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Impulsive integral BVP for nonlinear integro-differential equation with monotone homomorphism in Banach spaces

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Abstract
In this paper, we consider a class of integral boundary value problems for nonlinear third-order impulsive integro-differential equation with a monotone homomorphism in real Banach space. By employing fixed point index theory, some sufficient criteria are obtained to ensure the existence of positive solutions. An example is given to demonstrate the application of our main results.

Keywords: integral boundary value problems; impulsive integro-differential equation; monotone homomorphism; fixed point index theory; Banach spaces

1 Introduction

This paper deals with the existence of positive solutions for the following nonlinear third-order impulsive integro-differential equation with monotone homomorphism and integral boundary value conditions in real Banach space (abbreviated by BVP (1.1) throughout this paper):

$$\begin{cases} (\varphi(-x''(t)))' = f(t, x(t), x'(t), (Tx)(t), (Sx)(t)), \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = -I_k(x(t_k)), \quad k = 1, \ldots, m, \\ \Delta x'|_{t=t_k} = -\bar{I}_k(x(t_k), x'(t_k)), \quad k = 1, \ldots, m, \\ x'(0) = x''(0) = \theta, \qquad x(1) = \int_0^1 g(t)x(t)\,dt, \end{cases} \tag{1.1}$$

where $I = [0,1]$, $J = (0,1)$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \backslash \{t_1, t_2, \ldots, t_m\}$, $J_0 = (0, t_1]$, $J_k = (t_k, t_{k+1}]$ $(k = 1, \ldots, m-1)$, $J_m = (t_m, 1]$. $\mathbb{E}$ is a real Banach space with the norm $\|x\|$. $\mathbb{P}$ is a positive cone in $\mathbb{E}$. θ is a zero element of $\mathbb{E}$. $f \in C[J \times \mathbb{P}^4, \mathbb{P}]$, $I_k \in C[\mathbb{P}, \mathbb{P}]$, $\bar{I}_k \in C[\mathbb{P}^2, \mathbb{P}]$. $g \in L^1[0,1]$ is nonnegative. $\varphi : \mathbb{P} \to \mathbb{E}$ is an increasing and positive homomorphism (see Definition 2.3) and $\varphi(\theta) = \theta$. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$. $x(t_k^+)$, $x'(t_k^+)$ and $x(t_k^-)$, $x'(t_k^-)$ represent the right-hand limits and left-hand limits of $x(t)$ and $x'(t)$ at $t = t_k$, respectively. $(Tx)(t)$ and $(Sx)(t)$ are defined as

$$(Tx)(t) = \int_0^t K(t,s)x(s)\,ds, \qquad (Sx)(t) = \int_0^1 F(t,s)x(s)\,ds, \tag{1.2}$$

here $K \in C[D, J]$, $D = \{(t,s) \in J \times J : t \geq s\}$, $H \in C[J \times J, J]$.

It is worth noting that BVP (1.1) is the extension and generalization of many boundary value problems. For example, if $\varphi(x) = x$, then BVP (1.1) changes into a boundary value problem of the form

$$\begin{cases} -x'''(t) = f(t,x(t),x'(t),(Tx)(t),(Sx)(t)), \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = -I_k(x(t_k)), \quad k = 1,\ldots,m, \\ \Delta x'|_{t=t_k} = -\overline{I}_k(x(t_k),x'(t_k)), \quad k = 1,\ldots,m, \\ x'(0) = x''(0) = \theta, \qquad x(1) = \int_0^1 g(t)x(t)\,dt. \end{cases} \tag{1.3}$$

If $\varphi(x) = \Phi_p(x) := |x|^{p-2}x$ for some $p > 1$, then BVP (1.1) changes into the boundary value problem with p-Laplacian as follows:

$$\begin{cases} (\Phi_p(-x''(t)))' = f(t,x(t),x'(t),(Tx)(t),(Sx)(t)), \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = -I_k(x(t_k)), \quad k = 1,\ldots,m, \\ \Delta x'|_{t=t_k} = -\overline{I}_k(x(t_k),x'(t_k)), \quad k = 1,\ldots,m, \\ x'(0) = x''(0) = \theta, \qquad x(1) = \int_0^1 g(t)x(t)\,dt. \end{cases} \tag{1.4}$$

In recent years, the boundary value problems has aroused great attention due to its important applications in the fields of science and engineering such as chemical engineering, heat conduction, thermo-elasticity, plasma physics, underground water flow, and so on. Many scholars began with a study of the dynamics of boundary value problems by some nonlinear analysis methods such as fixed point theorems, shooting method, iterative method with upper and lower solutions, and so forth. The fixed point principle in cone is one of the important methods by applying to investigate the existence and multiplicity of positive solutions for boundary value problems.

In order to describe the dynamics of populations subject to abrupt changes and other phenomena such as harvesting, diseases, and so on, some authors have used an impulsive differential system to describe these kinds of phenomena since the last century. Some scholars have begun to study the boundary value problems of impulsive differential equations and obtained many good results (see [1–22]). The main tools of our study of the existence and multiplicity of positive solutions for this problem are the Schauder fixed point theorem, fixed point index theory, upper and lower solutions together with the monotone iterative technique, etc. In recent years, there have appeared some papers dealing with impulsive integro-differential equations in Banach spaces (see [4–18]).

In addition, the inspiration of this paper comes from the following two systems (see [4, 23]). In [23], Fu and Ding considered the existence of positive solutions of the boundary value problems with integral boundary conditions in Banach spaces of the form

$$\begin{cases} (\varphi(-x''(t)))' = f(t,x(t)), \quad t \in J, \\ x(0) = x''(0) = \theta, \qquad x(1) = \int_0^1 g(t)x(t)\,dt, \end{cases}$$

where θ is the zero element of $\mathbb{E}$. $g \in L^1[0,1]$ is nonnegative. $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing and positive homomorphism and $\varphi(0) = \theta_1$, $\mathbb{R} := (-\infty,+\infty)$.

Remark 1.1 In [23], the author defined the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ being an increasing and positive homomorphism and $\varphi(0) = \theta_1$. It seems that this definition is incorrect since $x(t)$

is an element of Banach space $\mathbb{E}$ but not $x(t) \in \mathbb{R}$. So we modify it as $\varphi : \mathbb{P} \to \mathbb{E}$ is an increasing and positive homomorphism (see Definition 2.3) and $\varphi(\theta) = \theta$.

In [4], Sun considered the positive solutions of the Sturm-Liouville boundary value problems for singular nonlinear second-order impulsive integro-differential equation in Banach spaces of the form

$$\begin{cases} y'' = f(t, y(t), y'(t), (Ty)(t), (Sy)(t)), \quad t \in J, t \neq t_k, \\ \Delta y|_{t=t_k} = I_k(y(t_k)), \quad k = 1, \dots, m, \\ -\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k), y'(t_k)), \quad k = 1, \dots, m, \\ \alpha y(0) - \beta y'(0) = 0, \qquad \gamma y(1) - \delta y'(1) = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta \geq 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $I = [0, 1]$, $J = (0, 1)$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \backslash \{t_1, t_2, \dots, t_m\}$, $\bar{J} = [0, 1]$, $J_0 = (0, t_1]$, $J_k = (t_k, t_k + 1]$ $(k = 1, \dots, m - 1)$, $J_m = (t_m, 1]$, $f \in C[J \times \mathbb{P}^4, \mathbb{P}]$. $\mathbb{P}$ is a positive cone in $\mathbb{E}$. θ is a zero element of E. $I_k \in C[\mathbb{P}, \mathbb{P}]$, $\bar{I}_k \in C[\mathbb{P}^2, \mathbb{P}]$.

To the best of our knowledge, there is less research dealing with integral BVPs for nonlinear third-order impulsive integro-differential equation with monotone homomorphism in Banach spaces. Therefore, we will investigate the existence of positive solutions for BVP (1.1) under some further conditions by making use of fixed point index theory. Our main results in essence improve and generalize the corresponding results of [4, 23].

The rest of this paper is organized as follows. In Section 2, we present some known results and introduce conditions to be used in the next section. We give some sufficient conditions for the existence of positive solutions for BVP (1.1) in Section 3. Finally, one example is also provided to illustrate the validity of our main results in Section 4.

2 Preliminaries and statements

In this section, we shall state some necessary definitions and preliminaries results.

Definition 2.1 Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. A nonempty closed convex set $\mathbb{P} \subset \mathbb{E}$ is said to be a cone provided that the following conditions are satisfied:

(i) $y \in \mathbb{P}$, $\lambda \geq 0$ implies $\lambda y \in \mathbb{P}$;
(ii) $y \in \mathbb{P}$, $-y \in \mathbb{P}$ implies $y = \theta$.

Let $\preceq$ be the partial order on $\mathbb{E}$ induced by the cone $\mathbb{P}$ in $\mathbb{E}$. That is, $x \preceq y$ if and only if $y - x \in \mathbb{P}$. If $x, y \in \mathbb{E}$, the notation $x \prec y$ means that $x \preceq y$ and $x \neq y$. $\mathbb{P}$ is said to be normal if there exists a positive constant N such that $\theta \preceq x \preceq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of $\mathbb{E}$, and N is called the normal constant of $\mathbb{P}$ (it is clear that $N \geq 1$).

Definition 2.2 Let $\mathbb{E}$ be a Banach space and $\mathbb{P} \subset \mathbb{E}$ be a cone in $\mathbb{E}$. Let $\mathbb{C}$ be a convex subset in $\mathbb{E}$. An operator $\Phi : \mathbb{C} \to \mathbb{E}$ is said to be an increasing operator if

$$x, y \in \mathbb{C}, \quad x \preceq y \quad \Rightarrow \quad \Phi x \preceq \Phi y.$$

Similarly, An operator $\Phi : \mathbb{C} \to \mathbb{E}$ is called a decreasing operator if

$$x, y \in \mathbb{C}, \quad x \preceq y \quad \Rightarrow \quad \Phi x \succeq \Phi y.$$

Definition 2.3 (see [24]) Let $\mathbb{E}$ be a Banach space and $\mathbb{P} \subset \mathbb{E}$ be a positive cone in $\mathbb{E}$. A projection $\varphi : \mathbb{P} \to \mathbb{E}$ is called an increasing and positive homomorphism if and only if

(i) if $x \preceq y$, then $\varphi(x) \preceq \varphi(y)$ for all $x, y \in \mathbb{P}$;
(ii) φ is a continuous bijection and its inverse mapping φ^{-1} is also continuous;
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{P}$.

It is easy to verify that φ^{-1} is also an increasing and positive homomorphism.

If $(\mathbb{E}, \|\cdot\|)$ is a real Banach space and $\mathbb{P} \subset \mathbb{E}$ is a cone in $\mathbb{E}$, then the common relationship of less than or equal to $\le$ decides a partial order on $\mathbb{E}$ induced by the cone $\mathbb{P}$ in $\mathbb{E}$. It is worth noting that the partial order $\le$ is used throughout the paper.

Definition 2.4 Let $\mathbb{E}$ be a metric space and $\mathbb{S}$ be a bounded subset of $\mathbb{E}$. The (Kuratowski) measure of noncompactness $\alpha(\mathbb{S})$ of $\mathbb{S}$ is defined by

$$\alpha(\mathbb{S}) = \inf\bigl\{\delta > 0 : \mathbb{S} \text{ admits a finite cover by subsets of } \mathbb{S}_i \subset \mathbb{S} \text{ such that } \operatorname{diam}(\mathbb{S}_i) \le \delta\bigr\},$$

where $\operatorname{diam}(\mathbb{S}_i)$ denotes the diameter of the set $\mathbb{S}_i$.

Definition 2.5 An operator $\Phi : \mathbb{D} \to \mathbb{E}$ is said to be completely continuous if it is continuous and compact.

Lemma 2.1 (Fixed-point index theorem; see [25]) *Let $\mathbb{E}$ be a Banach space, $\mathbb{P} \subset \mathbb{E}$ is a cone. For $r > 0$, define $\Omega_r = \{u \in \mathbb{P} : \|u\| < r\}$. Assume that $A : \overline{\Omega}_r \to \mathbb{P}$ is a completely continuous operator such that $Au \neq u$ for $u \in \partial\Omega_r = \{u \in \mathbb{P} : \|u\| = r\}$.*

(1) *If $\|Au\| \ge \|u\|$, for $u \in \partial\Omega_r$, then $i(A, \Omega_r, \mathbb{P}) = 0$.*
(2) *If $\|Au\| \le \|u\|$, for $u \in \partial\Omega_r$, then $i(A, \Omega_r, \mathbb{P}) = 1$.*

Let $(\mathbb{E}, \|\cdot\|)$ be a real Banach space and denote $\mathrm{PC}[J, \mathbb{E}] := \{x | x : J \to \mathbb{E}$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+)$ exists for $k = 1, \dots, m\}$ and $\mathrm{PC}[J, \mathbb{P}] := \{x \in \mathrm{PC}[J, \mathbb{E}] : x(t) \ge \theta\}$. It is easy to verify that $\mathrm{PC}[J, \mathbb{E}]$ is a Banach space with norm $\|x\|_{\mathrm{PC}} = \sup_{t \in J} \|x(t)\|$. Obviously, $\mathrm{PC}[J, \mathbb{P}]$ is a cone in Banach space $\mathrm{PC}[J, \mathbb{E}]$.

Denote $\mathrm{PC}^1[J, \mathbb{E}] := \{x \in \mathrm{PC}[J, \mathbb{E}] : x'(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x'(t_k^+)$ exists for $k = 1, \dots, m\}$ and $\mathrm{PC}^1[J, \mathbb{P}] := \{x \in \mathrm{PC}^1[J, \mathbb{E}] : x(t) \ge \theta, x'(t) \ge \theta\}$. Clearly, $\mathrm{PC}[J, \mathbb{E}]$ is a Banach space equipped with the norm $\|x\|_{\mathrm{PC}^1} = \max\{\|x\|_{\mathrm{PC}}, \|x'\|_{\mathrm{PC}}\}$ and $\|x\|_{\mathrm{PC}^1} \le \|x\|_{\mathrm{PC}} + \|x'\|_{\mathrm{PC}}$. $\mathrm{PC}^1[J, \mathbb{P}]$ is a cone in $\mathrm{PC}^1[J, \mathbb{E}]$.

Let $C^3[J', \mathbb{E}] := \{x : J' \to \mathbb{E} | x'''(t)$ is continuous in $J'\}$. A functional $x \in \mathrm{PC}^1[J, \mathbb{E}] \cap C^3[J', \mathbb{E}]$ is called a nonnegative solution of (1.1) if $x(t) \ge \theta$, $x'(t) \ge \theta$ for $t \in J$ and $x(t)$ satisfies BVP (1.1). A functional $x \in \mathrm{PC}^1[J, \mathbb{E}] \cap C^3[J', \mathbb{E}]$ is called a positive solution of BVP (1.1) if x is a nonnegative solution of BVP (1.1) and $x(t) \neq \theta$.

Next we state the integral equation associated with BVP (1.1).

Lemma 2.2 *Denote $\sigma := \int_0^1 g(s)\,ds$. If $\sigma \neq 1$, then $x \in \mathrm{PC}^1[\bar{J}, \mathbb{E}] \cap C^3[J', \mathbb{E}]$ is a solution of* (1.1) *if and only if $x \in C^3[\bar{J}, \mathbb{E}]$ is a solution of the impulsive integral equation of the form*

$$\begin{aligned} x(t) = {}& \int_0^1 H(t,s)\varphi^{-1}\left(\int_0^s f\bigl(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\bigr)\,d\tau\right) ds + \phi(t) + \psi(t) \\ & + \frac{1}{1-\sigma}\int_0^1 g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\psi(s)\,ds, \end{aligned} \tag{2.1}$$

where

$$H(t,s)=G(t,s)+\frac{1}{1-\sigma}\int_0^1 g(\tau)G(\tau,s)\,d\tau, \tag{2.2}$$

$$G(t,s)=\begin{cases}1-t, & 0\le s\le t\le 1;\\ 1-s, & 0\le t\le s\le 1,\end{cases} \tag{2.3}$$

$$\phi(t)=(1-t)\sum_{0<t_k<t}\bar{I}_k\big(x(t_k),x'(t_k)\big), \tag{2.4}$$

$$\psi(t)=\sum_{t\le t_k<1}\big[I_k\big(x(t_k)\big)+(1-t_k)\bar{I}_k\big(x(t_k),x'(t_k)\big)\big]. \tag{2.5}$$

Proof Now we prove the necessity of Lemma 2.2. Suppose that $x\in \mathrm{PC}^1[\bar{J},\mathbb{E}]\cap C^3[J',\mathbb{E}]$ is a solution of BVP (1.1). Integrating both sides of the first equation of (1.1) from 0 to t, we have

$$\varphi\big(-x''(t)\big)-\varphi\big(-x''(0)\big)=\int_0^t f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau.$$

By $\varphi(\theta)=\theta$ and the boundary value condition $x''(0)=\theta$, we get

$$x''(t)=-\varphi^{-1}\left(\int_0^t f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right). \tag{2.6}$$

By taking the integral of (2.6) on $[0,t]$ and noting that $x'(0)=\theta$, we obtain

$$\begin{aligned}x'(t)&=-\int_0^t\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds+\sum_{0<t_k<t}\big[x'\big(t_k^+\big)-x'\big(t_k^-\big)\big]\\&=-\int_0^t\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds\\&\quad-\sum_{0<t_k<t}\bar{I}_k\big(x(t_k),x'(t_k)\big).\end{aligned} \tag{2.7}$$

Taking the integral of (2.7) on $[0,t]$, we have

$$\begin{aligned}x(t)&=x(0)-\int_0^t(t-s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds\\&\quad-\sum_{0<t_k<t}I_k\big(x(t_k)\big)-\sum_{0<t_k<t}(t-t_k)\bar{I}_k\big(x(t_k),x'(t_k)\big).\end{aligned} \tag{2.8}$$

Letting $t=1$, (2.8) shows that

$$\begin{aligned}x(1)&=x(0)-\int_0^1(1-s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds\\&\quad-\sum_{k=1}^m I_k\big(x(t_k)\big)-\sum_{k=1}^m(1-t_k)\bar{I}_k\big(x(t_k),x'(t_k)\big),\end{aligned}$$

that is,

$$x(0) = x(1) + \int_0^1 (1-s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds \\ + \sum_{k=1}^m I_k(x(t_k) + \sum_{k=1}^m (1-t_k)\bar{I}_k\big(x(t_k), x'(t_k)\big). \tag{2.9}$$

Substituting $x(1) = \int_0^1 g(t)x(t)\,dt$ into (2.9), we obtain

$$x(0) = \int_0^1 g(s)x(s)\,ds + \int_0^1 (1-s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds \\ + \sum_{k=1}^m I_k(x(t_k) + \sum_{k=1}^m (1-t_k)\bar{I}_k\big(x(t_k), x'(t_k)\big). \tag{2.10}$$

Substituting (2.10) into (2.8), we have

$$\begin{aligned} x(t) = {} & \int_0^1 g(s)x(s)\,ds + \int_0^1 (1-s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds \\ & + \sum_{k=1}^m I_k(x(t_k) + \sum_{k=1}^m (1-t_k)\bar{I}_k\big(x(t_k), x'(t_k)\big) \\ & - \int_0^t (t-s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds \\ & - \sum_{0<t_k<t} I_k(x(t_k) - \sum_{0<t_k<t} (t-t_k)\bar{I}_k\big(x(t_k), x'(t_k)\big) \\ = {} & \left(\int_0^t + \int_t^1\right)\left[(1-s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right)\right] ds \\ & - \int_0^t (t-s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds + \int_0^1 g(s)x(s)\,ds \\ & + \sum_{t\le t_k<1} I_k\big(x(t_k)\big) + \sum_{t\le t_k<1} (1-t_k)\bar{I}_k\big(x(t_k), x'(t_k)\big) + (1-t)\sum_{0<t_k<t} \bar{I}_k\big(x(t_k), x'(t_k)\big) \\ = {} & \left(\int_0^t (1-t) + \int_t^1 (1-s)\right)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds \\ & + \int_0^1 g(s)x(s)\,ds + (1-t)\sum_{0<t_k<t} \bar{I}_k\big(x(t_k), x'(t_k)\big) + \sum_{t<t_k<1} \big[I_k\big(x(t_k)\big) + \bar{I}_k\big(x(t_k), x'(t_k)\big)\big] \\ = {} & \int_0^1 G(t,s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right) ds + \int_0^1 g(s)x(s)\,ds \\ & + \sum_{t\le t_k<1} I_k\big(x(t_k)\big) + \sum_{t\le t_k<1} (1-t_k)\bar{I}_k\big(x(t_k), x'(t_k)\big) \\ & + (1-t)\sum_{0<t_k<t} \bar{I}_k\big(x(t_k), x'(t_k)\big), \end{aligned} \tag{2.11}$$

where $G(t,s)$ is defined as (2.3).

Integrating both sides of (2.11) from 0 to 1, we have

$$\begin{aligned}
&\int_0^1 g(s)x(s)\,ds \\
&\quad= \int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)\varphi^{-1}\biggl(\int_0^\tau f\bigl(\eta,x(\eta),x'(\eta),(Tx)(\eta),(Sx)(\eta)\bigr)\,d\eta\biggr)\,d\tau\biggr]\,ds \\
&\qquad+ \int_0^1 g(s)\,ds \times \int_0^1 g(s)x(s)\,ds + \int_0^1 (1-s)g(s)\biggl(\sum_{0<t_k<s}\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\biggr)\,ds \\
&\qquad+ \int_0^1 g(s)\biggl(\sum_{s\le t_k<1}\bigl[I_k\bigl(x(t_k)\bigr)+(1-t_k)\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\bigr]\biggr)\,ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_0^1 g(s)x(s)\,ds \\
&\quad= \frac{1}{1-\sigma}\int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)\varphi^{-1}\biggl(\int_0^\tau f\bigl(\eta,x(\eta),x'(\eta),(Tx)(\eta),(Sx)(\eta)\bigr)\,d\eta\biggr)\,d\tau \\
&\qquad+ (1-s)\sum_{0<t_k<s}\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr) \\
&\qquad+ \sum_{s\le t_k<1}\bigl[I_k\bigl(x(t_k)\bigr)+(1-t_k)\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\bigr]\biggr]\,ds. \qquad (2.12)
\end{aligned}$$

According to (2.11) and (2.12), we derive

$$\begin{aligned}
x(t) &= \int_0^1 G(t,s)\varphi^{-1}\biggl(\int_0^s f\bigl(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad+ (1-t)\sum_{0<t_k<t}\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr) + \sum_{t\le t_k<1}\bigl[I_k\bigl(x(t_k)\bigr)+(1-t_k)\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\bigr] \\
&\quad+ \frac{1}{1-\sigma}\int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)\varphi^{-1}\biggl(\int_0^\tau f\bigl(\eta,x(\eta),x'(\eta),(Tx)(\eta),(Sx)(\eta)\bigr)\,d\eta\biggr)\,d\tau \\
&\quad+ (1-s)\sum_{0<t_k<s}\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr) + \sum_{s\le t_k<1}\bigl[I_k\bigl(x(t_k)\bigr)+(1-t_k)\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\bigr]\biggr]\,ds \\
&= \int_0^1 G(t,s)\varphi^{-1}\biggl(\int_0^s f\bigl(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad+ \frac{1}{1-\sigma}\int_0^1\biggl[\int_0^1 g(\tau)G(\tau,s)\,d\tau\biggr]\varphi^{-1}\biggl(\int_0^s f\bigl(\eta,x(\eta),x'(\eta),(Tx)(\eta),(Sx)(\eta)\bigr)\,d\eta\biggr)\,ds \\
&\quad+ (1-t)\sum_{0<t_k<t}\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr) + \sum_{t\le t_k<1}\bigl[I_k\bigl(x(t_k)\bigr)+(1-t_k)\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\bigr] \\
&\quad+ \frac{1}{1-\sigma}\int_0^1 g(s)\biggl[(1-s)\sum_{0<t_k<s}\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr) \\
&\quad+ \sum_{s\le t_k<1}\bigl[I_k\bigl(x(t_k)\bigr)+(1-t_k)\bar{I}_k\bigl(x(t_k),x'(t_k)\bigr)\bigr]\biggr]\,ds
\end{aligned}$$

$$= \int_0^1 H(t,s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds + \phi(t) + \psi(t)$$
$$+ \frac{1}{1-\sigma}\int_0^1 g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\psi(s)\,ds, \tag{2.13}$$

where $H(t,s)$, $\phi(s)$ and $\psi(s)$ are defined as (2.2), (2.4), and (2.5), respectively. To sum up, we know that x is a solution of the impulsive integral equation (2.1).

Next, we show the sufficiency of Lemma 2.2. Let $x \in C^3[\bar{J},\mathbb{E}]$ is a solution of impulsive integral equation (2.1). Taking the derivative at both sides of (2.1), we have

$$x'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds + \phi'(t) + \psi'(t)$$
$$= -\int_0^t \varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds$$
$$- \sum_{0<t_k<t} \bar{I}_k\big(x(t_k),x'(t_k)\big). \tag{2.14}$$

Equation (2.14) gives $x'(0) = \theta$. Taking the derivative at both sides of (2.14), we get

$$x''(t) = -\varphi^{-1}\left(\int_0^t f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right),$$

which implies that $x''(0) = \theta$ and

$$\big(\varphi\big(-x''(t)\big)\big)' = f\big(t,x(t),x'(t),(Tx)(t),(Sx)(t)\big).$$

From (2.11) and noting that $G(0,s) = G(1,s) = 0$, we obtain

$$x(1) = \int_0^1 g(t)x(t)\,dt.$$

In addition, it follows from (2.1) that

$$x\big(t_i^-\big) = x(t_i)$$
$$= \int_0^1 H(t_i,s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds + \phi(t_i) + \psi(t_i)$$
$$+ \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\psi(s)\,ds$$
$$= \int_0^1 H(t_i,s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds$$
$$+ (1-t_i)\sum_{k=1}^{i-1}\bar{I}_k\big(x(t_k),x'(t_k)\big) + \sum_{k=i}^{m}\big[I_k\big(x(t_k)\big) + (1-t_k)\bar{I}_k\big(x(t_k),x'(t_k)\big)\big]$$
$$+ \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\psi(s)\,ds$$

and

$$\begin{aligned}
x(t_i^+) &= \int_0^1 H(t_i,s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds + \phi\big(t_i^+\big) + \psi\big(t_i^+\big) \\
&\quad + \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\psi(s)\,ds \\
&= \int_0^1 H(t_i,s)\varphi^{-1}\left(\int_0^s f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\,d\tau\right)ds \\
&\quad + (1-t_i)\sum_{k=1}^{i}\overline{I}_k\big(x(t_k),x'(t_k)\big) + \sum_{k=i+1}^{m}\big[I_k\big(x(t_k)\big) + (1-t_k)\overline{I}_k\big(x(t_k),x'(t_k)\big)\big] \\
&\quad + \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\psi(s)\,ds.
\end{aligned}$$

Thus we derive

$$\Delta x|_{t=t_i} = x\big(t_i^+\big) - x\big(t_i^-\big) = -I_k\big(x(t_k)\big), \quad k = 1,\ldots,m.$$

Similarly, by means of (2.14), we have

$$\Delta x'|_{t=t_k} = x'\big(t_i^+\big) - x'\big(t_i^-\big) = -\overline{I}_k\big(x(t_k),x'(t_k)\big), \quad k = 1,\ldots,m.$$

From the above discussions, we know that $x(t)$ is a solution of (1.1). Thus the proof of Lemma 2.2 is complete. □

Lemma 2.3 *Denote* $\sigma := \int_0^1 g(s)\,ds$. *If* $0 \le \sigma < 1$ *and* $g \in L^1[0,1]$ *is nonnegative, for any* $\delta \in (0,\frac{1}{2})$, *then the functions* $H(t,s)$ *and* $G(t,s)$ *defined by* (2.2) *and* (2.3) *have the following properties*:

(i) $0 \le G(t,s) \le G(s,s) = 1-s$ *and* $0 \le H(t,s) \le \frac{1}{1-\sigma}G(s,s)$, *for all* $t,s \in [0,1]$;

(ii) $0 \le H_t'(t,s) \le 1-s = G(s,s)$, *for all* $t,s \in [0,1]$;

(iii) $G(t,s) \ge \delta G(s,s)$ *and* $H(t,s) \ge \frac{\delta}{1-\sigma}G(s,s)$, *for all* $t,s \in [\delta,1-\delta] \subset [0,1]$.

Proof (i) When $0 \le t \le s \le 1$, $G(t,s) = G(s,s) \equiv 1-s$. When $0 \le s \le t \le 1$, $G_t'(t,s) = -1 < 0$ implies $0 \le G(t,s) = 1-t \le 1-s = G(s,s)$. Thus we have $0 \le G(t,s) \le G(s,s) = 1-s$, $\forall s,t \in [0,1]$ and for all $s,t \in [0,1]$

$$\begin{aligned}
0 \le H(t,s) &= G(s,t) + \frac{1}{1-\sigma}\int_0^1 g(\tau)G(\tau,s)\,d\tau \\
&\le G(s,s) + \frac{1}{1-\sigma}\int_0^1 g(\tau)G(s,s)\,d\tau \\
&\le G(s,s) + \frac{1}{1-\sigma}\int_0^1 g(\tau)G(s,s)\,d\tau = \frac{1}{1-\sigma}G(s,s).
\end{aligned}$$

(ii) When $0 \le s \le t \le 1$, $H_t'(t,s) = -1 < 1-s$. When $0 \le t \le s \le 1$, $H_t'(t,s) = 0 < 1-s$.

(iii) When $0 < \delta \le t \le s \le 1-\delta \le 1$, $G(t,s) \equiv 1-s \ge \delta(1-s) = \delta G(s,s)$. When $0 \le \delta \le s \le t \le 1-\delta \le 1$, $G_t'(t,s) = -1 \le 0$ implies $G(t,s) = 1-t \ge 1-(1-\delta) = \delta \ge \delta(1-s) = \delta G(s,s)$.

Thus we have $G(t,s) \geq \delta G(s,s)$, $\forall s,t \in [\delta, 1-\delta] \subset [0,1]$ and for all $s,t \in [\delta, 1-\delta] \subset [0,1]$

$$\begin{aligned} H(t,s) &= G(s,t) + \frac{1}{1-\sigma}\int_0^1 g(\tau)G(\tau,s)\,d\tau \\ &\geq \delta G(s,s) + \frac{1}{1-\sigma}\int_0^1 g(\tau)\delta G(s,s)\,d\tau = \frac{\delta}{1-\sigma}G(s,s). \end{aligned}$$

Thus the proof of Lemma 2.3 is complete. □

Denote $K := \{x \in \mathrm{PC}[\overline{J},\mathbb{P}] : x \geq \delta x, t \in [0,1]\}$, $0 < \delta < \frac{1}{2}$. For any $0 < r < +\infty$, let $K_r := \{x \in K : \|x\| < r\}$, $\partial K_r := \{x \in K : \|x\| = r\}$, $\overline{K}_r := \{x \in K : \|x\| \leq r\}$.

According to Lemma 2.2, we define an operator $A : K \to K$ as

$$\begin{aligned} (Ax)(t) &= \int_0^1 H(t,s)\varphi^{-1}\left(\int_0^s f\big(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)\big)\,d\tau\right)ds + \phi(t) + \varphi(t) \\ &\quad + \frac{1}{1-\sigma}\int_0^1 g(s)\phi(s)\,ds + \frac{1}{1-\sigma}\int_0^1 g(s)\varphi(s)\,ds. \end{aligned} \tag{2.15}$$

It is easy to derive that x is a positive solution of BVP (1.1) if x is a nontrivial fixed point of operator A defined as (2.15).

For convenience and simplicity, we introduce some notations and assumptions as follows:

$$\begin{aligned} f_\nu &= \liminf_{\sum_{i=1}^4 \|x_i\| \to \nu} \min_{t\in[0,1]} \frac{\|f(t,x_1,x_2,x_3,x_4)\|}{\varphi(\sum_{i=1}^4 \|x_i\|)}, \\ f^\nu &= \limsup_{\sum_{i=1}^4 \|x_i\| \to \nu} \max_{t\in[0,1]} \frac{\|f(t,x_1,x_2,x_3,x_4)\|}{\varphi(\sum_{i=1}^4 \|x_i\|)}. \end{aligned}$$

Here ν denotes 0 or ∞.

(H_1) $0 \leq \sigma = \int_0^1 g(s)\,ds < 1$ and $g \in L^1[0,1]$ is nonnegative.

(H_2) For any bounded set $B_i \subset \mathbb{E}$ ($i = 1,2,3,4$), $f(t,B_1,B_2,B_3,B_4)$ and $I_k(B_1)$ together with $\overline{I}_k(B_1,B_2)$ ($k = 1,2,\ldots,m$) are relatively compact sets, where $f(t,B_1,B_2,B_3,B_4) := \{f(t,w_1,w_2,w_3,w_4) : w_i \in B_i, i = 1,2,3,4\}$, $I_k(B_1) := \{I_k(w_1) : w_i \in B_1\}$ and $\overline{I}_k(B_1,B_2) := \{\overline{I}_k(w_1,w_2) : w_i \in B_i, i = 1,2\}$.

(H_3) For $I_k \in C(\mathbb{P},\mathbb{P}^+)$, $\overline{I}_k \in C(\mathbb{P}^2,\mathbb{P}^+)$ ($\mathbb{P}^+$ represents a positive cone in a real Banach space $\mathbb{E}$, there exist some positive constants ν_k, ν_k^* and $\overline{\nu}_k$ ($k = 1,\ldots,m$) such that

$$\big\|I_k(x)\big\| \leq \nu_k\|x\|, \qquad \big\|\overline{I}_k(x_1,x_2)\big\| \leq \nu_k^*\|x_1\| + \overline{\nu}_k\|x_2\|, \quad \forall t \in J.$$

Lemma 2.4 *Suppose that* (H_1)-(H_3) *hold.* $A : K \to K$ *defined by* (2.15) *is completely continuous.*

Proof (1) Now we show that $A(K) \subset K$, that is, $A : K \to K$ is well defined. Clearly, $(Ax)(t) \geq \delta(Ax)(t)$. Hence, $A(K) \subset K$.

(2) We need to show that $A : K \to K$ is continuous. In fact, assume that $x_n, x_0 \in K$ and $\|x_n - x_0\| \to 0$, $\|x_n' - x_0'\| \to 0$ ($n \to \infty$). Since $f \in C(\overline{J} \times \mathbb{P}^4, \mathbb{P}^+)$, $I_k \in C(\mathbb{P},\mathbb{P})$, $\overline{I}_k \in C(\mathbb{P},\mathbb{P}^+)$

and $\varphi : \mathbb{P} \to \mathbb{E}$ is an increasing and positive homomorphism, then

$$\begin{aligned}
&\lim_{n\to\infty} \bigl\| f\bigl(t, x_n, x_n', (Tx_n)(t), (Sx_n)(t)\bigr) - f\bigl(t, x_0, x_0', (Tx_0)(t), (Sx_0)(t)\bigr) \bigr\| = 0, \\
&\lim_{n\to\infty} \bigl\| \varphi^{-1}\bigl(f\bigl(t, x_n, x_n', (Tx_n)(t), (Sx_n)(t)\bigr)\bigr) - \varphi^{-1}\bigl(f\bigl(t, x_0, x_0', (Tx_0)(t), (Sx_0)(t)\bigr)\bigr) \bigr\| = 0, \\
&\lim_{n\to\infty} \bigl\| I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x_0(t_k)\bigr) \bigr\| = 0, \quad k = 1, \ldots, m, \\
&\lim_{n\to\infty} \bigl\| \overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr) \bigr\| = 0, \quad k = 1, \ldots, m.
\end{aligned}$$

Thus, for any $t \in J$, from the Lebesgue dominate convergence theorem and Lemma 2.3 together with (2.15), we have

$$\begin{aligned}
&\bigl\|(Ax_n)(t) - (Ax_0)(t)\bigr\|_{\mathrm{PC}} \\
&\quad\leq \biggl\| \frac{1}{1-\sigma} \int_0^1 G(s,s)\biggl[\varphi^{-1}\biggl(\int_0^s f\bigl(\tau, x_n(\tau), x_n'(\tau), (Tx_n)(\tau), (Sx_n)(\tau)\bigr)\,d\tau\biggr) \\
&\qquad - \varphi^{-1}\biggl(\int_0^s f\bigl(\tau, x_0(\tau), x_0'(\tau), (Tx_0)(\tau), (Sx_0)(\tau)\bigr)\,d\tau\biggr)\biggr]\,ds \\
&\qquad + (1-t) \sum_{0<t_k<t} \bigl[\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr] \\
&\qquad + \sum_{t\leq t_k<1} \bigl[\bigl(I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x_0(t_k)\bigr)\bigr) + (1-t_k)\bigl(\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr)\bigr] \\
&\qquad + \frac{1}{1-\sigma} \int_0^1 (1-s)g(s) \sum_{0<t_k<s} \bigl[\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr]\,ds \\
&\qquad + \frac{1}{1-\sigma} \int_0^1 g(s) \sum_{t\leq t_k<1} \bigl[\bigl(I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x_0(t_k)\bigr)\bigr) \\
&\qquad + (1-t_k)\bigl(\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr)\bigr]\,ds \biggr\| \\
&\quad\leq \frac{1}{1-\sigma} \int_0^1 G(s,s)\biggl\| \varphi^{-1}\biggl(\int_0^s f\bigl(\tau, x_n(\tau), x_n'(\tau), (Tx_n)(\tau), (Sx_n)(\tau)\bigr)\,d\tau\biggr) \\
&\qquad - \varphi^{-1}\biggl(\int_0^s f\bigl(\tau, x_0(\tau), x_0'(\tau), (Tx_0)(\tau), (Sx_0)(\tau)\bigr)\,d\tau\biggr)\biggr\|\,ds \\
&\qquad + (1-t) \sum_{0<t_k<t} \bigl\|\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr\| \\
&\qquad + \sum_{t\leq t_k<1} \bigl[\bigl\|I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x_0(t_k)\bigr)\bigr\| + (1-t_k)\bigl\|\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr\|\bigr] \\
&\qquad + \frac{1}{1-\sigma} \int_0^1 (1-s)g(s) \sum_{0<t_k<s} \bigl\|\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr\|\,ds \\
&\qquad + \frac{1}{1-\sigma} \int_0^1 g(s) \sum_{t\leq t_k<1} \bigl[\bigl\|I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x_0(t_k)\bigr)\bigr\| \\
&\qquad + (1-t_k)\bigl\|\overline{I}_k\bigl(x_n(t_k), x_n'(t_k)\bigr) - \overline{I}_k\bigl(x_0(t_k), x_0'(t_k)\bigr)\bigr\|\bigr]\,ds \to 0, \quad \text{as } n \to \infty
\end{aligned}$$

and

$$
\begin{aligned}
&\left\|(Ax_n)'(t)-(Ax_0)'(t)\right\|_{\mathrm{PC}}\\
&\quad\le\left\|\frac{1}{1-\sigma}\int_0^1 H_t'(t,s)\left[\varphi^{-1}\left(\int_0^s f\left(\tau,x_n(\tau),x_n'(\tau),(Tx_n)(\tau),(Sx_n)(\tau)\right)d\tau\right)\right.\right.\\
&\qquad\left.-\varphi^{-1}\left(\int_0^s f\left(\tau,x_0(\tau),x_0'(\tau),(Tx_0)(\tau),(Sx_0)(\tau)\right)d\tau\right)\right]ds\\
&\qquad\left.-\sum_{0<t_k<t}\left[\overline{I}_k\left(x_n(t_k),x_n'(t_k)\right)-\overline{I}_k\left(x_0(t_k),x_0'(t_k)\right)\right]\right\|\\
&\quad\le\frac{1}{1-\sigma}\int_0^1 G(s,s)\left\|\varphi^{-1}\left(\int_0^s f\left(\tau,x_n(\tau),x_n'(\tau),(Tx_n)(\tau),(Sx_n)(\tau)\right)d\tau\right)\right.\\
&\qquad\left.-\varphi^{-1}\left(\int_0^s f\left(\tau,x_0(\tau),x_0'(\tau),(Tx_0)(\tau),(Sx_0)(\tau)\right)d\tau\right)\right\|ds\\
&\qquad+\sum_{0<t_k<t}\left\|\overline{I}_k\left(x_n(t_k),x_n'(t_k)\right)-\overline{I}_k\left(x_0(t_k),x_0'(t_k)\right)\right\|\to 0,\quad\text{as } n\to\infty.
\end{aligned}
$$

Hence, $A:K\to K$ is continuous.

(3) Now we are going to prove that A is compact by the Kuratowski's measure of noncompactness. Let $\Omega\subset K$ be any bounded subset in $\mathrm{PC}(J,\mathbb{E})$. For any $x\in\Omega$, $t,s\in[0,1]$, there exist some constants $L_i>0$ ($i=1,2,3,4$) such that

$$
\begin{aligned}
&\max_{t,s\in[0,1]}\left|H(t,s)\right|\le L_1,\qquad \max_{x\in\Omega,t\in[0,1]}\left\|f\left(t,x(t),x'(t),(Sx)(t),(Tx)(t)\right)\right\|\le L_2,\\
&\max_{1\le k\le m}\max_{x\in\Omega}\left\|I_k\left(x(t_k)\right)\right\|\le L_3,\qquad \max_{1\le k\le m}\max_{x\in\Omega}\left\|\overline{I}_k\left(x(t_k),x'(t_k)\right)\right\|\le L_4.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\left\|\phi(t)\right\|_{\mathrm{PC}}\le|1-t|\sum_{0<t_k<t}\left\|\overline{I}_k\left(x(t_k),x'(t_k)\right)\right\|\le\sum_{k=0}^{m}\left\|\overline{I}_k\left(x(t_k),x'(t_k)\right)\right\|\le mL_4,\\
&\left\|\psi(t)\right\|_{\mathrm{PC}}\le\sum_{t\le t_k<1}\left[\left\|I_k\left(x(t_k)\right)\right\|+|1-t_k|\left\|\overline{I}_k\left(x(t_k),x'(t_k)\right)\right\|\right]\\
&\qquad\qquad\le\sum_{k=0}^{m}\left[\left\|I_k\left(x(t_k)\right)\right\|+|1-t_k|\left\|\overline{I}_k\left(x(t_k),x'(t_k)\right)\right\|\right]\le m(L_3+L_4),\\
&\left\|(Ax)(t)\right\|_{\mathrm{PC}}\le\int_0^1\left|H(t,s)\right|\varphi^{-1}\left(\int_0^s\left\|f\left(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\right)\right\|d\tau\right)ds\\
&\qquad\qquad+\left\|\phi(t)\right\|_{\mathrm{PC}}+\left\|\psi(t)\right\|_{\mathrm{PC}}\\
&\qquad\qquad+\frac{1}{1-\sigma}\int_0^1 g(s)\left\|\phi(s)\right\|ds+\frac{1}{1-\sigma}\int_0^1 g(s)\left\|\varphi(s)\right\|ds\\
&\qquad\qquad\le L_1\varphi^{-1}(L_2)+mL_4+m(L_3+L_4)+\frac{mL_4\sigma}{1-\sigma}+\frac{m\sigma(L_3+L4)}{1-\sigma}\\
&\qquad\qquad=L_1\varphi^{-1}(L_2)+mL_3+2mL_4+\frac{m\sigma(L_3+2L_4)}{1-\sigma}:=M_1,
\end{aligned}
$$

and

$$\begin{aligned}
&\big\|(Ax)'(t)\big\|_{\mathrm{PC}} \\
&\quad \le \int_0^1 \big|H_t'(t,s)\big|\varphi^{-1}\left(\int_0^s \big\|f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\big\|\,d\tau\right)ds + \big\|\phi'(t)\big\| \\
&\quad \le \varphi^{-1}(L_1) + mL_4 := M_2.
\end{aligned}$$

Therefore, $A(\Omega)$ is uniformly bounded in $\mathrm{PC}(J,\mathbb{E})$.

Next we verify that $A : K \to K$ is equicontinuous. In fact, since $H(t,s)$ are continuous on $[0,1] \times [0,1]$, it is uniformly continuous on $[0,1] \times [0,1]$. Thus, for fixed $s \in [0,1]$ and for any $\epsilon > 0$, there exists a constant $\delta_1 > 0$ such that for any $\bar{t}_1, \bar{t}_2 \in [0,1]$ with $|\bar{t}_2 - \bar{t}_1| < \delta_1$, we have $|H(\bar{t}_2,s) - H(\bar{t}_1,s)| < \epsilon$. Without loss of the generality, assume that $\bar{t}_1, \bar{t}_2 \in [0,1]$ with $\bar{t}_1 \le \bar{t}_1$, then

$$\begin{aligned}
&\big\|(Ax)(\bar{t}_2) - (Ax)(\bar{t}_1)\big\|_{\mathrm{PC}} \\
&\quad \le \int_0^1 \big|H(\bar{t}_2,s) - H(\bar{t}_1,s)\big|\varphi^{-1}\left(\int_0^s \big\|f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\big\|\,d\tau\right)ds \\
&\qquad + \sum_{\bar{t}_1 \le t_k < \bar{t}_2} \big\|\bar{I}_k\big(x(t_k),x'(t_k)\big)\big\| + |t_2 - t_1| \sum_{0 < t_k < \bar{t}_1} \big\|\bar{I}_k\big(x(t_k),x'(t_k)\big)\big\| \\
&\qquad + \sum_{\bar{t}_1 \le t_k < \bar{t}_2} \big[\big\|I_k\big(x(t_k)\big)\big\| + |1 - t_k|\big\|\bar{I}_k\big(x(t_k),x'(t_k)\big)\big\|\big] \\
&\quad \le |\bar{t}_2 - \bar{t}_1|\varphi^{-1}(L_2) + |\bar{t}_2 - \bar{t}_1|L_3 + |\bar{t}_2 - \bar{t}_1|mL_4 + |\bar{t}_2 - \bar{t}_1|(L_3 + L_4) \\
&\quad = \big[\varphi^{-1}(L_2) + 2L_3 + (m+1)L_4\big]|\bar{t}_2 - \bar{t}_1| := N|\bar{t}_2 - \bar{t}_1|,
\end{aligned}$$

and noting that

$$H_t'(t,s) = \begin{cases} -1, & 0 \le t \le s \le 1, \\ 0, & 0 \le s \le t \le 1, \end{cases}$$

we have

$$\begin{aligned}
&\big\|(Ax)'(\bar{t}_2) - (Ax)'(\bar{t}_1)\big\|_{\mathrm{PC}} \\
&\quad \le \int_0^1 \big|H_t'(\bar{t}_2,s) - H_t'(\bar{t}_1,s)\big|\varphi^{-1}\left(\int_0^s \big\|f\big(\tau,x(\tau),x'(\tau),(Tx)(\tau),(Sx)(\tau)\big)\big\|\,d\tau\right)ds \\
&\qquad + \sum_{\bar{t}_1 \le t_k < \bar{t}_2} \big\|\bar{I}_k\big(x(t_k),x'(t_k)\big)\big\| \\
&\quad \le L_4|\bar{t}_2 - \bar{t}_1|,
\end{aligned}$$

where $N := \varphi^{-1}(L_2) + 2L_3 + (m+1)L_4$. Take $\delta = \min\{\frac{\epsilon}{N}, \frac{\epsilon}{L_4},\} = \frac{\epsilon}{N}$. Thus, for fixed $s \in [0,1]$ and for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that, for any $\bar{t}_1, \bar{t}_2 \in [0,1]$ with $|\bar{t}_2 - \bar{t}_1| < \delta$, we have $\|(Ax)(\bar{t}_2) - (Ax)(\bar{t}_1)\|_{\mathrm{PC}^1} < \epsilon$, which means that $A : K \to K$ is equicontinuous. Hence, by [11], Lemma 7, we get

$$\alpha_{\mathrm{PC}^1}\big(A(\Omega)\big) = \max\Big\{\sup_{t \in J} \alpha\big(A(\Omega)\big)(t), \sup_{t \in J} \alpha\big(A'(\Omega)\big)(t)\Big\}, \tag{2.16}$$

where $A(\Omega) := \{(Ax)(t) : x \in \Omega\}$ and $A'(\Omega) := \{(Ax)'(t) : x \in \Omega\}$, α, α_{PC^1} denote the Kuratowski measures of noncompactness of bounded sets in $\mathbb{E}$ and $\mathrm{PC}^1(J,\mathbb{E})$, respectively (see [26], Section 1.2).

It follows from (2.15) and [26], Theorem 1.2.3, that

$$\begin{aligned}
&\alpha\bigl((A\Omega)\bigr)(t)\\
&\quad\le \frac{1}{1-\sigma}\int_0^1 (1-s)\varphi^{-1}\biggl(2\int_0^s \alpha\bigl(f\bigl(\tau,\Omega(\tau),\Omega'(\tau),(T\Omega)(\tau),(S\Omega)(\tau)\bigr)\,d\tau\bigr)\biggr)\,ds\\
&\qquad + 2\sum_{0<t_k<1}\bigl[\alpha\bigl(I_k\bigl(\Omega(t_k)\bigr)\bigr) + (1-t_k)\alpha\bigl(\overline{I}_k\bigl(\Omega(t_k),x'(t_k)\bigr)\bigr)\bigr]\\
&\qquad + 2(1-t)\sum_{0<t_k<1}\alpha\bigl(\overline{I}_k\bigl(\Omega(t_k),\Omega'(t_k)\bigr)\bigr)\\
&\qquad + \frac{2}{1-\sigma}\int_0^1 g(s)\sum_{0<t_k<1}\bigl[\alpha\bigl(I_k\bigl(\Omega(t_k)\bigr)\bigr) + (1-t_k)\alpha\bigl(\overline{I}_k\bigl(\Omega(t_k),x'(t_k)\bigr)\bigr)\bigr]\,ds\\
&\qquad + \frac{2}{1-\sigma}\int_0^1 (1-s)g(s)\sum_{0<t_k<1}\alpha\bigl(\overline{I}_k\bigl(\Omega(t_k),\Omega'(t_k)\bigr)\bigr)\,ds
\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}
\alpha\bigl((A\Omega)\bigr)'(t) \le{}& \frac{1}{1-\sigma}\int_0^1 (1-s)\varphi^{-1}\biggl(2\int_0^s \alpha\bigl(f\bigl(\tau,\Omega(\tau),\Omega'(\tau),(T\Omega)(\tau),(S\Omega)(\tau)\bigr)\,d\tau\bigr)\biggr)\,ds\\
&+ 2\sum_{0<t_k<1}\alpha\bigl(\overline{I}_k\bigl(\Omega(t_k),\Omega'(t_k)\bigr)\bigr).
\end{aligned}\tag{2.18}$$

According to condition (H_2), we have

$$\alpha\bigl(f\bigl(\tau,\Omega(\tau),\Omega'(\tau),(T\Omega)(\tau),(S\Omega)(\tau)\bigr)\,d\tau\bigr) = 0,\quad \forall\tau\in J,\tag{2.19}$$

and

$$\alpha\bigl(I_k\bigl(\Omega(t_k)\bigr)\bigr) = 0,\qquad \alpha\bigl(\overline{I}_k\bigl(\Omega(t_k),\Omega'(t_k)\bigr)\bigr) = 0,\quad k = 1,2,\ldots,m.\tag{2.20}$$

It follows from (2.16)-(2.20) that

$$\alpha_{\mathrm{PC}^1}\bigl(A(\Omega)\bigr) = \max\Bigl\{\sup_{t\in J}\alpha\bigl(A(\Omega)\bigr)(t), \sup_{t\in J}\alpha\bigl(A'(\Omega)\bigr)(t)\Bigr\} = 0.$$

Thus the compactness of A is proved. Therefore, $A : K \to K$ defined by (2.15) is completely continuous. The proof is complete. □

3 Main results

In this section, we show that there exists at least one positive solution for BVP (1.1).

Theorem 3.1 *Assume that* (H_1)-(H_3) *hold. If* $0 < f^\infty < \varphi(1-\sigma)$ *and* $\lambda < \frac{1}{2}$, *then BVP* (1.1) *has at least one positive solution, where* $\lambda = \frac{1}{1-\sigma}\sum_{k=1}^m[\nu_k + 2\nu_k^* + 2\overline{\nu}_k]$.

Proof According to $0 < f^\infty < \varphi(1-\sigma)$, there exists $\varepsilon_2 > 0$ such that $f^\infty < M - \varepsilon_2$. From the definition of f^∞, there exists $R_1 > 1$ such that $\|f(t,x_1,x_2,x_3,x_4)\| \le f^\infty \varphi(\sum_{i=1}^4 \|x_i\|)$ for all $\sum_{i=1}^4 \|x_i\| \ge R_1$, $t \in [0,1]$. Let $K_{R_1} := \{x \in K : \|x\|_{\mathrm{PC}} < R_1\}$, $\partial K_{R_1} := \{x \in K : \|x\|_{\mathrm{PC}} = R_1\}$. Define an operator $A : K_{R_1} \to K_{R_1}$ by (2.15). By Lemma 2.4, we know that $A : K_{R_1} \to K_{R_1}$ is completely continuous. For any $x \in \partial K_{R_1}$, that is, $\|x\|_{\mathrm{PC}} = R_1$, it follows from ($\mathrm{H}_1$) and ($\mathrm{H}_3$) that

$$\begin{aligned}
&\bigl\|(Ax)(t)\bigr\|_{\mathrm{PC}} \\
&\quad \le \frac{1}{1-\sigma}\int_0^1 G(s,s) \\
&\qquad \times \varphi^{-1}\biggl(\int_0^s \bigl(\varphi(1-\sigma)-\varepsilon_2\bigr)\varphi\bigl(\|x\|_{\mathrm{PC}} + \|x'\|_{\mathrm{PC}} + \|Tx\|_{\mathrm{PC}} + \|Sx\|_{\mathrm{PC}}\bigr)\,d\tau\biggr)\,ds \\
&\qquad + \sum_{0<t_k<t}\bigl[v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr] + \sum_{t<t_k<1}\bigl[v_k\|x\|_{\mathrm{PC}} + v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr] \\
&\qquad + \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\sum_{0<t_k<s}\bigl[v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr]\,ds \\
&\qquad + \frac{1}{1-\sigma}\int_0^1 g(s)\sum_{s<t_k<1}\bigl[v_k\|x\|_{\mathrm{PC}} + v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr]\,ds \\
&\quad \le \frac{1}{1-\sigma}\varphi^{-1}\bigl(\varphi(1-\sigma)-\varepsilon_2\bigr)\int_0^1 G(s,s)\,ds \\
&\qquad \times \varphi^{-1}\bigl(\varphi\bigl(\|x\|_{\mathrm{PC}} + \|x'\|_{\mathrm{PC}} + \|Tx\|_{\mathrm{PC}} + \|Sx\|_{\mathrm{PC}}\bigr)\bigr) \\
&\qquad + \sum_{k=1}^m\bigl[v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr] + \sum_{k=1}^m\bigl[v_k\|x\|_{\mathrm{PC}} + v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr] \\
&\qquad + \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\sum_{k=1}^m\bigl[v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr]\,ds \\
&\qquad + \frac{1}{1-\sigma}\int_0^1 g(s)\sum_{k=1}^m\bigl[v_k\|x\|_{\mathrm{PC}} + v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr]\,ds \\
&\quad \le \frac{1}{1-\sigma}\varphi^{-1}\bigl(\varphi(1-\sigma)\bigr)\int_0^1 (1-s)\,ds \times \varphi^{-1}\bigl(\varphi\bigl(\|x\|_{\mathrm{PC}} + \|x'\|_{\mathrm{PC}} + \|Tx\|_{\mathrm{PC}} + \|Sx\|_{\mathrm{PC}}\bigr)\bigr) \\
&\qquad + \sum_{k=1}^m\bigl[v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr] + \sum_{k=1}^m\bigl[v_k\|x\|_{\mathrm{PC}} + v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr] \\
&\qquad + \frac{1}{1-\sigma}\int_0^1 (1-s)g(s)\sum_{k=1}^m\bigl[v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr]\,ds \\
&\qquad + \frac{1}{1-\sigma}\int_0^1 g(s)\sum_{k=1}^m\bigl[v_k\|x\|_{\mathrm{PC}} + v_k^*\|x\|_{\mathrm{PC}} + \bar{v}_k\|x'\|_{\mathrm{PC}}\bigr]\,ds \\
&\quad \le \frac{1}{1-\sigma}\times(1-\sigma)\times\frac{1}{2}\times\|x\|_{\mathrm{PC}} + \|x\|_{\mathrm{PC}}\sum_{k=1}^m\bigl[v_k^* + \bar{v}_k\bigr] + \|x\|_{\mathrm{PC}}\sum_{k=1}^m\bigl[v_k + v_k^* + \bar{v}_k\bigr] \\
&\qquad + \frac{\sigma\|x\|_{\mathrm{PC}}}{1-\sigma}\sum_{k=1}^m\bigl[v_k^* + \bar{v}_k\bigr] + \frac{\sigma\|x\|_{\mathrm{PC}}}{1-\sigma}\sum_{k=1}^m\bigl[v_k + v_k^* + \bar{v}_k\bigr] \\
&\quad \le \frac{1}{2}\|x\|_{\mathrm{PC}} + \lambda\|x\|_{\mathrm{PC}} < \frac{1}{2}\|x\|_{\mathrm{PC}} + \frac{1}{2}\|x\|_{\mathrm{PC}} = \|x\|_{\mathrm{PC}} = R_1.
\end{aligned} \tag{3.1}$$

It is worth noticing that the inequality of (3.1) is strict. So $Ax \neq x$ for all $\|x\|_{\mathrm{PC}} = R_1$. By way of (2) of Lemma 2.1, we obtain

$$i(A, K_{R_1} \cap K, K) = 1. \tag{3.2}$$

Therefore, A has at least one positive fixed point on $x \in K_{R_1} \cap K$, namely, $\|x\|_{\mathrm{PC}} \leq R_1$ and $\|Ax\|_{\mathrm{PC}} > \delta \|A\|_{\mathrm{PC}}$. This positive fixed point x of A is a solution of BVP (1.1). □

Similar to the proof of 3.1, we have Theorem 3.2.

Theorem 3.2 *Assume that* (H_1)-(H_3) *hold. If* $0 < \lambda < 1$ *and* $0 < f^0 < \varphi(\frac{2(1-\lambda)}{\rho})$, *for all* $\rho > \frac{1}{1-\lambda}$, *then BVP* (1.1) *has at least one positive solution, where* $\lambda = \frac{1}{1-\sigma} \sum_{k=1}^{m} [\nu_k + 2\nu_k^* + 2\overline{\nu}_k]$.

Next similarly we prove some corollaries for BVP (1.3) and (1.4).

Corollary 3.1 *Assume that* (H_1')-(H_3') *hold.*

(H_1') $0 \leq \sigma = \int_0^1 g(s)\,ds < 1$ *and* $g \in L^1[0,1]$ *is nonnegative.*

(H_2') *For any bounded set* $B_i \subset \mathbb{E}$ $(i = 1,2,3,4)$, $f(t, B_1, B_2, B_3, B_4)$ *and* $I_k(B_1)$ *together with* $\overline{I}_k(B_1, B_2)$ $(k = 1, 2, \ldots, m)$ *are relatively compact sets, where* $f(t, B_1, B_2, B_3, B_4) := \{f(t, w_1, w_2, w_3, w_4) : w_i \in B_i, i = 1,2,3,4\}$, $I_k(B_1) := \{I_k(w_1) : w_i \in B_1\}$ *and* $\overline{I}_k(B_1, B_2) := \{\overline{I}_k(w_1, w_2) : w_i \in B_i, i = 1,2\}$.

(H_3') *For* $I_k \in C(\mathbb{P}, \mathbb{P}^+)$, $\overline{I}_k \in C(\mathbb{P}^2, \mathbb{P}^+)$ ($\mathbb{P}^+$ *represents a positive cone in a real Banach space* $\mathbb{E}$, *there exist some positive constants* ν_k, ν_k^* *and* $\overline{\nu}_k$ $(k = 1, \ldots, m)$ *such that*

$$\|I_k(x)\| \leq \nu_k \|x\|, \qquad \|\overline{I}_k(x_1, x_2)\| \leq \nu_k^* \|x_1\| + \overline{\nu}_k \|x_2\|, \quad \forall t \in J.$$

If $0 < f^\infty < 1 - \sigma$ *and* $\lambda < \frac{1}{2}$, *then BVP* (1.3) *has at least one positive solution, where* $\lambda = \frac{1}{1-\sigma} \sum_{k=1}^{m} [\nu_k + 2\nu_k^* + 2\overline{\nu}_k]$.

Corollary 3.2 *Assume that* (H_1')-(H_3') *hold. If* $0 < \lambda < 1$ *and* $0 < f^0 < \frac{2(1-\lambda)}{\rho}$, *for all* $\rho > \frac{1}{1-\lambda}$, *then BVP* (1.3) *has at least one positive solution, where* $\lambda = \frac{1}{1-\sigma} \sum_{k=1}^{m} [\nu_k + 2\nu_k^* + 2\overline{\nu}_k]$.

Corollary 3.3 *Assume that* (H_1'')-(H_3'') *hold.*

(H_1'') $0 \leq \sigma = \int_0^1 g(s)\,ds < 1$ *and* $g \in L^1[0,1]$ *is nonnegative.*

(H_2'') *For any bounded set* $B_i \subset \mathbb{E}$ $(i = 1,2,3,4)$, $f(t, B_1, B_2, B_3, B_4)$ *and* $I_k(B_1)$ *together with* $\overline{I}_k(B_1, B_2)$ $(k = 1, 2, \ldots, m)$ *are relatively compact sets, where* $f(t, B_1, B_2, B_3, B_4) \triangleq \{f(t, w_1, w_2, w_3, w_4) : w_i \in B_i, i = 1,2,3,4\}$, $I_k(B_1) := \{I_k(w_1) : w_i \in B_1\}$ *and* $\overline{I}_k(B_1, B_2) := \{\overline{I}_k(w_1, w_2) : w_i \in B_i, i = 1,2\}$.

(H_3'') *For* $I_k \in C(\mathbb{P}, \mathbb{P}^+)$, $\overline{I}_k \in C(\mathbb{P}^2, \mathbb{P}^+)$ ($\mathbb{P}^+$ *represents a positive cone in a real Banach space* $\mathbb{E}$, *there exist some positive constants* ν_k, ν_k^* *and* $\overline{\nu}_k$ $(k = 1, \ldots, m)$ *such that*

$$\|I_k(x)\| \leq \nu_k \|x\|, \qquad \|\overline{I}_k(x_1, x_2)\| \leq \nu_k^* \|x_1\| + \overline{\nu}_k \|x_2\|, \quad \forall t \in J.$$

If $0 < f^\infty < (1 - \sigma)^{p-1}$ *and* $\lambda < \frac{1}{2}$, *then BVP* (1.4) *has at least one positive solution, where* $\lambda = \frac{1}{1-\sigma} \sum_{k=1}^{m} [\nu_k + 2\nu_k^* + 2\overline{\nu}_k]$.

Corollary 3.4 *Assume that* (H_1'')-(H_3'') *hold. If* $0 < \lambda < 1$ *and* $0 < f^0 < (\frac{2(1-\lambda)}{\rho})^{p-1}$, *for all* $\rho > \frac{1}{1-\lambda}$, *then BVP* (1.4) *has at least one positive solution, where* $\lambda = \frac{1}{1-\sigma} \sum_{k=1}^{m} [\nu_k + 2\nu_k^* + 2\overline{\nu}_k]$.

4 Illustrative example

Consider the following boundary value problem:

$$\begin{cases} (\varphi(-x''(t)))' = f(t,x(t),x'(t),(Tx)(t),(Sx)(t)), & t \in J, t \neq \frac{1}{4}, \\ \Delta x|_{t_1=\frac{1}{4}} = I_k(x(\frac{1}{4})), \quad k = 1,\dots,m, \\ -\Delta x'|_{t_1=\frac{1}{4}} = \overline{I}_k(x(\frac{1}{4}),x'(\frac{1}{4})), \quad k = 1,\dots,m, \\ x(0) = x''(0) = \theta, \qquad x(1) = \int_0^1 g(t)x(t)\,dt, \end{cases} \tag{4.1}$$

where $m = 1$, $t_1 = \frac{1}{4}$, $I_1(u) = \frac{u}{120}$, $\overline{I}_1(u,v) = \frac{(u^2e^{-v}+v^2e^{-u})}{120(1+|uv|)}$, $g(t) = t$, $K(t,s) = e^{-ts}$, $F(t,s) = e^{-2s}$, $f(t,x,u,v,w) = \frac{\ln(3+t^2)+(t^3|x|+t|u|+|v|+|w|)^3}{1+26(|x|+|u|+|v|+|w|)}$,

$$\varphi(u) = \begin{cases} u^3, & u < 0, \\ u^2, & u > 0. \end{cases}$$

Let $\mathbb{E} = \mathbb{R}$ with the norm $\|x\| = |x|$ and $\mathbb{P} = \{x \in \mathbb{R} : x \geq 0\}$.

Choose $\delta = \frac{1}{5} \in (0, \frac{1}{2})$. By a simple computation, we get $0 < \sigma = \int_0^1 g(s)\,ds = \frac{1}{2} < 1$, $|I_1(u)| = \frac{|u|}{120}$, $|\overline{I}_1(u,v)| \leq \frac{|u|+|v|}{120}$, $v_1 = v_1^* = \overline{v}_1 = \frac{1}{120}$. For all $t, s \in [0,1]$, $0 < K(s,t) < 1$, $0 < F(t,s) < 1$,

$$\varphi^{-1}(u) = \begin{cases} \sqrt[3]{u}, & u < 0, \\ \sqrt{u}, & u > 0, \end{cases}$$

$$\begin{aligned} f^\infty &= \limsup_{|x|+|u|+|v|+|w|\to v} \max_{t\in[0,1]} \frac{|f(t,x,u,v,w)|}{\varphi(|x|+|u|+|v|+|w|)} \\ &= \limsup_{|x|+|u|+|v|+|w|\to\infty} \max_{t\in[0,1]} \frac{\ln(3+t^2)+(t^3|x|+t|u|+|v|+|w|)^3}{[1+26(|x|+|u|+|v|+|w|)](|x|+|u|+|v|+|w|)^2} \\ &= \frac{1}{26} < \frac{1}{4} = \varphi(1-\sigma), \end{aligned}$$

$$\lambda = \frac{1}{1-\sigma}\sum_{k=1}^m \big[v_k + 2v_k^* + 2\overline{v}_k\big] = \frac{1}{1-\sigma}\big[v_1 + 2v_1^* + 2\overline{v}_1\big] = \frac{1}{12} < \frac{1}{2}.$$

Thus all conditions of Theorem 3.1 are satisfied. It follows from Theorem 3.1 that for BVP (4.1) there exists at least one positive solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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