)} + \|u_{xx}\|^2_{L^2(\Omega_R)}\right)\mathrm{d}t \\ &\quad + \sup_{0\le t\le T_0} t\|u_{xx}\|^2_{L^2(\Omega_R)} \le M,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}K_0 \triangleq{}& \left\|\rho_0^{1/2}u_0\right\|_{L^2(\Omega_R)} + \|u_{0x}\|_{L^2(\Omega_R)} + \|u_{0x}\|_{L^p(\Omega_R)} + \|u_0\|_{L^2(\Omega_R)} \\ &+ \|\Phi\rho_0\|_{L^1(\Omega_R)\cap H^1(\Omega_R)\cap W^{1,q}(\Omega_R)}.\end{aligned}$$

The proof of Proposition 3.1 will be postponed to the end of this section; we begin with the following standard energy estimate for (ρ, u).

Lemma 3.2 *Let* (ρ, u) *be a smooth solution to the initial-boundary value problem* (2.2). *Then there exist a* $T^* = T^*(R_0, K_0) > 0$ *and a positive constant* $\beta = \beta(\gamma, q) > 1$ *such that, for* *all* $t \in (0, T^*]$,

$$\sup_{0\le s\le t}\left(\int_{-R}^{R}|u|^2\,\mathrm{d}x+\int_{-R}^{R}|u_x|^2\,\mathrm{d}x+\int_{-R}^{R}|u_x|^p\,\mathrm{d}x\right)+\int_0^t\int_{-R}^{R}\rho|u_t|^2\,\mathrm{d}x\,\mathrm{d}s \le C+C\int_0^t\psi^\beta\,\mathrm{d}s, \tag{3.4}$$

where (*as in the following*) C *denotes a generic positive constant depending only on* γ, q, ζ_0, μ_0, R_0 *and* K_0, *under the conditions of Proposition* 3.1.

Proof First, applying standard energy estimate to (2.2) gives

$$\sup_{0\le s\le t}\left(\|\sqrt{\rho}u\|^2_{L^2(\Omega_R)}+\|\rho\|^\gamma_{L^\gamma(\Omega_R)}\right)+\int_0^t\left(\int_{-R}^{R}|u_x|^p\,\mathrm{d}x+\int_{-R}^{R}|u|^2\,\mathrm{d}x\right)\mathrm{d}s\le C. \tag{3.5}$$

Next, for $R > 1$ and $\tilde{\xi}_R \in C_0^\infty(\Omega_R)$ such that

$$\begin{aligned}&0\le\tilde{\xi}_R\le 1,\qquad \tilde{\xi}_R(x)=1,\qquad |x|\le\frac{R}{2},\\ &|\tilde{\xi}_{Rx}|\le\frac{C}{R},\qquad |\tilde{\xi}_{Rxx}|\le\frac{C}{R^2},\end{aligned} \tag{3.6}$$

it follows from (3.5) and (3.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R}\rho\tilde{\xi}_{2R_0}\,\mathrm{d}x&=\int_{-R}^{R}\rho u\tilde{\xi}_{2R_0x}\,\mathrm{d}x\\ &\ge -C(R_0)^{-1}\left(\int_{-R}^{R}\rho\,\mathrm{d}x\right)^{1/2}\left(\int_{-R}^{R}\rho|u|^2\,\mathrm{d}x\right)^{1/2}\\ &\ge -\tilde{B},\end{aligned} \tag{3.7}$$

where in the last inequality we have used

$$\int_{-R}^{R}\rho\,\mathrm{d}x=\int_{-R}^{R}\rho_0\,\mathrm{d}x,$$

due to $(2.2)_1$ and $(2.2)_5$. Integrating (3.7) gives

$$\begin{aligned}\inf_{0\le t\le T^*}\int_{-R}^{R}\rho\,\mathrm{d}x&\ge\inf_{0\le t\le T^*}\int_{-R}^{R}\rho\tilde{\xi}_{2R_0}\,\mathrm{d}x\\ &\ge\int_{-R}^{R}\rho_0\tilde{\xi}_{2R_0}\,\mathrm{d}x-\tilde{B}T^*\\ &\ge 1/4,\end{aligned} \tag{3.8}$$

where $T^* \triangleq \min\{1, (4\tilde{B})^{-1}\}$.

From now on, we will always assume that $t \le T^*$. The combination of (3.8), (3.5), and (2.26) shows, for $\varepsilon > 0$, $\zeta > 0$, $\forall v \in W_0^{1,2}(\Omega_R) \cap W_0^{1,p}(\Omega_R)$ satisfies

$$\|v\Phi^{-\zeta}\|^2_{L^{(s+\varepsilon)/\tilde{\zeta}}(\Omega_R)}\le C(\varepsilon,\zeta)\int_{-R}^{R}\rho|v|^2\,\mathrm{d}x+C(\varepsilon,\zeta)\|v_x\|^2_{L^s(\Omega_R)}, \tag{3.9}$$

with $\tilde{\zeta} = \min\{1, \zeta\}$. In particular, we have

$$\|\rho^{\zeta} v\|_{L^{(s+\varepsilon)/\tilde{\zeta}}(\Omega_R)} + \|v\Phi^{-\zeta}\|_{L^{(s+\varepsilon)/\tilde{\zeta}}(\Omega_R)} \leq C(\varepsilon, \zeta)\psi^{1+\zeta}. \tag{3.10}$$

Multiplying $(2.2)_2$ by u_t, and integrating it over $(-R, R)$ on x and integrating over $(0, t)$ on the time variable

$$\begin{aligned} &\int_0^t \int_{-R}^R \rho|u_t|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{-R}^R \bigl[\bigl((u_x)^2 + \mu_0\bigr)^{\frac{p-2}{2}} u_x\bigr] u_{xt} \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{-R}^R f u_t \,\mathrm{d}x\,\mathrm{d}s \\ &\quad = -\int_{-R}^R \pi u_x(0) \,\mathrm{d}x + \int_{-R}^R \pi u_x(t) \,\mathrm{d}x + \int_0^t \int_{-R}^R \bigl|(-\rho u u_x) u_t - \pi_t u_x\bigr| \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.11}$$

We first compute the second term of (3.11), and we get

$$\begin{aligned} &\int_{-R}^R \bigl(u_x^2 + \mu_0\bigr)^{\frac{p-2}{2}} u_x u_{xt} \,\mathrm{d}x \\ &\quad = \frac{1}{2} \int_{-R}^R \bigl(u_x^2 + \mu_0\bigr)^{\frac{p-2}{2}} (u_x)_t^2 \,\mathrm{d}x \\ &\quad = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-R}^R \biggl(\int_0^{u_x^2} (s + \mu_0)^{\frac{p-2}{2}} \,\mathrm{d}s \biggr) \,\mathrm{d}x \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\int_0^{u_x^2} (s + \mu_0)^{\frac{p-2}{2}} \,\mathrm{d}s \\ &\quad \geq \int_0^{u_x^2} s^{\frac{p-2}{2}} \,\mathrm{d}s \\ &\quad = \frac{2}{p} |u_x|^p. \end{aligned} \tag{3.13}$$

Substituting (3.12), and (3.13) into (3.11), and by using (1.3), we obtain

$$\begin{aligned} &\int_0^t \bigl\|\sqrt{\rho} u_t(t)\bigr\|_{L^2(\Omega_R)}^2 \,\mathrm{d}s + \frac{1}{p} \int_{-R}^R \bigl|u_x(t)\bigr|^p \,\mathrm{d}x + \frac{1}{2} \int_{-R}^R \bigl|u(t)\bigr|^2 \,\mathrm{d}x \\ &\quad \leq \frac{2}{p} \bigl\|u_x(0)\bigr\|_{L^p(\Omega_R)}^p + \bigl\|\pi(\rho_0)\bigr\|_{L^2(\Omega_R)}^2 + \int_{-R}^R \bigl|\pi u_x(t)\bigr| \,\mathrm{d}x \\ &\quad\quad + \int_0^t \int_{-R}^R \bigl|(-\rho u u_x) u_t - \pi_t u_x\bigr| \,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leq C + \int_{-R}^R \bigl|\pi u_x(t)\bigr| \,\mathrm{d}x + \int_0^t \int_{-R}^R \bigl|(-\rho u u_x) u_t - \pi_t u_x\bigr| \,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned} &\int_0^t \bigl\|\sqrt{\rho} u_t(t)\bigr\|_{L^2(\Omega_R)}^2 \,\mathrm{d}s + \int_{-R}^R \bigl|u_x(t)\bigr|^p \,\mathrm{d}x + \int_{-R}^R \bigl|u(t)\bigr|^2 \,\mathrm{d}x \\ &\quad \leq C + C \int_0^t \|\sqrt{\rho} u u_x\|_{L^2(\Omega_R)}^2 \,\mathrm{d}s + C \int_0^t \int_{-R}^R |\pi_x u u_x| \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{-R}^R \bigl|\pi u_x^2\bigr| \,\mathrm{d}x\,\mathrm{d}s \\ &\quad\quad + C\|\pi\|_{L^{\frac{p}{p-1}}(\Omega_R)}^{\frac{p}{p-1}}. \end{aligned} \tag{3.14}$$

First, the Gagliardo-Nirenberg inequality implies that, for all $k \in (p, +\infty)$,

$$\|u_x\|_{L^k(\Omega_R)} \le C\|u_x\|^{\theta}_{L^p(\Omega_R)}\|u_{xx}\|^{1-\theta}_{L^2(\Omega_R)}, \tag{3.15}$$

where $\theta = \frac{2k-2p}{2k+kp}$.

Next, we estimate each term on the right-hand side of (3.14) as follows: We deduce from (3.10), (3.2), and Hölder's inequality that

$$\begin{aligned}\|\sqrt{\rho}uu_x\|_{L^2(\Omega_R)} &\le \|u_x\|_{L^p(\Omega_R)}\|\rho\Phi\|^{\frac{1}{2}}_{L^\infty(\Omega_R)}\big\|\Phi^{-\frac{1}{4}}u\big\|_{L^{2p}(\Omega_R)}\big\|\Phi^{-\frac{1}{4}}\big\|_{L^{\frac{2p}{p-3}}(\Omega_R)} \\ &\le C\psi^{\beta},\end{aligned} \tag{3.16}$$

where (as in the following) we use $\beta = \beta(\gamma, q) > 1$ to denote a generic constant depending only on γ and q, which may be different from line to line.

By (3.9),we have

$$\begin{aligned}\int_{-R}^{R}|\pi_x uu_x|\,\mathrm{d}x &\le C\|\rho\|^{\gamma-1}_{L^\infty(\Omega_R)}\|\Phi\rho_x\|_{L^2(\Omega_R)}\|u_x\|_{L^p(\Omega_R)}\big\|\Phi^{-\frac{1}{2}}u\big\|_{L^{2p}(\Omega_R)}\big\|\Phi^{-\frac{1}{2}}\big\|_{L^{\frac{2p}{p-3}}(\Omega_R)} \\ &\le C\psi^{\beta}.\end{aligned} \tag{3.17}$$

Using $(2.2)_2$, we get

$$\big[\big((u_x)^2 + \mu_0\big)^{\frac{p-2}{2}}u_x\big]_x = \rho u_t + \rho uu_x + \pi_x + f.$$

By virtue of

$$\mu_0^{\frac{p-2}{2}}|u_{xx}| \le \big[\big((u_x)^2 + \mu_0\big)^{\frac{p-2}{2}}u_x\big]_x,$$

we have

$$|u_{xx}| \le C|\rho u_t + \rho uu_x + \pi_x + f|.$$

Taking the above inequality by L_2 norm and L_p norm, we obtain

$$\begin{aligned}&\|u_{xx}\|_{L^2(\Omega_R)} \\ &\quad\le C\big\|\big(\big(u_x^2 + \mu_0\big)^{\frac{p-2}{2}}u_x\big)_x\big\|_{L^2(\Omega_R)} \\ &\quad\le C\|\rho u_t\|_{L^2(\Omega_R)} + C\|\rho uu_x\|_{L^2(\Omega_R)} + C\|\pi_x\|_{L^2(\Omega_R)} + \|f\|_{L^2(\Omega_R)} \\ &\quad\le C\psi^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)} \\ &\qquad + C\|u_x\|_{L^p(\Omega_R)}\|\rho\Phi\|_{L^q(\Omega_R)}\big\|\Phi^{-\frac{1}{2}}u\big\|_{L^{\frac{2pq}{\varepsilon_0}}(\Omega_R)}\big\|\Phi^{-\frac{1}{2}}\big\|_{L^{\frac{2pq}{pq-2p-2q-\varepsilon_0}}(\Omega_R)} + C\psi^{\beta} \\ &\quad\le C\psi^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)} + C\psi^{\beta},\end{aligned} \tag{3.18}$$

where $0 < \varepsilon_0 < \min\{1, pq - 2p - 2q\}$, and

$$\|u_{xx}\|_{L^q(\Omega_R)} \le C\big(\|\rho u_t\|_{L^q(\Omega_R)} + \|\rho uu_x\|_{L^q(\Omega_R)} + \|\pi_x\|_{L^q(\Omega_R)} + \|f\|_{L^q(\Omega_R)}\big). \tag{3.19}$$

By (3.15), we have

$$\begin{aligned}\int_{-R}^{R}\left|\pi u_x^2\right|\mathrm{d}x &\leq \|\rho\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\Phi\rho\|_{L^2(\Omega_R)}\|u_x\|_{L^p(\Omega_R)}\|u_x\|_{L^{\frac{p^2}{p-2}}(\Omega_R)}\left\|\Phi^{-1}\right\|_{L^{\frac{2p^2}{(p-2)^2}}(\Omega_R)} \\ &\leq C\psi^\beta\|u_{xx}\|_{L^2(\Omega_R)}^\theta \\ &\leq C(\varepsilon)\psi^\beta + C\varepsilon\psi^{-1}\|u_{xx}\|_{L^2(\Omega_R)}^2, \end{aligned} \tag{3.20}$$

where $\theta = \frac{4}{p^2+2p}$. We use (3.10) to get

$$\begin{aligned}&\int_{-R}^{R}|\pi|^{\frac{p}{p-1}}\,\mathrm{d}x \\ &\quad= \int_{-R}^{R}\left|\pi(0)\right|^{\frac{p}{p-1}}\mathrm{d}x + \int_0^t\frac{\partial}{\partial s}\left(\int_{-R}^{R}\left(\pi(s)\right)^{\frac{p}{p-1}}\mathrm{d}x\right)\mathrm{d}s \\ &\quad= \int_{-R}^{R}\left|\pi(0)\right|^{\frac{p}{p-1}}\mathrm{d}x + \frac{p}{p-1}\int_0^t\int_{-R}^{R}\gamma(\rho)^{\gamma-1}\pi^{\frac{1}{p-1}}(-\rho_x u-\rho u_x)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\leq C + C\int_0^t\|\rho\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\pi\|_{L^\infty(\Omega_R)}^{\frac{1}{p-1}}\int_{-R}^{R}\left(\left|\rho_x\Phi\Phi^{-1}u\right| + \left|\rho\Phi\Phi^{-1}u_x\right|\right)\mathrm{d}x\,\mathrm{d}s \\ &\quad\leq C + C\int_0^t\|\rho\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\pi\|_{L^\infty(\Omega_R)}^{\frac{1}{p-1}}\left(\|\rho_x\Phi\|_{L^2(\Omega_R)}\left\|\Phi^{-\frac{1}{2}}u\right\|_{L^{2p}(\Omega_R)}\left\|\Phi^{-\frac{1}{2}}\right\|_{L^{\frac{2p}{p-1}}(\Omega_R)}\right. \\ &\qquad\left. + \|\rho\Phi\|_{L^2(\Omega_R)}\left\|\Phi^{-1}\right\|_{L^{\frac{2p}{p-2}}(\Omega_R)}\|u_x\|_{L^p(\Omega_R)}\right)\mathrm{d}s \\ &\quad\leq C + \int_0^t\psi^\beta\,\mathrm{d}s. \end{aligned} \tag{3.21}$$

By virtue of

$$\begin{aligned}&\int_0^{u_x^2}(s+\mu_0)^{\frac{p-2}{2}}\,\mathrm{d}s \\ &\quad\geq \int_0^{u_x^2}\mu_0^{\frac{p-2}{2}}\,\mathrm{d}s \\ &\quad\geq \frac{1}{2}\mu_0^{\frac{p-2}{2}}|u_x|^2, \end{aligned} \tag{3.22}$$

and the arguments as in [8], we have

$$\sup_{0\leq s\leq t}\|u_x\|_{L^2(\Omega_R)}^2 \leq C + C\int_0^t\psi^\beta\,\mathrm{d}s. \tag{3.23}$$

Putting (3.16), (3.17), (3.20), (3.21), and (3.23) into (3.14) and choosing ε suitably small yield

$$\int_{-R}^{R}|u|^2\,\mathrm{d}x + \int_{-R}^{R}|u_x|^2\,\mathrm{d}x + \int_{-R}^{R}|u_x|^p\,\mathrm{d}x + \int_0^t\int_{-R}^{R}\rho|u_t|^2\,\mathrm{d}x\,\mathrm{d}s \leq C + \int_0^t\psi^\beta\,\mathrm{d}s.$$

The proof of Lemma 3.2 is finished. □

Lemma 3.3 *Let* (ρ, u) *and* T^* *be as in Lemma* 3.2. *Then, for all* $t \in (0, T^*]$,

$$\sup_{0\le s\le t} s\int_{-R}^{R}\rho|u_t|^2\,\mathrm{d}x+\int_0^t s\int_{-R}^{R}\bigl(|u_{xt}|^2+|u_t|^2\bigr)\,\mathrm{d}x\,\mathrm{d}s\le C\exp\biggl\{C\int_0^t\psi^\beta\,\mathrm{d}s\biggr\}. \tag{3.24}$$

Proof We differentiate equation $(2.2)_2$ with respect to t, and multiply it by u_t, by using (1.3), and integrating it over $(-R, R)$ with respect to x we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R}\rho|u_t|^2\,\mathrm{d}x+\int_{-R}^{R}|u_{xt}|^2\,\mathrm{d}x+\int_{-R}^{R}|u_t|^2\,\mathrm{d}x\\
&\quad\le C\int_{-R}^{R}|\pi_x||u||u_{xt}|\,\mathrm{d}x+C\int_{-R}^{R}|\pi||u_x||u_{xt}|\,\mathrm{d}x+C\int_{-R}^{R}\rho|u||u_t||u_{xt}|\,\mathrm{d}x\\
&\qquad+C\int_{-R}^{R}\rho|u||u_x|^2|u_t|\,\mathrm{d}x+C\int_{-R}^{R}\rho|u|^2|u_{xx}||u_t|\,\mathrm{d}x+C\int_{-R}^{R}\rho|u|^2|u_x||u_{xt}|\,\mathrm{d}x\\
&\qquad+C\int_{-R}^{R}\rho|u_x||u_t|^2\,\mathrm{d}x\\
&\quad=C\sum_{i=1}^{7}I_j.
\end{aligned} \tag{3.25}$$

We estimate each term I_j. We deduce from the Gagliardo-Nirenberg inequality and (3.10) that

$$\begin{aligned}
I_1&=\int_{-R}^{R}|\pi_x||u||u_{xt}|\,\mathrm{d}x\\
&\le C\|\rho\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\Phi\rho_x\|_{L^q(\Omega_R)}\|u_{xt}\|_{L^2(\Omega_R)}\bigl\|\Phi^{-\frac{1}{2}}u\bigr\|_{L^{2q}(\Omega_R)}\bigl\|\Phi^{-\frac{1}{2}}\bigr\|_{L^{\frac{2q}{q-3}}(\Omega_R)}\\
&\le\varepsilon\|u_{xt}\|_{L^2(\Omega_R)}^2+C(\varepsilon)\psi^\beta
\end{aligned} \tag{3.26}$$

and that

$$\begin{aligned}
I_2&=\int_{-R}^{R}|\pi||u_x||u_{xt}|\,\mathrm{d}x\\
&\le C\|\rho\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\Phi\rho\|_{L^q(\Omega_R)}\|u_{xt}\|_{L^2(\Omega_R)}\|u_x\|_{L^{\frac{q^2}{q-2}}(\Omega_R)}\bigl\|\Phi^{-1}\bigr\|_{L^{\frac{2q^2}{(q-2)^2}}(\Omega_R)}\\
&\le C\psi^\beta\psi^{1-\theta}\|u_{xx}\|_{L^2(\Omega_R)}^\theta\|u_{xt}\|_{L^2(\Omega_R)}\\
&\le C\psi^\beta\bigl(\psi+\|u_{xx}\|_{L^2(\Omega_R)}\bigr)\|u_{xt}\|_{L^2(\Omega_R)}\\
&\le\varepsilon\|u_{xt}\|_{L^2(\Omega_R)}^2+C(\varepsilon)\psi^\beta\|u_{xx}\|_{L^2(\Omega_R)}^2+C\psi^\beta,
\end{aligned} \tag{3.27}$$

where $\theta=\frac{2q^2-2qp+4p}{q^2(2+p)}$.

Using the Gagliardo-Nirenberg inequality and (3.9), we have

$$\begin{aligned}
I_3&=\int_{-R}^{R}\rho|u||u_t||u_{xt}|\,\mathrm{d}x\\
&\le C\bigl\|\rho^{\frac{1}{2}}u\bigr\|_{L^{2p}(\Omega_R)}\bigl\|\rho^{\frac{1}{2}}u_t\bigr\|_{L^2(\Omega_R)}^{\frac{1}{2}}\bigl\|\rho^{\frac{1}{2}}u_t\bigr\|_{L^{\frac{4p}{p-2}}(\Omega_R)}^{\frac{1}{2}}\|u_{xt}\|_{L^2(\Omega_R)}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\|u_{xt}\|_{L^2(\Omega_R)}\psi^{\beta}\left(\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}+\|u_{xt}\|_{L^2(\Omega_R)}\right)^{\frac{1}{2}} \\
&\le C\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\|u_{xt}\|_{L^2(\Omega_R)}\psi^{\beta}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}} \\
&\quad + C\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\|u_{xt}\|_{L^2(\Omega_R)}\psi^{\beta}\|u_{xt}\|_{L^2(\Omega_R)}^{\frac{1}{2}} \\
&\le \varepsilon\|u_{xt}\|_{L^2(\Omega_R)}^2 + C\psi^{\beta}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^2
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
I_4 &= \int_{-R}^{R}\rho|u||u_x|^2|u_t|\,\mathrm{d}x \\
&\le \|\rho\Phi\|_{L^\infty(\Omega_R)}^{\frac{1}{2}}\left\|\Phi^{-\frac{1}{4}}u\right\|_{L^{2p}(\Omega_R)}\left\|\Phi^{-\frac{1}{4}}\right\|_{L^{\frac{2p}{p-3}}(\Omega_R)} \\
&\quad \times \left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^{\frac{2p}{p-2}}(\Omega_R)}^{\frac{1}{2}}\|u_x\|_{L^p(\Omega_R)}\|u_x\|_{L^{2p}(\Omega_R)} \\
&\le \psi^{\beta}\psi^{1-\theta}\|u_{xx}\|_{L^2(\Omega_R)}^{\theta}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\left(\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}+\|u_{xt}\|_{L^2(\Omega_R)}\right)^{\frac{1}{2}} \\
&\le \psi^{\beta}\left(\psi+\|u_{xx}\|_{L^2(\Omega_R)}\right)\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\left(\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}+\|u_{xt}\|_{L^2(\Omega_R)}^{\frac{1}{2}}\right) \\
&\le \psi^{\beta}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}+\psi^{\beta}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\|u_{xt}\|_{L^2(\Omega_R)}^{\frac{1}{2}} \\
&\quad + \psi^{\beta}\|u_{xx}\|_{L^2(\Omega_R)}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\|u_{xt}\|_{L^2(\Omega_R)}^{\frac{1}{2}} \\
&\quad + \psi^{\beta}\|u_{xx}\|_{L^2(\Omega_R)}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)} \\
&\le \varepsilon\|u_{xt}\|_{L^2(\Omega_R)}^2 + C(\varepsilon)\psi^{\beta}\left(\|u_{xx}\|_{L^2(\Omega_R)}^2+\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^2+1\right),
\end{aligned}
\tag{3.29}
$$

where $\theta = \frac{1}{p+2}$. By (3.9) and (3.10), we get

$$
\begin{aligned}
I_5 &= \int_0^1 \rho|u|^2|u_{xx}||u_t|\,\mathrm{d}x \\
&\le C\left\|\rho^{\frac{1}{4}}u\right\|_{L^{4p}(\Omega_R)}^2\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^{\frac{2p}{p-2}}(\Omega_R)}^{\frac{1}{2}}\|u_{xx}\|_{L^2(\Omega_R)} \\
&\le C\psi\|u_{xx}\|_{L^2(\Omega_R)}\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}\left(\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^{\frac{1}{2}}+\|u_{xt}\|_{L^2(\Omega_R)}^{\frac{1}{2}}\right) \\
&\le C\psi^{\beta}\|u_{xx}\|_{L^2(\Omega_R)}^2+\psi^2\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^2+\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}\|u_{xt}\|_{L^2(\Omega_R)} \\
&\le \varepsilon\|u_{xt}\|_{L^2(\Omega_R)}^2 + C(\varepsilon)\psi^{\beta}\left(\|u_{xx}\|_{L^2(\Omega_R)}^2+\left\|\rho^{\frac{1}{2}}u_t\right\|_{L^2(\Omega_R)}^2+1\right).
\end{aligned}
\tag{3.30}
$$

Young's inequality together with (3.9), (3.10) yields

$$
\begin{aligned}
I_6 &= \int_{-R}^{R}\rho|u|^2|u_x||u_{xt}|\,\mathrm{d}x \\
&\le C\|\rho\Phi\|_{L^q(\Omega_R)}\left\|\Phi^{-\frac{1}{4}}u\right\|_{L^{\frac{4pq}{\varepsilon_0}}(\Omega_R)}^2\|u_x\|_{L^p(\Omega_R)}\|u_{xt}\|_{L^2(\Omega_R)}\left\|\Phi^{-\frac{1}{2}}\right\|_{L^{\frac{2pq}{pq-2p-2q-\varepsilon_0}}(\Omega_R)} \\
&\le \varepsilon\|u_{xt}\|_{L^2(\Omega_R)}^2 + C(\varepsilon)\psi^{\beta},
\end{aligned}
\tag{3.31}
$$

where $0 < \varepsilon_0 < \min\{1, pq - 2p - 2q\}$, and

$$\begin{aligned} I_7 &= \int_{-R}^{R} \rho |u_x||u_t|^2 \,\mathrm{d}x \\ &\le \|u_x\|_{L^p(\Omega_R)} \|\rho^{\frac{1}{2}} u_t\|_{L^2(\Omega_R)}^{\frac{1}{2}} \|\rho^{\frac{1}{2}} u_t\|_{L^{\frac{6p}{3p-4}}(\Omega_R)}^{\frac{3}{2}} \\ &\le \psi^\beta \|\rho^{\frac{1}{2}} u_t\|_{L^2(\Omega_R)}^{\frac{1}{2}} \big(\|\rho^{\frac{1}{2}} u_t\|_{L^2(\Omega_R)}^{\frac{3}{2}} + \|u_{xt}\|_{L^2(\Omega_R)}^{\frac{3}{2}}\big) \\ &\le \varepsilon \|u_{xt}\|_{L^2(\Omega_R)}^2 + C(\varepsilon)\psi^\beta \|\rho^{\frac{1}{2}} u_t\|_{L^2(\Omega_R)}^2. \end{aligned} \tag{3.32}$$

Substituting (3.26)-(3.32) into (3.25) and choosing ε suitably small lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R} \rho|u_t|^2 \,\mathrm{d}x + \int_{-R}^{R} \big(|u_{xt}|^2 + |u_t|^2\big)\,\mathrm{d}x \le C\psi^\beta \|\rho^{1/2} u_t\|_{L^2(\Omega_R)}^2 + C\psi^\beta, \tag{3.33}$$

where in the last inequality we have used (3.18). Multiplying (3.33) by t, we obtain (3.24) after using Gronwall's inequality and (3.4). The proof of Lemma 3.3 is completed. □

Lemma 3.4 *Let* (ρ, u) *and* T^* *be as in Lemma* 3.2. *Then, for all* $t \in (0, T^*]$,

$$\sup_{0\le s\le t} \big(\|\rho\Phi\|_{L^1(\Omega_R)\cap H^1(\Omega_R)\cap W^{1,q}(\Omega_R)}\big) \le \exp\left\{C\exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}x\right\}\right\}. \tag{3.34}$$

Proof First, multiplying $(2.2)_1$ by Φ and integrating the resulting equality over Ω_R, we obtain after integration by parts and using (3.10)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R} \rho\Phi \,\mathrm{d}x &\le \int_{-R}^{R} \rho|u|\Phi_x \,\mathrm{d}x \\ &\le C\int_{-R}^{R} \rho|u|\big(e+x^2\big)^{\zeta_0} x \,\mathrm{d}x \\ &\le C\int_{-R}^{R} \rho|u|\big(e+x^2\big)^{\zeta_0}\big(e+x^2\big)^{\frac{1}{2}} \,\mathrm{d}x \\ &\le C\|\rho\Phi\|_{L^2(\Omega_R)} \big\|u\big(e+x^2\big)^{-\frac{1}{2}}\big\|_{L^2(\Omega_R)} \\ &\le C\|\rho\Phi\|_{L^2(\Omega_R)} \big\|u\big(e+x^2\big)^{-\frac{1}{4}}\big(e+x^2\big)^{-\frac{1}{4}}\big\|_{L^2(\Omega_R)} \\ &\le C\|\rho\Phi\|_{L^2(\Omega_R)} \big\|u\Phi^{-\frac{1}{4(1+\zeta_0)}}\big\|_{L^{2p}(\Omega_R)} \big\|\big(e+x^2\big)^{-\frac{1}{4}}\big\|_{L^{\frac{2p}{p-1}}(\Omega_R)} \le C\psi^\beta, \end{aligned}$$

which gives

$$\sup_{0\le s\le t}\int_{-R}^{R} \rho\Phi \,\mathrm{d}x \le C\exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}x\right\}. \tag{3.35}$$

Next, it follows from the Gagliardo-Nirenberg inequality and (3.10) that, for $0 < \delta < 1$,

$$\begin{aligned} \big\|u\Phi^{-\delta}\big\|_{L^\infty(\Omega_R)} &\le C\big\|u\Phi^{-\delta}\big\|_{L^k(\Omega_R)}^{1-\theta} \big\|\big(u\Phi^{-\delta}\big)_x\big\|_{L^k(\Omega_R)}^{\theta} \\ &\le C\big(\big\|u\Phi^{-\delta}\big\|_{L^k(\Omega_R)} + \big\|u_x\Phi^{-\delta}\big\|_{L^k(\Omega_R)} + \big\|u\Phi_x^{-\delta}\big\|_{L^k(\Omega_R)}\big) \end{aligned}$$

$$\begin{aligned}&\le C\big(\psi^{\beta}+\|u_{xx}\|_{L^2(\Omega_R)}+\big\|u\Phi^{-\delta}\big\|_{L^{2k}(\Omega_R)}\big\|\Phi^{-1}\Phi_x\big\|_{L^{2k}(\Omega_R)}\big)\\&\le C\big(\psi^{\beta}+\|u_{xx}\|_{L^2(\Omega_R)}\big),\end{aligned}\tag{3.36}$$

where $\theta=\frac{1}{k}$.

One derives from $(1.1)_1$ that $g\triangleq\rho\Phi$ satisfies

$$g_t+ug_x-agu(\ln\Phi)_x+gu_x=0,$$

which together with (3.36) gives

$$\begin{aligned}&\big(\|g_x\|_{L^2(\Omega_R)}\big)_t\\&\quad\le C\big(1+\|u_x\|_{L^\infty(\Omega_R)}+\big\|u(\ln\Phi)_x\big\|_{L^\infty(\Omega_R)}\big)\|g_x\|_{L^2(\Omega_R)}\\&\qquad+C\big(\big\||u_x|\big|(\ln\Phi)_x\big|\big\|_{L^2(\Omega_R)}+\big\||u|\big|(\ln\Phi)_{xx}\big|\big\|_{L^2(\Omega_R)}+\|u_{xx}\|_{L^2(\Omega_R)}\big)\|g\|_{L^\infty(\Omega_R)}\\&\quad\le C\big(\psi^{\beta}+\|u_x\|_{W^{1,q}(\Omega_R)}+\big\|u\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^\infty(\Omega_R)}\big)\|g_x\|_{L^2(\Omega_R)}\\&\qquad+C\big(\|u_{xx}\|_{L^2(\Omega_R)}+\|u_x\|_{L^p(\Omega_R)}\big\|\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^{\frac{2p}{p-2}}(\Omega_R)}\\&\qquad+\big\|u\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^{2p}(\Omega_R)}\big\|\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^{\frac{2p}{p-1}}(\Omega_R)}\big)\|g\|_{L^\infty(\Omega_R)}\\&\quad\le C\big(\psi^{\beta}+\|u_{xx}\|_{L^q(\Omega_R)}\big)\|g_x\|_{L^2(\Omega_R)}+C\big(\psi^{\beta}+\|u_{xx}\|_{L^2(\Omega_R)}\big)\|g\|_{L^\infty(\Omega_R)}\\&\quad\le C\big(\psi^{\beta}+\|u_{xx}\|_{L^2(\Omega_R)\cap L^q(\Omega_R)}\big)\big(1+\|g_x\|_{L^2(\Omega_R)}+\|g_x\|_{L^q(\Omega_R)}\big)\end{aligned}\tag{3.37}$$

and

$$\begin{aligned}&\big(\|g_x\|_{L^q(\Omega_R)}\big)_t\\&\quad\le C\big(1+\|u_x\|_{L^\infty(\Omega_R)}+\big\|u(\ln\Phi)_x\big\|_{L^\infty(\Omega_R)}\big)\|g_x\|_{L^q(\Omega_R)}\\&\qquad+C\big(\big\||u_x|\big|(\ln\Phi)_x\big|\big\|_{L^q(\Omega_R)}+\big\||u|\big|(\ln\Phi)_{xx}\big|\big\|_{L^q(\Omega_R)}+\|u_{xx}\|_{L^q(\Omega_R)}\big)\|g\|_{L^\infty(\Omega_R)}\\&\quad\le C\big(\psi^{\beta}+\|u_x\|_{W^{1,q}(\Omega_R)}+\big\|u\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^\infty(\Omega_R)}\big)\|g_x\|_{L^q(\Omega_R)}\\&\qquad+C\big(\|u_x\|_{L^q(\Omega_R)}+\big\|u\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^{2q}(\Omega_R)}\big\|\Phi^{-\frac{1}{2(1+\zeta_0)}}\big\|_{L^{2q}(\Omega_R)}+\|u_{xx}\|_{L^q(\Omega_R)}\big)\|g\|_{L^\infty(\Omega_R)}\\&\quad\le C\big(\psi^{\beta}+\|u_{xx}\|_{L^q(\Omega_R)}\big)\|g_x\|_{L^q(\Omega_R)}+C\big(\psi^{\beta}+\|u_{xx}\|_{L^2(\Omega_R)\cap L^q(\Omega_R)}\big)\|g\|_{L^\infty(\Omega_R)}\\&\quad\le C\big(\psi^{\beta}+\|u_{xx}\|_{L^2(\Omega_R)\cap L^q(\Omega_R)}\big)\big(1+\|g_x\|_{L^q(\Omega_R)}\big),\end{aligned}\tag{3.38}$$

where in the last inequality we have used (3.35).

Next, we claim that

$$\int_0^t\big(\|u_{xx}\|_{L^2(\Omega_R)\cap L^q(\Omega_R)}^{(q+1)/q}+t\|u_{xx}\|_{L^2(\Omega_R)\cap L^q(\Omega_R)}^2\big)\,\mathrm{d}x\le C\exp\bigg\{C\int_0^t\psi^{\beta}\,\mathrm{d}t\bigg\},\tag{3.39}$$

which together with (3.35), (3.37), (3.38), and the Gronwall inequality yields

$$\sup_{0\le s\le t}\|\rho\Phi\|_{L^1(\Omega_R)\cap H^1(\Omega_R)\cap W^{1,q}(\Omega_R)}\le\exp\bigg\{C\exp\bigg\{C\int_0^t\psi^{\beta}\,\mathrm{d}x\bigg\}\bigg\}.$$

Finally, it only remains to prove (3.39). In fact, on the one hand, it follows from (3.18), (3.4), and (3.24) that

$$\begin{aligned}
&\int_0^t \left(\|u_{xx}\|_{L^2(\Omega_R)}^{\frac{q+1}{q}} + \|u_{xx}\|_{L^2(\Omega_R)}^2\right) \mathrm{d}s \\
&\quad\le \int_0^t \left[\left(C\psi^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)} + C\psi^\beta\right)^{\frac{q+1}{q}} + s\left(C\psi^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)} + C\psi^\beta\right)^2\right] \mathrm{d}s \\
&\quad\le \int_0^t \left[C\psi^{\frac{q+1}{2q}}\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)}^{\frac{q+1}{q}} + C\left(\psi^\beta\right)^{\frac{q+1}{q}} + C\psi s\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)}^2 + Cs\left(\psi^\beta\right)^2\right] \mathrm{d}s \\
&\quad\le \int_0^t \left[C\psi^{\frac{q+1}{q-1}} + \frac{q+1}{2q}\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)}^2 + C\psi \exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}s\right\} + Cs\psi^\beta\right] \mathrm{d}s \\
&\quad\le \int_0^t \left(\|\sqrt{\rho}u_t\|_{L^2(\Omega_R)}^2 + \psi^\beta\right) \mathrm{d}s + C\exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}s\right\}\int_0^t \psi^\beta \,\mathrm{d}s \\
&\quad\le C\exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}s\right\}.
\end{aligned} \tag{3.40}$$

On the other hand, we denote

$$\dot{u} \triangleq u_t + u\cdot u_x.$$

By (3.19) and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\|u_{xx}\|_{L^q(\Omega_R)} &\le C\left(\|\rho\dot{u}\|_{L^q(\Omega_R)} + \|\pi_x\|_{L^q(\Omega_R)} + \|f\|_{L^q(\Omega_R)}\right) \\
&\le C\psi^\beta + C\|\rho\dot{u}\|_{L^q(\Omega_R)} + \|f\|_{L^2(\Omega_R)}^{1-\theta}\|f_x\|_{L^2(\Omega_R)}^\theta \\
&\le C\psi^\beta + C\|\rho\dot{u}\|_{L^q(\Omega_R)},
\end{aligned} \tag{3.41}$$

where $\theta = \frac{q-2}{2q}$.

By (3.9), (3.10), and (3.15), the last term on the right-hand side of (3.41) can be estimated as follows:

$$\begin{aligned}
\|\rho\dot{u}\|_{L^q(\Omega_R)} &\le \|\rho u_t\|_{L^q(\Omega_R)} + \|\rho u u_x\|_{L^q(\Omega_R)} \\
&\le \|\rho u_t\|_{L^2(\Omega_R)}^{2(q-1)/(q^2-2)}\|\rho u_t\|_{L^{q^2}(\Omega_R)}^{(q^2-2q)/(q^2-2)} + \|\rho u\|_{L^{2q}(\Omega_R)}\|u_x\|_{L^{2q}(\Omega_R)} \\
&\le C\psi^\beta\left(\left\|\rho^{1/2}u_t\right\|_{L^2(\Omega_R)}^{2(q-1)/(q^2-2)}\|u_{xt}\|_{L^2(\Omega_R)}^{(q^2-2q)/(q^2-2)} + \left\|\rho^{1/2}u_t\right\|_{L^2(\Omega_R)}\right) \\
&\quad + C\psi^\beta\|u_{xx}\|_{L^2(\Omega_R)}^{\frac{2q-p}{q(p+2)}}.
\end{aligned}$$

This combined with (3.40), (3.24), and (3.4) yields

$$\begin{aligned}
\int_0^t \|\rho\dot{u}\|_{L^q(\Omega_R)}^{\frac{q+1}{q}} &\le C\int_0^t \psi^\beta t^{-\frac{q+1}{2q}}\left(t\left\|\rho^{1/2}u_t\right\|_{L^2(\Omega_R)}^2\right)^{\frac{q^2-1}{q(q^2-2)}}\left(t\|u_{xt}\|_{L^2(\Omega_R)}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}} \,\mathrm{d}t \\
&\quad + C\int_0^t \left\|\rho^{1/2}u_t\right\|_{L^2(\Omega_R)}^2 \,\mathrm{d}t + C\exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}s\right\} \\
&\le C\exp\left\{C\int_0^t \psi^\beta \,\mathrm{d}s\right\}\int_0^t \left(\psi^\beta + t^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}} + t\|u_{xt}\|_{L^2(\Omega_R)}^2\right) \mathrm{d}t
\end{aligned}$$

$$\begin{aligned}&+C\exp\left\{C\int_0^t\psi^\beta\,\mathrm{d}s\right\}\\&\le C\exp\left\{C\int_0^t\psi^\beta\,\mathrm{d}s\right\},\end{aligned} \tag{3.42}$$

$$\begin{aligned}t^{\frac{1}{2}}\|u_{xx}\|_{L^2(\Omega_R)}&\le C\psi^\beta+Ct\|\sqrt{\rho}u_t\|^2_{L^2(\Omega_R)}\\&\le C\psi^\beta+C\exp\left\{C\int_0^t\psi^\beta\,\mathrm{d}s\right\},\end{aligned} \tag{3.43}$$

and that

$$\int_0^t t\|\rho\dot{u}\|^2_{L^q(\Omega_R)}\le C\exp\left\{C\int_0^t\psi^\beta\,\mathrm{d}s\right\}. \tag{3.44}$$

One thus obtains (3.39) from (3.40)-(3.44) and finishes the proof of Lemma 3.4. □

Proof of Proposition 3.1 It follows from (3.5), (3.4), and (3.33) that, for all $\forall t\in(0,T^*]$,

$$\psi(t)\le\exp\left\{C\exp\left\{C\int_0^t\psi^\beta\,\mathrm{d}x\right\}\right\}.$$

Standard arguments thus yield, for $\tilde{K}\triangleq e^{Ce}$ and $T_0\triangleq\min\{T^*,(CM^\beta)^{-1}\}$,

$$\sup_{0\le t\le T_0}\psi(t)\le\tilde{K},$$

which together with (3.18), (3.39), and (3.4) gives (3.3). The proof of Proposition 3.1 is thus completed. □

4 Proof of Theorem 1.2

Let (ρ_0,u_0) be as in Theorem 1.2. Without loss of generality, assume that

$$\int_{-\infty}^{+\infty}\rho_0\,\mathrm{d}x=1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{-N_0}^{N_0}\rho_0\,\mathrm{d}x\ge\frac{3}{4}\int_{-\infty}^{+\infty}\rho_0\,\mathrm{d}x=\frac{3}{4}. \tag{4.1}$$

We construct $\rho_0^R=j_{\frac{1}{R}}*\rho_0+R^{-1}e^{-x^2}$ where $j_{\frac{1}{R}}*\rho_0$ satisfies

$$\int_{-N_0}^{N_0}j_{\frac{1}{R}}*\rho_0\,\mathrm{d}x\ge 1/2, \tag{4.2}$$

and

$$\Phi(j_{\frac{1}{R}}*\rho_0)\to\Phi\rho_0,\quad\text{in } L^1(\mathbb{R})\cap H^1(\mathbb{R})\cap W^{1,q}(\mathbb{R}), \tag{4.3}$$

as $R \to \infty$, since $u_0 \in L^2(\mathbb{R})$ and $u_{0x} \in L^2(\mathbb{R})$, choosing $v^R \in C_0^\infty(\Omega_R)$ such that

$$\begin{cases} \lim_{R\to\infty} \|v^R - u_0\|_{L^2(\mathbb{R})} = 0, \\ \lim_{R\to\infty} \|v^R_x - u_{0x}\|_{L^2(\mathbb{R})} = 0, \\ \|v^R_{xx}\|_{L^2(\mathbb{R})} \le C, \end{cases} \tag{4.4}$$

we consider a smooth solution u_0^R of the following elliptic problem (see [21]):

$$-u^R_{0xx} + u_0^R = -\rho_0^R u_0^R + \sqrt{\rho_0^R} h^R - v^R_{xx} + v^R, \tag{4.5}$$

$$u_0^R(-R) = u_0^R(R) = 0, \tag{4.6}$$

where $d^R = (\sqrt{\rho_0} u_0) * j_{1/R}$, with j_δ being the standard mollifying kernel of width δ. Extending u_0^R to $\mathbb{R}$ by defining 0 outside Ω_R and denoting $w_0^R \triangleq u_0^R \tilde{\xi}_R$ with $\tilde{\xi}_R$ as in (3.6), we claim that

$$\lim_{R\to\infty} \Big(\big\| (w_0^R - u_0) \big\|_{L^2(\mathbb{R})} + \big\| (w^R_{0x} - u_{0x}) \big\|_{L^2(\mathbb{R})} + \Big\| \sqrt{\rho_0^R} w_0^R - \sqrt{\rho_0} u_0 \Big\|_{L^2(\mathbb{R})} \Big) = 0. \tag{4.7}$$

In fact, multiplying (4.5) by u_0^R and integrating the resulting equation over Ω_R lead to

$$\begin{aligned} &\int_{\Omega_R} (\rho_0^R + 1) |u_0^R|^2 \,\mathrm{d}x + \int_{\Omega_R} |u^R_{0x}|^2 \,\mathrm{d}x \\ &\quad \le C \|v^R_x\|_{L^2(\Omega_R)} \|u^R_{0x}\|_{L^2(\Omega_R)} \\ &\qquad + \Big\| \sqrt{\rho_0^R} u_0^R \Big\|_{L^2(\Omega_R)} \|d^R\|_{L^2(\Omega_R)} + \|v^R\|_{L^2(\Omega_R)} \|u_0^R\|_{L^2(\Omega_R)} \\ &\quad \le \varepsilon \|u_0^R\|^2_{L^2(\Omega_R)} + \varepsilon \|u^R_{0x}\|^2_{L^2(\Omega_R)} + \varepsilon \int_{\Omega_R} \rho_0^R |u_0^R|^2 \,\mathrm{d}x + C(\varepsilon), \end{aligned}$$

which implies

$$\int_{\Omega_R} |u_0^R|^2 \,\mathrm{d}x + \int_{\Omega_R} \rho_0^R |u_0^R|^2 \,\mathrm{d}x + \int_{\Omega_R} |u^R_{0x}|^2 \,\mathrm{d}x \le C, \tag{4.8}$$

for some C independent of R.

By the Gagliardo-Nirenberg inequality and (4.3), we obtain

$$\begin{aligned} \|\rho_0^R\|_{L^\infty(\Omega_R)} &= \|j_{\frac{1}{R}} * \rho_0^R + R^{-1} e^{-x^2}\|_{L^\infty(\Omega_R)} \\ &\le 1 + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\Omega_R)} \\ &\le 1 + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\mathbb{R})} \\ &\le 1 + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\mathbb{R})} \\ &\le 1 + C\|\Phi j_{\frac{1}{R}} * \rho_0\|_{L^2(\mathbb{R})} + C\big\|(j_{\frac{1}{R}} * \rho_0)_x\big\|_{L^2(\mathbb{R})} \\ &\le C. \end{aligned} \tag{4.9}$$

Combining (4.4), (4.8), and (4.9), we get

$$\begin{aligned}
&\left\|u_{0xx}^R\right\|_{L^2(\Omega_R)} \\
&\quad \le \left\|u_0^R\right\|_{L^2(\Omega_R)} + \left\|\rho_0^R\right\|_{L^\infty(\Omega_R)}^{\frac{1}{2}} \left\|\sqrt{\rho_0^R} u_0^R\right\|_{L^2(\Omega_R)} \\
&\qquad + \left\|\rho_0^R\right\|_{L^\infty(\Omega_R)}^{\frac{1}{2}} \left\|d^R\right\|_{L^2(\Omega_R)} + \left\|v_{xx}^R\right\|_{L^2(\Omega_R)} + \left\|v^R\right\|_{L^2(\Omega_R)} \\
&\quad \le C.
\end{aligned} \tag{4.10}$$

We deduce from (4.8) and (4.13) that there exist a subsequence $R_j \to \infty$ and a function $w_0 \in \{w_0 \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}) | \sqrt{\rho_0} w_0 \in L^2(\mathbb{R})\}$ such that

$$\begin{cases}
\sqrt{\rho_0^{R_j}} w_0^{R_j} \rightharpoonup \sqrt{\rho_0} w_0, & \text{weakly in } L^2(\mathbb{R}), \\
w_0^{R_j} \to w_0, & \text{weakly in } L^2(\mathbb{R}), \\
w_{0x}^{R_j} \to w_{0x}, & \text{weakly in } L^2(\mathbb{R}).
\end{cases} \tag{4.11}$$

It follows from (4.5), (4.6), and (4.8) that w_0^R satisfies

$$-w_{0xx}^R + w_0^R = -\rho_0^R w_0^R + \sqrt{\rho_0^R} h^R \tilde{\xi}_R - v_{xx}^R \tilde{\xi}_R + v^R \tilde{\xi}_R + R^{-1} F_R, \tag{4.12}$$

with $\int_{\mathbb{R}} (F_R)^2 \,\mathrm{d}x \le C$.

Thus, one can deduce from (4.21)-(4.24), for any $\psi \in C_0^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} (w_0 - u_0) \cdot \psi \,\mathrm{d}x + \int_{\mathbb{R}} (w_{0x} - u_{0x}) \cdot \psi_x \,\mathrm{d}x + \int_{\mathbb{R}} \rho_0 (w_0 - u_0) \cdot \psi \,\mathrm{d}x = 0,$$

which yields

$$w_0 = u_0. \tag{4.13}$$

We get from (4.12)

$$\limsup_{R_j \to \infty} \int_{\mathbb{R}} \left(\left|w_0^{R_j}\right|^2 + \left|w_{0x}^{R_j}\right|^2 + \rho_0^{R_j} \left|w_0^{R_j}\right|^2\right) \mathrm{d}x \le \int_{\mathbb{R}} \left(|u_0|^2 + |u_{0x}|^2 + \rho_0 |u_0|^2\right) \mathrm{d}x,$$

which combined with (4.11) implies

$$\lim_{R_j \to \infty} \int_{\mathbb{R}} \left|w_0^{R_j}\right|^2 \mathrm{d}x = \int_{\mathbb{R}} |u_0|^2 \,\mathrm{d}x,$$

$$\lim_{R_j \to \infty} \int_{\mathbb{R}} \left|w_{0x}^{R_j}\right|^2 \mathrm{d}x = \int_{\mathbb{R}} |u_{0x}|^2 \,\mathrm{d}x,$$

and

$$\lim_{R_j \to \infty} \int_{\mathbb{R}} \rho_0^{R_j} \left|w_0^{R_j}\right|^2 \mathrm{d}x = \int_{\mathbb{R}} \rho_0 |u_0|^2 \,\mathrm{d}x.$$

This, along with (4.13) and (4.8), gives (4.7).

Then, in terms of Lemma 2.1, the initial-boundary value problem (2.2) with the initial data (ρ_0^R, w_0^R) has a classical solution (ρ^R, u^R) on $\Omega_R \times [0, T_R]$. Moreover, Proposition 3.1 shows that there exists a T_0 independent of R such that (3.2) holds for (ρ^R, u^R). We first deduce from (3.3) that

$$\begin{aligned}
&\int_0^{T_0} \|\Phi^{\frac{1}{2}}\rho_t^R\|_{L^2(\Omega_R)}^2 \, dt \\
&\leq C\int_0^{T_0} \left(\|\Phi^{\frac{1}{2}}u^R\rho_x^R\|_{L^2(\Omega_R)}^2 + \|\Phi^{\frac{1}{2}}\rho^R u_x^R\|_{L^2(\Omega_R)}^2\right) dt \\
&\leq C\int_0^{T_0} \|\Phi^{-\frac{1}{2}}u^R\|_{L^\infty(\Omega_R)}^2 \|\Phi\rho_x^R\|_{L^2(\Omega_R)}^2 \, dt + C \\
&\leq C
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&\int_0^{T_0} \|\Phi^{\frac{1}{2}}\rho_t^R\|_{L^q(\Omega_R)}^2 \, dt \\
&\leq C\int_0^{T_0} \left(\|\Phi^{\frac{1}{2}}u^R\rho_x^R\|_{L^q(\Omega_R)}^2 + \|\Phi^{\frac{1}{2}}\rho^R u_x^R\|_{L^q(\Omega_R)}^2\right) dt \\
&\leq C\int_0^{T_0} \|\Phi^{-\frac{1}{2}}u^R\|_{L^\infty(\Omega_R)}^2 \|\Phi\rho_x^R\|_{L^q(\Omega_R)}^2 \, dt + C \\
&\leq C.
\end{aligned} \tag{4.15}$$

By using (3.3), we have

$$\begin{aligned}
&\int_0^{T_0} \left\|\left((u_x^R)^2 + \mu_0\right)^{\frac{p-2}{2}} u_x^R\right\|_{L^2(\Omega_R)} dt \\
&\leq C\int_0^{T_0} \|u_x^R\|_{L^\infty(\Omega_R)}^{p-2} \|u_x^R\|_{L^2(\Omega_R)} \, dt + C\int_0^{T_0} \mu_0^{\frac{p-2}{2}} \|u_x^R\|_{L^2(\Omega_R)} \, dt \\
&\leq C\int_0^{T_0} \left(\|u_x^R\|_{L^p(\Omega_R)}^{1-\theta} \|u_{xx}^R\|_{L^2(\Omega_R)}^{\theta}\right)^{p-2} \|u_x^R\|_{L^2(\Omega_R)} \, dt + C \\
&\leq C\int_0^{T_0} \left(\|u_x^R\|_{L^p(\Omega_R)}^{1-\theta} t^{-\frac{\theta}{2}} t^{\frac{\theta}{2}} \|u_{xx}^R\|_{L^2(\Omega_R)}^{\theta}\right)^{p-2} \|u_x^R\|_{L^2(\Omega_R)} \, dt + C \\
&\leq C\int_0^{T_0} t^{-\frac{p-2}{p+2}} \, dt + C \\
&\leq C,
\end{aligned} \tag{4.16}$$

where $\theta = \frac{2}{p+2}$.

With all these estimates (4.14)-(4.16), (3.3), (3.4), (3.24), and (3.34) at hand, we find that the sequence (ρ^R, u^R) converges, up to the extraction of subsequences, to some limit (ρ^R, u^R) in the obvious weak sense, that is, as $R \to \infty$, we have

$$u^R \to u, \quad \text{weakly}^* \text{ in } L^\infty\left(0, T_0; L^2(\mathbb{R})\right), \tag{4.17}$$

$$\Phi^{\frac{1}{2}}\rho^R \to \Phi^{\frac{1}{2}}\rho, \quad \text{in } C\left(\overline{\Omega}_N \times [0, T_0]\right), \forall N > 0, \tag{4.18}$$

$$\Phi\rho^R \rightharpoonup \Phi\rho, \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;H^1(\mathbb{R})\cap W^{1,q}(\mathbb{R})\big), \tag{4.19}$$

$$\Phi\rho^R \rightharpoonup \Phi\rho, \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;L^1(\mathbb{R})\big), \tag{4.20}$$

$$\sqrt{\rho^R}u^R \rightharpoonup \sqrt{\rho}u, \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;L^2(\mathbb{R})\big), \tag{4.21}$$

$$u_x^R \rightharpoonup u_x, \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;L^2(\mathbb{R})\cap L^p(\mathbb{R})\big), \tag{4.22}$$

$$u_{xx}^R \rightharpoonup u_{xx}, \quad \text{weakly in } L^{(q+1)/q}\big(0,T_0;L^q(\mathbb{R})\big)\cap L^2\big(0,T_0;L^2(\mathbb{R})\big), \tag{4.23}$$

$$t^{1/2}u_{xx}^R \rightharpoonup t^{1/2}u_{xx}, \quad \text{weakly in } L^2\big(0,T_0;L^2(\mathbb{R})\cap L^q(\mathbb{R})\big), \tag{4.24}$$

$$t^{1/2}u_{xx}^R \rightharpoonup t^{1/2}u_{xx}, \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;L^2(\mathbb{R})\big), \tag{4.25}$$

$$\sqrt{t}\sqrt{\rho^R}u_t^R \rightharpoonup \sqrt{t}\sqrt{\rho}u_t, \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;L^2(\mathbb{R})\big), \tag{4.26}$$

$$\sqrt{t}u_{tx}^R \rightharpoonup \sqrt{t}u_{tx}, \quad \text{weakly in } L^2\big(\mathbb{R}\times(0,T_0)\big), \tag{4.27}$$

$$\inf_{0\le t\le T_0}\int_{\Omega_{2N_0}} \rho(x,t)\,\mathrm{d}x \ge \frac{1}{4}. \tag{4.28}$$

Next, for $1 < L < R$ and $\xi_L \in C^1(\Omega_R)$ such that

$$\begin{cases} 0 \le \xi_L(x) \le 1, \quad |\xi_{Lx}| \le \frac{C}{L}, \\ \xi_L(x) = 0, \quad \text{if } |x| \le \frac{L}{2}, \\ \xi_L(x) = 1, \quad \text{if } |x| > L, \end{cases} \tag{4.29}$$

multiplying (1.1) by ξ_L, and integrating over Ω_R, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^R \xi_L\rho^R\,\mathrm{d}x &= \int_{-R}^R \xi_{Lx}\rho^R u^R\,\mathrm{d}x \\ &\le \frac{C}{L}\|\rho^R\|_{L^2(\Omega_R)}\|u^R\|_{L^2(\Omega_R)} \\ &\le \frac{C}{L}. \end{aligned} \tag{4.30}$$

By virtue of ξ_L, we have

$$\begin{aligned} &\xi_L(x)(j_{\frac{1}{R}} * \rho_0)(x) - j_{\frac{1}{R}} * (\xi_{\frac{L}{2}}\rho_0)(x) \\ &\quad\le \xi_L(x)\int_{-\infty}^{+\infty} j_{\frac{1}{R}}(x-y)\rho_0(y)\,\mathrm{d}y - \int_{-\infty}^{+\infty} j_{\frac{1}{R}}(x-y)(\xi_{\frac{L}{2}}\rho_0)(y)\,\mathrm{d}y \\ &\quad\le \int_{-\infty}^{+\infty} j_{\frac{1}{R}}(x-y)\big(\xi_L(x)-\xi_{\frac{L}{2}}(y)\big)\rho_0(y)\,\mathrm{d}y \\ &\quad\le 0. \end{aligned} \tag{4.31}$$

Integrating the above inequality with respect to the time variable over $(0,t)$, we get

$$\begin{aligned} \int_{-R}^R \xi_L\rho^R\,\mathrm{d}x &\le \int_{-R}^R \xi_L\rho_0^R(x)\,\mathrm{d}x + \frac{C}{L} \\ &\le \int_{-\infty}^{+\infty} \xi_L(j_{\frac{1}{R}} * \rho_0)(x)\,\mathrm{d}x + \frac{1}{R}\int_{-R}^R \xi_L e^{-x^2}\,\mathrm{d}x + \frac{C}{L} \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^{+\infty} \big(j_{\frac{1}{R}} * (\xi_{\frac{L}{2}}\rho_0)\big)(x)\,\mathrm{d}x + \frac{1}{R}\int_{-R}^{R} \xi_L e^{-x^2}\,\mathrm{d}x + \frac{C}{L} \\
&\leq \int_{-\infty}^{+\infty} (\xi_{\frac{L}{2}}\rho_0)(x)\,\mathrm{d}x + \frac{1}{R}\int_{-R}^{R} \xi_L e^{-x^2}\,\mathrm{d}x + \frac{C}{L} \\
&\leq \int_{\frac{L}{4}}^{+\infty} \rho_0(x)\,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0(x)\,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{2}} e^{-x^2}\,\mathrm{d}x \\
&\quad + \int_{\frac{L}{2}}^{+\infty} e^{-x^2}\,\mathrm{d}x + \frac{C}{L}.
\end{aligned} \tag{4.32}$$

Multiplying $(2.2)_1$ by $\xi_L^2\rho_0^R$, and integrating over Ω_R, it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R} \big(\xi_L\rho^R\big)^2\,\mathrm{d}x &= -\int_{-R}^{R} \rho^R u_x^R \xi_L^2 \rho^R\,\mathrm{d}x + 2\int_{-R}^{R} \rho^R u^R \xi_L \xi_{Lx}\rho^R\,\mathrm{d}x \\
&\leq \big\|\xi_L\rho^R\big\|_{L^\infty(\Omega_R)}\big\|u^R\big\|_{L^2(\Omega_R)}\big\|\rho^R\big\|_{L^2(\Omega_R)} \\
&\quad + \frac{C}{L}\big\|\rho^R\big\|_{L^\infty(\Omega_R)}\big\|u^R\big\|_{L^2(\Omega_R)}\big\|\rho^R\big\|_{L^2(\Omega_R)} \\
&\leq \big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta}\big\|\rho_x^R\big\|_{L^2(\Omega_R)}^{1-\theta}\big\|u^R\big\|_{L^2(\Omega_R)}\big\|\rho^R\big\|_{L^2(\Omega_R)} \\
&\quad + \frac{C}{L}\big\|\rho^R\big\|_{L^\infty(\Omega_R)}\big\|u^R\big\|_{L^2(\Omega_R)}\big\|\rho^R\big\|_{L^2(\Omega_R)} \\
&\leq C\big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta} + \frac{C}{L},
\end{aligned} \tag{4.33}$$

where $\theta = \frac{2}{3}$.

Integrating the above inequality with respect to the time variable over $(0,t)$, we get

$$\begin{aligned}
&\int_{-R}^{R} \big(\xi_L\rho^R\big)^2\,\mathrm{d}x \\
&\quad\leq \int_{-R}^{R} \big(\xi_L\rho_0^R\big)^2\,\mathrm{d} + C\int_0^t \big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta}\,\mathrm{d}t + \frac{C}{L} \\
&\quad\leq \int_{-R}^{R} \bigg[\xi_L(j_{\frac{1}{R}} * \rho_0) + \frac{1}{R}e^{-x^2}\bigg]^2\,\mathrm{d}x + C\int_0^{T_1} \big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta}\,\mathrm{d}t + \frac{C}{L} \\
&\quad\leq \int_{-\infty}^{\infty} \big(j_{\frac{1}{R}} * (\xi_{\frac{L}{2}}\rho_0)\big)^2\,\mathrm{d}x + \frac{2}{R}\|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\Omega_R)}\int_{-R}^{R} \xi_L e^{-x^2}\,\mathrm{d}x + \frac{1}{R^2}\int_{-R}^{R} \xi_L\big(e^{-x^2}\big)^2\,\mathrm{d}x \\
&\qquad + C\int_0^t \big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta}\,\mathrm{d}t + \frac{C}{L} \\
&\quad\leq \int_{-\infty}^{\infty} (\xi_{\frac{L}{2}}\rho_0)^2\,\mathrm{d}x + \frac{2}{R}\|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\Omega_R)}\int_{-R}^{R} \xi_L e^{-x^2}\,\mathrm{d}x + \frac{1}{R^2}\int_{-R}^{R} \xi_L\big(e^{-x^2}\big)^2\,\mathrm{d}x \\
&\qquad + C\int_0^t \big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta}\,\mathrm{d}t + \frac{C}{L} \\
&\quad\leq \int_{\frac{L}{4}}^{\infty} \rho_0^2\,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0^2\,\mathrm{d}x + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\Omega_R)}\bigg(\int_{-\infty}^{-\frac{L}{2}} e^{-x^2}\,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} e^{-x^2}\,\mathrm{d}x\bigg) \\
&\qquad + \int_{-\infty}^{-\frac{L}{2}} \big(e^{-x^2}\big)^2\,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} \big(e^{-x^2}\big)^2\,\mathrm{d}x + C\int_0^t \big\|\xi_L\rho^R\big\|_{L^1(\Omega_R)}^{\theta}\,\mathrm{d}t + \frac{C}{L},
\end{aligned} \tag{4.34}$$

where $\theta = \frac{2}{3}$.

Next, by the fixed $\kappa \in (L,R)$ and by using (4.34), we have

$$\begin{aligned}
&\int_L^{\kappa} \rho^2 \,\mathrm{d}x \\
&\quad \le \int_{\frac{L}{4}}^{\infty} \rho_0^2 \,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0^2 \,\mathrm{d}x + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\Omega_R)} \left(\int_{-\infty}^{-\frac{L}{2}} e^{-x^2} \,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} e^{-x^2} \,\mathrm{d}x \right) \\
&\qquad + \int_{-\infty}^{-\frac{L}{2}} \left(e^{-x^2}\right)^2 \mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} \left(e^{-x^2}\right)^2 \mathrm{d}x + C \int_0^t \left\| \xi_L \rho^R \right\|_{L^1(\Omega_R)}^{\theta} \mathrm{d}t + \frac{C}{L},
\end{aligned} \tag{4.35}$$

where $\theta = \frac{2}{3}$.

Taking the limit on κ for inequality (4.35) we obtain, as $\kappa \to +\infty$,

$$\begin{aligned}
&\int_L^{+\infty} \rho^2 \,\mathrm{d}x \\
&\quad \le \int_{\frac{L}{4}}^{\infty} \rho_0^2 \,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0^2 \,\mathrm{d}x + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\mathbb{R})} \left(\int_{-\infty}^{-\frac{L}{2}} e^{-x^2} \,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} e^{-x^2} \,\mathrm{d}x \right) \\
&\qquad + \int_{-\infty}^{-\frac{L}{2}} \left(e^{-x^2}\right)^2 \mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} \left(e^{-x^2}\right)^2 \mathrm{d}x + \frac{C}{L} \\
&\qquad + C \int_0^t \left(\int_{\frac{L}{4}}^{+\infty} \rho_0(x) \,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0(x) \,\mathrm{d}x \right. \\
&\qquad \left. + \int_{-\infty}^{-\frac{L}{2}} e^{-x^2} \,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} e^{-x^2} \,\mathrm{d}x + \frac{C}{L} \right)^{\theta} \mathrm{d}t,
\end{aligned} \tag{4.36}$$

where $\theta = \frac{2}{3}$.

Thus, we deduce that

$$\begin{aligned}
&\int_L^{R} \rho^2 \,\mathrm{d}x \\
&\quad \le \int_{\frac{L}{4}}^{\infty} \rho_0^2 \,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0^2 \,\mathrm{d}x + \|j_{\frac{1}{R}} * \rho_0\|_{L^\infty(\mathbb{R})} \left(\int_{-\infty}^{-\frac{L}{2}} e^{-x^2} \,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} e^{-x^2} \,\mathrm{d}x \right) \\
&\qquad + \int_{-\infty}^{-\frac{L}{2}} \left(e^{-x^2}\right)^2 \mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} \left(e^{-x^2}\right)^2 \mathrm{d}x + \frac{C}{L} \\
&\qquad + C \int_0^t \left(\int_{\frac{L}{4}}^{+\infty} \rho_0(x) \,\mathrm{d}x + \int_{-\infty}^{-\frac{L}{4}} \rho_0(x) \,\mathrm{d}x \right. \\
&\qquad \left. + \int_{-\infty}^{-\frac{L}{2}} e^{-x^2} \,\mathrm{d}x + \int_{\frac{L}{2}}^{+\infty} e^{-x^2} \,\mathrm{d}x + \frac{C}{L} \right)^{\theta} \mathrm{d}t,
\end{aligned} \tag{4.37}$$

where $\theta = \frac{2}{3}$. In the same way, we can also handle $\int_{-R}^{-L} \rho^2 \,\mathrm{d}x$.

By the Sobolev embedding theorem, we have

$$\int_{-L}^{L} \left(\rho^R - \rho\right)^2 \mathrm{d}x \to 0, \quad R \to +\infty. \tag{4.38}$$

By virtue of

$$\begin{aligned}
&\int_{-R}^{R}(\rho^R-\rho)^2\,\mathrm{d}x \\
&\quad\le \int_{-L}^{L}(\rho^R-\rho)^2\,\mathrm{d}x+\int_{L}^{R}(\rho^R)^2\,\mathrm{d}x+\int_{L}^{R}\rho^2\,\mathrm{d}x+\int_{-R}^{-L}(\rho^R)^2\,\mathrm{d}x+\int_{-R}^{-L}\rho^2\,\mathrm{d}x \\
&\quad\le \int_{-L}^{L}(\rho^R-\rho)^2\,\mathrm{d}x+\int_{-R}^{R}(\xi_L\rho^R)^2\,\mathrm{d}x+\int_{L}^{R}\rho^2\,\mathrm{d}x+\int_{-R}^{-L}\rho^2\,\mathrm{d}x,
\end{aligned} \tag{4.39}$$

and by using (4.34), (4.37), and (4.38), letting $R\to\infty$, and then letting $L\to\infty$, we have

$$\lim_{R\to\infty}\int_{-R}^{R}(\rho^R-\rho)^2\,\mathrm{d}x=0. \tag{4.40}$$

By using (4.22) and (4.25), we have

$$\begin{aligned}
&\int_0^{T_0}\int_{-R}^{R}\bigl((u_x^R)^2+\mu_0\bigr)^{\frac{p-2}{2}}(u_x^R)^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le C\int_0^{T_0}\|u_x^R\|_{L^\infty(\Omega_R)}^{p-2}\int_{-R}^{R}(u_x^R)^2\,\mathrm{d}x\,\mathrm{d}t+C\int_0^{T_0}\mu_0^{\frac{p-2}{2}}\int_{-R}^{R}(u_x^R)^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le C\int_0^{T_0}\bigl(\|u_x^R\|_{L^p(\Omega_R)}^{1-\theta}\|u_{xx}^R\|_{L^2(\Omega_R)}^{\theta}\bigr)^{p-2}\int_{-R}^{R}(u_x^R)^2\,\mathrm{d}x\,\mathrm{d}t+C \\
&\quad\le C\int_0^{T_0}\bigl(\|u_x^R\|_{L^p(\Omega_R)}^{1-\theta}t^{-\frac{\theta}{2}}t^{\frac{\theta}{2}}\|u_{xx}^R\|_{L^2(\Omega_R)}^{\theta}\bigr)^{p-2}\int_{-R}^{R}(u_x^R)^2\,\mathrm{d}x\,\mathrm{d}t+C \\
&\quad\le C\int_0^{T_0}t^{-\frac{p-2}{p+2}}\,\mathrm{d}t+C \\
&\quad\le C,
\end{aligned} \tag{4.41}$$

where $\theta=\frac{2}{P+2}$. Multiplying $(2.2)_2$ by $\xi_L u^R$, and integrating over Ω_R, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R}\xi_L\rho^R(u^R)^2\,\mathrm{d}x+\int_{-R}^{R}\xi_L(u^R)^2\,\mathrm{d}x+\int_{-R}^{R}\xi_L\bigl((u_x^R)^2+\mu_0\bigr)^{\frac{p-2}{2}}(u_x^R)^2\,\mathrm{d}x \\
&\quad\le -\frac{1}{2}\int_{-R}^{R}\xi_L\rho^R u_x^R(u^R)^2\,\mathrm{d}x+\frac{1}{2}\int_{-R}^{R}\xi_L\rho_x^R(u^R)^3\,\mathrm{d}x \\
&\qquad+\int_{-R}^{R}\xi_{Lx}\bigl((u_x^R)^2+\mu_0\bigr)^{\frac{p-2}{2}}u_x^R u^R\,\mathrm{d}x+\int_{-R}^{R}P_x\xi_L u^R\,\mathrm{d}x \\
&\le 2\|u^R\|_{L^\infty(\Omega_R)}^2\|\xi_L\rho^R\|_{L^2(\Omega_R)}\|u_x^R\|_{L^2(\Omega_R)}+\frac{C}{L}\|u^R\|_{L^\infty(\Omega_R)}^2\|\rho^R\|_{L^2(\Omega_R)}\|u^R\|_{L^2(\Omega_R)} \\
&\qquad+\frac{C}{L}\bigl\|\bigl((u_x^R)^2+\mu_0\bigr)^{\frac{p-2}{2}}u_x^R\bigr\|_{L^2(\Omega_R)}\|u^R\|_{L^2(\Omega_R)} \\
&\qquad+\|\rho^R\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\xi_L\rho^R\|_{L^2(\Omega_R)}\|u_x^R\|_{L^2(\Omega_R)} \\
&\quad\le C\|\xi_L\rho^R\|_{L^2(\Omega_R)}+\frac{C}{L}.
\end{aligned} \tag{4.42}$$

Integrating the above inequality with respect to the time variable over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{-R}^{R}\xi_L\rho^R\left(u^R\right)^2\,\mathrm{d}x+\int_0^t\int_{-R}^{R}\xi_L\left(u^R\right)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{-R}^{R}\xi_L\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}\left(u_x^R\right)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\frac{1}{2}\int_{-R}^{R}\xi_L\rho_0^R\left(u_0^R\right)^2\,\mathrm{d}x+C\int_0^t\left\|\xi_L\rho^R\right\|_{L^2(\Omega_R)}\,\mathrm{d}t+\int_0^t\frac{C}{L}\,\mathrm{d}t\\
&\quad\le C\int_{-R}^{R}\xi_L\rho_0^R\,\mathrm{d}x+C\int_0^t\left\|\xi_L\rho^R\right\|_{L^2(\Omega_R)}\,\mathrm{d}t+\int_0^t\frac{C}{L}\,\mathrm{d}t.
\end{aligned}\tag{4.43}$$

It is easy to see that the following inequality is established:

$$\int_{-R}^{R}\xi_L\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}\left(u_x^R\right)^2\,\mathrm{d}x\ge\frac{1}{2}\int_{-R}^{R}\xi_L\left|u_x^R\right|^p\,\mathrm{d}x+\frac{\mu_0^{\frac{p-2}{2}}}{2}\int_{-R}^{R}\xi_L\left|u_x^R\right|^2\,\mathrm{d}x.\tag{4.44}$$

By virtue of

$$\begin{aligned}
&\int_0^t\int_{-R}^{R}\left(u^R-u\right)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\int_0^t\int_{-L}^{L}\left(u^R-u\right)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{L}^{R}\left(u^R\right)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{L}^{R}u^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^t\int_{-R}^{-L}\left(u^R\right)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{-R}^{-L}u^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^t\int_{-L}^{L}\left(u^R-u\right)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{-R}^{R}\left(\xi_Lu^R\right)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^t\int_{L}^{R}u^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{-R}^{-L}u^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{4.45}$$

and by using (4.42) and by a similar approach to (4.40), letting $R\to\infty$, and then letting $L\to\infty$, we have

$$\lim_{R\to\infty}\int_0^t\int_{-R}^{R}\left(u^R-u\right)^2\,\mathrm{d}x\,\mathrm{d}t=0.\tag{4.46}$$

Subtracting both sides of the inequality $(2.2)_2$ from $[(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x]_x$, multiplying by $\xi_L(u_x^R-u_x)$, and integrating over Ω_R, it follows that

$$\begin{aligned}
&\int_0^t\int_{-R}^{R}\xi_L\left(u_x^R-u_x\right)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\frac{C}{L}\int_0^t\left\|\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x\right\|_{L^2(\Omega_R)}\left(\left\|u^R\right\|_{L^2(\Omega_R)}+\|u\|_{L^2(\Omega_R)}\right)\,\mathrm{d}t\\
&\qquad+\int_0^t\int_{-R}^{R}\xi_L^2\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x\left(u_x^R-u_x\right)\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\|f\|_{L^2(\Omega_R)}\left\|u^R-u\right\|_{L^2(\Omega_R)}\,\mathrm{d}t\\
&\qquad+\int_0^t\left\|\rho^R\right\|_{L^\infty(\Omega_R)}^{\frac{1}{2}}\left\|\sqrt{\rho^R}u_t^R\right\|_{L^2(\Omega_R)}\left\|u^R-u\right\|_{L^2(\Omega_R)}\,\mathrm{d}t\\
&\qquad+\int_0^t\left\|\rho^R\right\|_{L^\infty(\Omega_R)}\left\|u^R\right\|_{L^\infty(\Omega_R)}\left\|u_x^R\right\|_{L^2(\Omega_R)}\left\|u^R-u\right\|_{L^2(\Omega_R)}\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C}{L}\int_0^t \|\rho^R\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\rho^R\|_{L^2(\Omega_R)}\big(\|u^R\|_{L^2(\Omega_R)}+\|u\|_{L^2(\Omega_R)}\big)\,\mathrm{d}t\\
&+\int_0^t \|\rho^R\|_{L^\infty(\Omega_R)}^{\gamma-1}\|\xi_L\rho^R\|_{L^2(\Omega_R)}\big(\|u_x^R\|_{L^2(\Omega_R)}+\|u_x\|_{L^2(\Omega_R)}\big)\,\mathrm{d}t\\
\le{}& C\int_0^t \|u^R-u\|_{L^2(\Omega_R)}^2\,\mathrm{d}t+\frac{C}{L}+C\|\xi_L\rho^R\|_{L^2(\Omega_R)}\\
&+C\int_0^t\int_{-R}^R \xi_L\big((u_x^R)^2+\mu_0\big)^{\frac{p-2}{2}}\big(u_x^R\big)^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^t\int_{-R}^R \xi_L|u_x|^{p-2}\big(u_x^R\big)^2\,\mathrm{d}x\,\mathrm{d}t+C\int_0^t\int_{-R}^R \xi_L\big(u_x^R\big)^2\,\mathrm{d}x\,\mathrm{d}t\\
\le{}& C\int_0^t \|u^R-u\|_{L^2(\Omega_R)}^2\,\mathrm{d}t+\frac{C}{L}+C\|\xi_L\rho^R\|_{L^2(\Omega_R)}\\
&+C\int_0^t\int_{-R}^R \xi_L\big((u_x^R)^2+\mu_0\big)^{\frac{p-2}{2}}\big(u_x^R\big)^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^t\left(\int_{-R}^R |u_x|^p\,\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{-R}^R \xi_L\big(u_x^R\big)^p\,\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}t+C\int_0^t\int_{-R}^R \xi_L\big(u_x^R\big)^2\,\mathrm{d}x\,\mathrm{d}t, \qquad (4.47)
\end{aligned}
$$

where in the second inequality we have used (4.44).

By virtue of

$$
\int_0^t\int_{-R}^R \big(u_x^R-u_x\big)^2\,\mathrm{d}x\,\mathrm{d}t \le \int_0^t\int_{-L}^L \big(u_x^R-u_x\big)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{-R}^R \xi_L\big(u_x^R-u_x\big)^2\,\mathrm{d}x\,\mathrm{d}t, \qquad (4.48)
$$

and by (4.47) and by the lower semi-continuity of various norms, letting $R \to \infty$, and then letting $L \to \infty$, we have

$$
\lim_{R\to\infty}\int_0^t\int_{-R}^R \big(u_x^R-u_x\big)^2\,\mathrm{d}x\,\mathrm{d}t=0. \qquad (4.49)
$$

By virtue of the nonlinear term, we have

$$
\begin{aligned}
\big[\big(s^2+\mu_0\big)^{\frac{p-2}{2}}s\big]' &= \big(s^2+\mu_0\big)^{\frac{p-4}{2}}\big[(p-1)s^2+\mu_0\big]\\
&\ge \big(s^2+\mu_0\big)^{\frac{p-2}{2}}\\
&\ge (\mu_0)^{\frac{p-2}{2}}.
\end{aligned}
$$

First, we begin with the case $u^R u \ge 0$ for fixed (t,x), we get

$$
\begin{aligned}
&\big[\big((u_x^R)^2+\mu_0\big)^{\frac{p-2}{2}}u_x^R-\big(u_x^2+\mu_0\big)^{\frac{p-2}{2}}u_x\big]\big(u_x^R-u_x\big)\\
&\ge \int_0^1 \big|\theta u_x^R+(1-\theta)u_x\big|^{p-2}\,\mathrm{d}\theta\big(u_x^R-u_x\big)^2\\
&\ge \int_0^1 \big|\theta u_x^R\big|^{p-2}\,\mathrm{d}\theta\big(u_x^R-u_x\big)^2+\int_0^1 \big|(1-\theta)u_x\big|^{p-2}\,\mathrm{d}\theta\big(u_x^R-u_x\big)^2\\
&\ge \frac{1}{p-1}\big(\big|u_x^R\big|^{p-2}+|u_x|^{p-2}\big)\big(u_x^R-u_x\big)^2
\end{aligned}
$$

$$\begin{aligned}&\geq C\left(\left|u_x^R\right|+|u_x|\right)^{p-2}\left(u_x^R-u_x\right)^2\\&\geq C\left|u_x^R-u_x\right|^p.\end{aligned}\tag{4.50}$$

Next, we discuss the case $u^R u < 0$ for fixed (t,x), we have

$$\begin{aligned}&\left[\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}u_x^R-\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x\right]\left(u_x^R-u_x\right)\\&\geq\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}\left(u_x^R\right)^2-\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}u_x^R u_x-\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x u_x^R\\&\quad+\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x^2\\&\geq\left|u_x^R\right|^p+|u_x|^p\\&\geq 2^{1-p}\left(\left|u_x^R\right|+|u_x|\right)^p\\&=2^{1-p}\left|u_x^R-u_x\right|^p.\end{aligned}\tag{4.51}$$

Consequently, we obtain after using (4.50) and (4.51)

$$\begin{aligned}&\int_{-R}^{R}\left[\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}u_x-\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x\right]\left(u_x^R-u_x\right)\,\mathrm{d}x\\&\geq 2^{1-p}\int_{-R}^{R}\left|u_x^R-u_x\right|^p\,\mathrm{d}x.\end{aligned}\tag{4.52}$$

By (4.52) and (4.47), we know

$$\int_0^t\int_{-R}^{R}\left|u_x^R-u_x\right|^p\,\mathrm{d}x\,\mathrm{d}t\leq\int_0^t\int_{-L}^{L}\left|u_x^R-u_x\right|^p\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_{-R}^{R}\xi_L\left|u_x^R-u_x\right|^p\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\lim_{R\to\infty}\int_0^t\int_{-R}^{R}\left|u_x^R-u_x\right|^p\,\mathrm{d}x\,\mathrm{d}t=0.\tag{4.53}$$

Next, for any function $\Psi\in C_0^\infty(\mathbb{R}\times[0,T_0))$, we take Ψ as a test function in the initial-boundary value problem (2.2) with the initial data (ρ_0^R,w_0^R). By standard arguments, letting $R\to\infty$, it follows from (4.17)-(4.28), (4.40), (4.46), (4.49), and (4.53) that (ρ,u) is a strong solution of (1.1)-(1.2) on $\mathbb{R}\times(0,T_0]$ satisfying (1.6) and (1.7). We only give a proof of the nonlinear term as follows:

$$\begin{aligned}&\int_0^{T_0}\int_{-R}^{R}\left[\left(\left(u_x^R\right)^2+\mu_0\right)^{\frac{p-2}{2}}u_x^R-\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x\right]\Psi\,\mathrm{d}x\,\mathrm{d}t\\&\leq\int_0^{T_0}\int_{-R}^{R}\left|\int_0^1\left[\mu_0+\left(\theta u_x^R+(1-\theta)u_x\right)^2\right]^{\frac{p-2}{2}}\,\mathrm{d}\theta\right|\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t\\&\leq\int_0^{T_0}\int_{-R}^{R}\left|\left[\mu_0+\left(\left|u_x^R\right|+|u_x|\right)^2\right]^{\frac{p-2}{2}}\right|\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t\\&\leq C\int_0^{T_0}\int_{-R}^{R}\left(1+\mu_0+\left|u_x^R\right|^2+|u_x|^2\right)^{\frac{p-1}{2}}\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t\\&\leq C\int_0^{T_0}\int_{-R}^{R}\left(1+\mu_0^{\frac{p-1}{2}}+\left|u_x^R\right|^{p-1}+|u_x|^{p-1}\right)\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{T_0}\int_{-R}^{R}\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t+C\int_0^{T_0}\int_{-R}^{R}\left|u_x^R\right|^{p-1}\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C\int_0^{T_0}\int_{-R}^{R}|u_x|^{p-1}\left|u_x^R-u_x\right||\Psi|\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\left(\int_0^{T_0}\int_{-R}^{R}\left|u_x^R-u_x\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^{T_0}\int_{-R}^{R}|\Psi|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad+C\int_0^{T_0}\left\|u_x^R\right\|_{L^p(\Omega_R)}^{p-1}\left\|\left(u_x^R-u_x\right)\Psi\right\|_{L^p(\Omega_R)}\mathrm{d}t\\
&\quad+C\int_0^{T_0}\|u_x\|_{L^p(\Omega_R)}^{p-1}\left\|\left(u_x^R-u_x\right)\Psi\right\|_{L^p(\Omega_R)}\mathrm{d}t\\
&\le C\left(\int_0^{T_0}\int_{-R}^{R}\left|u_x^R-u_x\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}+C\int_0^{T_0}\|\Psi\|_{L^\infty(\Omega_R)}\left\|\left(u_x^R-u_x\right)\right\|_{L^p(\Omega_R)}\mathrm{d}t\\
&\le C\left(\int_0^{T_0}\int_{-R}^{R}\left|u_x^R-u_x\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}+C\left(\int_0^{T_0}\int_{-R}^{R}\left|u_x^R-u_x\right|^p\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}. \qquad (4.54)
\end{aligned}
$$

The proof of the existence part of Theorem 1.2 is finished.

Finally, to finish the proof of the main theorem, we only need to prove the uniqueness of the solution of the problem (1.1). Let (ρ,u) and $(\bar\rho,\bar u)$ be two solutions that satisfy the same initial condition. Denote

$$
Z\triangleq\rho-\bar\rho,\qquad U\triangleq u-\bar u,\qquad \Theta\triangleq\pi(\rho)-\pi(\bar\rho).
$$

Subtracting the momentum equations satisfied by (ρ,u) and $(\bar\rho,\bar u)$ yields

$$
\begin{aligned}
&\rho U_t+\rho u\cdot U_x-\left\{\left[\left(u_x^2+\mu_0\right)^{\frac{p-2}{2}}u_x\right]_x-\left[\left(\bar u_x^2+\mu_0\right)^{\frac{p-2}{2}}\bar u_x\right]_x\right\}\\
&\quad=-\rho U\cdot\bar u_x-Z(\bar u_t+\bar u\cdot\bar u_x)-\Theta_x+f(u)-f(\bar u). \qquad (4.55)
\end{aligned}
$$

By virtue of f, we have

$$
f_s=\frac{\partial f(t,x,s)}{\partial s}. \qquad (4.56)
$$

By (1.4), we know

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\left[f(u)-f(\bar u)\right]U\,\mathrm{d}x&=\int_{-\infty}^{+\infty}\int_0^1 f_s\left(\theta u+(1-\theta)\bar u\right)\mathrm{d}\theta\, U^2\,\mathrm{d}x\\
&\le -B\int_{-\infty}^{+\infty}U^2\,\mathrm{d}x. \qquad (4.57)
\end{aligned}
$$

By virtue of

$$
\begin{aligned}
\left[\left(s^2+\mu_0\right)^{\frac{p-2}{2}}s\right]'&=\left(s^2+\mu_0\right)^{\frac{p-4}{2}}\left[(p-1)s^2+\mu_0\right]\\
&\ge\left(s^2+\mu_0\right)^{\frac{p-2}{2}}\\
&\ge(\mu_0)^{\frac{p-2}{2}},
\end{aligned}
$$

multiplying (4.55) by U, by using (1.4) and integrating by parts, leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\rho|U|^2\,\mathrm{d}x+\int_{-\infty}^{+\infty}|U_x|^2\,\mathrm{d}x+B\int_{-\infty}^{+\infty}U^2\,\mathrm{d}x\\
&\quad\le C\|\bar{u}_x\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{+\infty}\rho|U|^2\,\mathrm{d}x+C\int_{-\infty}^{+\infty}|Z||U|\bigl(|\bar{u}_t|+|\bar{u}||\bar{u}_x|\bigr)\,\mathrm{d}x\\
&\qquad+C\|\Theta\|_{L^2(\mathbb{R})}\|U_x\|_{L^2(\mathbb{R})}\\
&\quad\triangleq C\|\bar{u}_x\|_{L^\infty}\int_{-\infty}^{+\infty}\rho|U|^2\,\mathrm{d}x+A_1+A_2.
\end{aligned}\tag{4.58}$$

Then subtracting the mass equation for (ρ,u) and $(\bar{\rho},\bar{u})$ gives

$$Z_t+\bar{u}\cdot Z_x+Z\bar{u}_x+\rho U_x+U\cdot\rho_x=0.\tag{4.59}$$

Multiplying (4.59) by $2H\Phi$ and integrating by parts lead to

$$\begin{aligned}
&\bigl(\|Z\Phi^{\frac{1}{2}}\|_{L^2(\mathbb{R})}^2\bigr)_t\\
&\quad\le C\bigl(\|\bar{u}_x\|_{L^\infty(\mathbb{R})}+\bigl\|\bar{u}\Phi^{-\frac{1}{2(1+\eta_0)}}\bigr\|_{L^\infty(\mathbb{R})}\bigr)\bigl\|Z\Phi^{\frac{1}{2}}\bigr\|_{L^2(\mathbb{R})}^2\\
&\qquad+C\bigl\|\rho\Phi^{\frac{1}{2}}\bigr\|_{L^\infty(\mathbb{R})}\|U_x\|_{L^2(\mathbb{R})}\bigl\|Z\Phi^{\frac{1}{2}}\bigr\|_{L^2(\mathbb{R})}\\
&\qquad+C\bigl\|Z\Phi^{\frac{1}{2}}\bigr\|_{L^2(\mathbb{R})}\bigl\|U\Phi^{-\frac{1}{2}}\bigr\|_{L^{\frac{2q}{q-2}}(\mathbb{R})}\|\Phi\rho_x\|_{L^q(\mathbb{R})}\\
&\quad\le C\bigl(1+\|\bar{u}_x\|_{W^{1,q}(\mathbb{R})}\bigr)\bigl\|Z\Phi^{\frac{1}{2}}\bigr\|_{L^2(\mathbb{R})}^2\\
&\qquad+C\bigl\|Z\Phi^{\frac{1}{2}}\bigr\|_{L^2(\mathbb{R})}\bigl(\|U_x\|_{L^2(\mathbb{R})}+\|\sqrt{\rho}U\|_{L^2(\mathbb{R})}\bigr),
\end{aligned}$$

where in the second inequality we have used Sobolev's inequality, (1.7), (2.26), and (1.8). This combined with Gronwall's inequality yields, for all $t\in(0,T^*]$,

$$\bigl\|Z\Phi^{\frac{1}{2}}\bigr\|_{L^2(\mathbb{R})}\le C\int_0^t\bigl(\|U_x\|_{L^2(\mathbb{R})}+\|\sqrt{\rho}U\|_{L^2(\mathbb{R})}\bigr)\,\mathrm{d}s.\tag{4.60}$$

As observed by Germain [21], putting (4.53) into (4.55) leads to

$$\begin{aligned}
A_1&\le C(\varepsilon)\bigl(1+t\|\bar{u}_{xt}\|_{L^2(\mathbb{R})}^2+t\|\bar{u}_{xx}\|_{L^q(\mathbb{R})}^2\bigr)\int\bigl(\|U_x\|_{L^2(\mathbb{R})}^2+\|\sqrt{\rho}U\|_{L^2(\mathbb{R})}^2\bigr)\,\mathrm{d}s\\
&\quad+\bigl(\|\sqrt{\rho}U\|_{L^2(\mathbb{R})}^2+\|U_x\|_{L^2(\mathbb{R})}^2\bigr).
\end{aligned}\tag{4.61}$$

Next, we will estimate A_2. In fact, one deduces from $(2.2)_1$ that

$$\Theta_t+u\Theta_x+U\pi(\bar{\rho})_x+\gamma\Theta u_x+\gamma\pi(\bar{\rho})U_x=0,$$

which gives

$$\begin{aligned}
&\bigl(\|\Theta\|_{L^2(\mathbb{R})}\bigr)_t\\
&\quad\le C\bigl(1+\|u_x\|_{L^\infty(\mathbb{R})}\bigr)\|\Theta\|_{L^2(\mathbb{R})}+C\|U_x\|_{L^2(\mathbb{R})}
\end{aligned}$$

$$+ C\big\|U\Phi^{-1}\big\|_{L^{\frac{2q}{q-2}}(\mathbb{R})}\big\|\pi(\bar{\rho})_x\Phi\big\|_{L^q(\mathbb{R})}$$
$$\le C\big(1 + \|u_x\|_{L^\infty(\mathbb{R})}\big)\|\Theta\|_{L^2(\mathbb{R})} + C\|\sqrt{\rho}U\|^2_{L^2(\mathbb{R})} + C\|U_x\|^2_{L^2(\mathbb{R})}, \tag{4.62}$$

which together with Gronwall's inequality gives

$$\|\Theta\|_{L^2(\mathbb{R})} \le C\int_0^t \big(\|U_x\|_{L^2(\mathbb{R})} + \|\sqrt{\rho}U\|_{L^2(\mathbb{R})}\big)\,\mathrm{d}s,$$

which shows

$$A_2 \le \varepsilon\|U_x\|^2_{L^2(\mathbb{R})} + C(\varepsilon)\int_0^t \big(\|U_x\|^2_{L^2(\mathbb{R})} + \|\sqrt{\rho}U\|^2_{L^2(\mathbb{R})}\big)\,\mathrm{d}s. \tag{4.63}$$

Denoting

$$H(t) \triangleq \int_0^t \|U\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + \|\sqrt{\rho}U\|^2_{L^2(\mathbb{R})} + \int_0^t \big(\|\sqrt{\rho}U\|^2_{L^2(\mathbb{R})} + \|U_x\|^2_{L^2(\mathbb{R})}\big)\,\mathrm{d}s,$$

putting (4.61) and (4.63) into (4.58) and choosing ε suitably small lead to

$$H'(t) \le C\big(1 + \|\bar{u}_x\|_{L^\infty(\mathbb{R})} + t\|\bar{u}_{xx}\|^2_{L^q(\mathbb{R})} + t\|\bar{u}_{xt}\|^2_{L^2(\mathbb{R})}\big)G,$$

which together with Gronwall's inequality and (1.8) yields $H(t) = 0$. Hence, $U(x,t) = 0$ for almost everywhere $(x,t) \in \mathbb{R} \times (0, T_0)$. Then (4.60) implies that $Z(x,t) = 0$ for almost everywhere $(x,t) \in \mathbb{R} \times (0, T_0)$. The proof of Theorem 1.2 is completed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The research is supported by the National Natural Science Foundation of China Grant No. 11271153 and Specialized Research Fund for the Doctoral Program of Higher Education No. 20140101-20161231.

Received: 29 August 2016 Accepted: 8 November 2016 Published online: 15 November 2016

References
1. Böhme, G: Non-Newtonian Fluid Mechanics. Appl. Math. Mech. North-Holland, Amsterdam (1970)
2. Huilgol, RR: Continuum Mechanics of Viscoelastic Liquids. Hindustan Publishing Corporation, Delhi (1975)
3. Schwalter, WR: Mechanics of Non-Newtonian Fluid. Pergamon, New York (1978)
4. Lions, PL: Mathematical Topics in Fluid Mechanics. Volume 2: Compressible Models. Oxford University Press, New York (1998)
5. Li, J, Xin, Z: Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum. arXiv:1310.1673v1
6. Huang, X, Li, J: Global well-posedness of classical solutions to the Cauchy problem of two-dimensional baratropic compressible Navier-Stokes system with vacuum and large initial data. arXiv:1207.3746v1
7. Liang, Z, Lu, Y: Classical large solution for the viscous compressible fluids with initial vacuum in 1D. Bound. Value Probl. **2015**, 100 (2015)
8. Li, J, Liang, Z: On local classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum. J. Math. Pures Appl. **102**, 640-671 (2014)
9. Yuan, H, Xu, X: Existence and uniqueness of solutions for a class of non-Newtonian fluids with singularity and vacuum. J. Differ. Equ. **245**, 2871-2916 (2008)
10. Bloom, F, Hao, W: Regularization of a non-Newtonian system in an unbounded channel: existence of a maximal compact attractor. Nonlinear Anal. **43**, 743-766 (2001)
11. Xu, X, Yuan, H: Existence of the unique strong solution for a class of non-Newtonian fluids with vacuum. Q. Appl. Math. **66**, 249-279 (2008)

12. Wang, C, Yuan, H: Global strong solutions for a class of heat-conducting non-Newtonian fluids with vacuum. Nonlinear Anal., Real World Appl. **11**, 3680-3703 (2010)
13. Wang, C, Yuan, H: Global strong solutions for a class of compressible non-Newtonian fluids with vacuum. Math. Models Methods Appl. Sci. **34**, 397-417 (2011)
14. Yin, L, Xu, X, Yuan, H: Global existence and uniqueness of solution of the initial boundary value problem for a class of non-Newtonian fluids with vacuum. Z. Angew. Math. Phys. **59**, 457-474 (2008)
15. Yuan, H, Li, H: Existence and uniqueness of solutions for a class of non-Newtonian fluids with vacuum and damping. J. Math. Anal. Appl. **391**, 223-239 (2012)
16. Yuan, H, Wang, C: Unique solvability for a class of full non-Newtonian fluids of one dimension with vacuum. Z. Angew. Math. Phys. **60**, 868-898 (2009)
17. Cho, Y, Choe, HJ, Kim, H: Unique solvability of the initial boundary value problems for compressible viscous fluids. J. Math. Pures Appl. **83**, 243-275 (2004)
18. Choe, HJ: Strong solutions of the Navier-Stokes equations for isentropic compressible fluids. J. Differ. Equ. **190**, 504-523 (2003)
19. Cho, Y, Kim, H: On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. Manuscr. Math. **120**, 91-129 (2006)
20. Feireisl, E, Novotny, A: Dynamics of Viscous Compressible Fluids. Oxford University Press, London (2004)
21. Germain, P: Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. J. Math. Fluid Mech. **13**, 137-146 (2011)