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Impulsive boundary value problem for a fractional differential equation

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Abstract
In this paper, we discuss a boundary value problem for an impulsive fractional differential equation. By transforming the boundary value problem into an equivalent integral equation, and employing the Banach fixed point theorem and the Schauder fixed point theorem, existence results for the solutions are obtained. For application, we provide some examples to illustrate our main results.

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Keywords: boundary value problem; impulsive; fractional derivative; fixed point theorem

1 Introduction

Fractional differential equations have attracted great attention from many researchers because fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes in science and engineering, such as physics, control theory, electrochemistry, biology, viscoelasticity, signal processing, nuclear dynamics, etc. For details, see [1–4] and the references therein. Another important class of differential equations is known as impulsive differential equations. The interest in the study of them is that the impulsive differential system can model the processes which are subject to abrupt changes in their states, refer to [5–9]. Recently, boundary value problems for impulsive fractional differential equations have been attractive to many researchers; see [10–12].

Tian *et al.* [10] developed a sufficient condition for the existence of solutions to the impulsive boundary value problem involving the Caputo fractional derivative as follows:

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) = f(t,x(t)), \quad t \in J = [0,1], t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \qquad \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k)), \quad t_k \in (0,1), k = 1,2,\ldots,m, \\ x(0) = h(x), \qquad x(1) = g(x), \end{cases} \tag{1}$$

where ${}^cD_{0^+}^{\alpha}$ is the Caputo fractional derivative, $\alpha \in \mathbb{R}$, $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, \overline{I}_k$ are continuous functions, g, h are fixed continuous functionals. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, $x(t_k^+), x(t_k^-)$ are the right limit and the left limit of the function $x(t)$ at $t = t_k$, respectively. By applying the Banach fixed point theorem and Kransnoselskii fixed point theorem, the existence results for the solution are obtained.

Shah *et al.* [11] studied the coupled system of fractional impulsive boundary problems

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) = \Phi(t,x(t),y(t)), \quad t \in J = [0,1], t \neq t_i, \\ \Delta x|_{t=t_i} = I_i(x(t_i)), \qquad \Delta x'|_{t=t_i} = \bar{I}_i(x(t_i)), \quad t_i \in (0,1), i = 1,2,\ldots,m, \\ x(0) = h(x), \qquad x(1) = g(x), \\ {}^cD_{0^+}^{\beta}y(t) = \Psi(t,x(t),y(t)), \quad t \in J = [0,1], t \neq t_j, \\ \Delta y|_{t=t_j} = J_i(y(t_j)), \qquad \Delta y'|_{t=t_j} = \bar{J}_i(y(t_j)), \quad t_j \in (0,1), j = 1,2,\ldots,n, \\ y(0) = \kappa(x), \qquad y(1) = f(x), \end{cases} \tag{2}$$

where $1 < \alpha, \beta \leq 2$, $\Phi, \Psi, I_i, \bar{I}_i, J_j, \bar{J}_j$ are continuous functions, g, h, κ, f are fixed continuous functionals, and $\Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-), \Delta x'|_{t=t_i} = x'(t_i^+) - x'(t_i^-)$, $\Delta y|_{t=t_j} = y(t_j^+) - y(t_j^-), \Delta y'|_{t=t_j} = y'(t_j^+) - y'(t_j^-)$. By using the same methods as Tian *et al.* [10], the existence results for the solution are obtained.

Motivated by the above papers, in this paper, we shall transform the impulsive boundary value problem (1) into an equivalent integral equation which is different from the integral equation [10] and [11] have obtained, and we apply the Schauder and Banach fixed point theorems to prove the existence and uniqueness of solutions to the boundary value problem with some growth conditions and Lipschitz conditions. Furthermore, the coupled system like (2) can be investigated in the same way. We note that here $g, h \in PC(J, \mathbb{R})$ are any fixed functionals defined on the Banach space $PC(J, \mathbb{R})$, which will be defined in Section 2. Furthermore, g, h may be given by

$$g(x) = \max_j \frac{|x(\xi_j)|}{\lambda + |x(\xi_j)|}, \qquad h(x) = \min_j \frac{|x(\zeta_j)|}{\kappa + |x(\zeta_j)|},$$

where $0 < \xi_j, \zeta_j < 1$, $\xi_j, \zeta_j \neq t_i$, $j = 1,2,\ldots,n, i = 1,2,\ldots,m$, and λ, κ are given positive constants.

The rest of the paper is organized as follows. In Section 2, we will give some notations, recall some definitions, and introduce some lemmas which are essential to prove our main results. In Section 3, the main results are given, and some examples are presented to demonstrate our main results.

2 Preliminaries

In this section, we introduce notations, definitions, lemmas, and preliminary facts that will be used in the rest of this paper.

Definition 1 ([1]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \, \mathrm{d}s$$

provided that the integral exists.

Definition 2 ([1]) The Caputo fractional derivative of order $\alpha > 0$ of the function f : $(0, \infty) \to \mathbb{R}$ is given by

$${}^cD_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} \, \mathrm{d}s,$$

where $n = [\alpha] + 1$, and the notation $[\alpha]$ stands for the largest integer not greater than α, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 1 ([2]) *For $\alpha > 0, f(t) \in C[0,T] \cap L_1[0,T]$, the homogeneous fractional differential equation*

$$^{c}D^{\alpha}_{0+}f(t) = 0$$

has a solution

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$.

Lemma 2 ([2]) *Assume that $f(t) \in C[0,T] \cap L_1[0,T]$, with derivative of order n that belongs to $C[0,T] \cap L_1[0,T]$, then*

$$I^{\alpha}_{0+}\,{}^{c}D^{\alpha}_{0+}f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$.

We denote $t_0 = 0, t_{m+1} = 1, J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_m = (t_m, 1]$, and the Banach space

$$\begin{aligned} PC(J,\mathbb{R}) = \bigl\{ & x : J \to \mathbb{R}; x(t) \in C\bigl((t_k, t_{k+1}], \mathbb{R}\bigr), k = 0, 1, \ldots, m+1 \\ & \text{and } x\bigl(t_k^+\bigr), x\bigl(t_k^-\bigr) \text{ exist with } x\bigl(t_k^-\bigr) = x(t_k), k = 1, 2, \ldots, m\bigr\} \end{aligned}$$

with the norm $\|x\|_{PC} := \sup\{|x(t)| : t \in J\}$.

We have the following auxiliary lemmas which are useful in the following.

Lemma 3 *Let $1 < \alpha \leq 2$. Assume that $f \in C[J,\mathbb{R}]$. Then the problem*

$$\begin{cases} ^{c}D^{\alpha}_{0^+}x(t) = f(t), & t \in J = [0,1], t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \qquad \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k)), & t_k \in (0,1), k = 1, 2, \ldots, m, \\ x(0) = h(x), \qquad x(1) = g(x), \end{cases} \tag{3}$$

is equivalent to the following integral equation:

$$x(t) = \begin{cases} ct + h(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,\mathrm{d}s, \quad t \in J_0, \\ ct + h(x) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f(s)\,\mathrm{d}s \\ \quad + \sum_{j=1}^k \frac{1}{\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}f(s)\,\mathrm{d}s + \sum_{j=1}^k (t-t_j)\overline{I}_j(x(t_j)) \\ \quad + \sum_{j=1}^k \frac{t-t_j}{\Gamma(\alpha-1)}\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-2}f(s)\,\mathrm{d}s + \sum_{j=1}^k I_j(x(t_j)), \\ \quad t \in J_k, k = 1, 2, \ldots, m, \end{cases} \tag{4}$$

where

$$c = g(x) - h(x) - \sum_{j=1}^{m+1} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} f(s)\,\mathrm{d}s - \sum_{j=1}^{m} I_j\bigl(x(t_j)\bigr)$$
$$- \sum_{j=1}^{m} \frac{1 - t_j}{\Gamma(\alpha - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} f(s)\,\mathrm{d}s - \sum_{j=1}^{m} (1 - t_j)\bar{I}_j\bigl(x(t_j)\bigr). \tag{5}$$

In other words, every solution of (3) *is a solution of* (4) *and vice versa.*

Proof Assume that $x(t)$ is a solution of the impulsive boundary value problem (3), using Lemma 2 we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)\,\mathrm{d}s - c_0 - c_1 t, \quad t \in J_0,$$

for some constants $c_0, c_1 \in \mathbb{R}$. Then we find that

$$x'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha-2} f(s)\,\mathrm{d}s - c_1.$$

The boundary condition $x(0) = -h(x)$ implies that $c_0 = h(x)$ and thus

$$x(t) = ct + h(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in J_0,$$

with the constant $c = -c_1$.

If $t \in J_1$, then also by Lemma 2, we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha-1} f(s)\,\mathrm{d}s - d_0 - d_1(t - t_1)$$

and

$$x'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_{t_1}^t (t - s)^{\alpha-2} f(s)\,\mathrm{d}s - d_1.$$

Then by the conditions $\Delta x|_{t=t_1} = I_1(x(t_1))$ and $\Delta x'|_{t=t_1} = \bar{I}_1(x(t_1))$, we can obtain

$$-d_0 = ct_1 + h(x) + I_1\bigl(x(t_1)\bigr) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s)\,\mathrm{d}s,$$
$$-d_1 = c + \bar{I}_1\bigl(x(t_1)\bigr) + \frac{1}{\Gamma(\alpha - 1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} f(s)\,\mathrm{d}s.$$

Thus, for $t \in J_1$ we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha-1} f(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s)\,\mathrm{d}s$$
$$+ \frac{t - t_1}{\Gamma(\alpha - 1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} f(s)\,\mathrm{d}s + (t - t_1)\bar{I}_1\bigl(x(t_1)\bigr) + I_1\bigl(x(t_1)\bigr) + h(x) + ct.$$

Repeating in the same fashion, we obtain the expression for the solution $x(t)$ for $t \in J_k$ as follows:

$$\begin{aligned} x(t) &= ct + h(x) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s)\,\mathrm{d}s \\ &\quad + \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} f(s)\,\mathrm{d}s + \sum_{j=1}^{k} (t-t_j)\bar{I}_j\bigl(x(t_j)\bigr) \\ &\quad + \sum_{j=1}^{k} \frac{t-t_j}{\Gamma(\alpha-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} f(s)\,\mathrm{d}s + \sum_{j=1}^{k} I_j\bigl(x(t_j)\bigr). \end{aligned} \tag{6}$$

By the application of the boundary condition $x(1) = g(x)$, we can obtain equation (5).

Conversely, we assume that $x(t)$ is a solution of (4). If $t \in J_0$, then, using the fact that ${}^cD_{0^+}^{\alpha}$ is the left inverse of $I_{0^+}^{\alpha}$, we get ${}^cD_{0^+}^{\alpha} x(t) = f(t)$. If $t \in J_k, k = 0, 1, \ldots, m$, due to the fact that the Caputo fractional derivative of a constant is equal to zero, we can verify easily that $x(t)$ satisfies (3), therefore, $x(t)$ is a solution of (3). The lemma is proved. □

3 Main results

According to Lemma 3, we know that the solution of impulsive boundary value problem coincides with the fixed point of the operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined as follows:

$$\begin{aligned} Tx(t) &= c_0 t + (1-t)h(x) + tg(x) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f\bigl(x(s), s\bigr)\,\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f\bigl(x(s), s\bigr)\,\mathrm{d}s + \sum_{0<t_k<t} (t-t_k)\bar{I}_k\bigl(x(t_k)\bigr) \\ &\quad + \sum_{0<t_k<t} \frac{t-t_k}{\Gamma(\alpha-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} f\bigl(x(s), s\bigr)\,\mathrm{d}s + \sum_{0<t_k<t} I_k\bigl(x(t_k)\bigr), \end{aligned} \tag{7}$$

where

$$\begin{aligned} c_0 &= -\sum_{k=1}^{m+1} \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f\bigl(x(s), s\bigr)\,\mathrm{d}s - \sum_{k=1}^{m} I_k\bigl(x(t_k)\bigr) \\ &\quad - \sum_{k=1}^{m} \frac{1-t_k}{\Gamma(\alpha-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} f\bigl(x(s), s\bigr)\,\mathrm{d}s - \sum_{k=1}^{m} (1-t_k)\bar{I}_k\bigl(x(t_k)\bigr). \end{aligned} \tag{8}$$

Our first result is based on the Banach fixed point theorem. Before stating and proving the main result, we introduce the following hypotheses.

(H1) $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the Lipschitz condition

$$\bigl|f(x,t) - f(y,t)\bigr| \le L_1|x-y|$$

with a constant $L_1 > 0$ for any $x, y \in \mathbb{R}$, and $t \in J$.

(H2) $I_k, \bar{I}_k, g$, and h are continuous functions and satisfy the Lipschitz conditions with Lipschitz constants $L_2, L_3, L_4, L_5 > 0$.

Theorem 1 *Assume that* (H1) *and* (H2) *hold. If*

$$\frac{4m+2}{\Gamma(\alpha)}L_1 + 2m(L_2+L_3) + L_4 + L_5 < 1,$$

then the impulsive boundary value problem (1) *has a unique solution.*

Proof The proof is based on the Banach fixed point theorem. Let us denote

$$\sup_{t\in J}\bigl|f(0,t)\bigr| = M_1, \qquad \max_k\bigl|I_k(0)\bigr| = M_2, \qquad \max_k\bigl|\bar{I}_k(0)\bigr| = M_3,$$
$$\bigl|h(0)\bigr| = M_4, \qquad \bigl|g(0)\bigr| = M_5.$$

Consider

$$U_0 := \bigl\{x(t)\in PC(J,\mathbb{R}) : \|x\|_{PC}\le R_0\bigr\},$$

where

$$R_0 \ge \frac{(4m+2)M_1 + 2m\Gamma(\alpha)(M_2+M_3) + \Gamma(\alpha)(M_4+M_5)}{\Gamma(\alpha) - (4m+2)L_1 - 2m\Gamma(\alpha)(L_2+L_3) - \Gamma(\alpha)(L_4+L_5)}.$$

Our first goal in this context is to show that T maps U_0 into U_0. It is clear that T is well defined on $PC(J,\mathbb{R})$. Moreover, for any $x(t)\in U_0$ and $t\in J_k, k=0,1,\ldots,m$, we have

$$\begin{aligned}
|c_0| &\le \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\bigl|f\bigl(x(s),s\bigr)-f(0,s)\bigr|\,\mathrm{d}s + \sum_{k=1}^{m}\bigl|I_k\bigl(x(t_k)\bigr)-I_k(0)\bigr| \\
&\quad + \sum_{k=1}^{m}\bigl|I_k(0)\bigr| + \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\bigl|f\bigl(x(s),s\bigr)-f(0,s)\bigr|\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\bigl|f(0,s)\bigr|\,\mathrm{d}s + \sum_{k=1}^{m}\bigl|\bar{I}_k\bigl(x(t_k)\bigr)-\bar{I}_k(0)\bigr| \\
&\quad + \sum_{k=1}^{m}\bigl|\bar{I}_k(0)\bigr| + \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\bigl|f(0,s)\bigr|\,\mathrm{d}s \\
&\le \frac{m+1}{\Gamma(\alpha+1)}\bigl(L_1\|x\|_{PC}+M_1\bigr) + \frac{m}{\Gamma(\alpha)}\bigl(L_1\|x\|_{PC}+M_1\bigr) + m\bigl(L_2\|x\|_{PC}+M_2\bigr) \\
&\quad + m\bigl(L_3\|x\|_{PC}+M_3\bigr) \\
&\le \frac{2m+1}{\Gamma(\alpha)}\bigl(L_1\|x\|_{PC}+M_1\bigr) + m\bigl(L_2\|x\|_{PC}+M_2\bigr) + m\bigl(L_3\|x\|_{PC}+M_3\bigr)
\end{aligned}$$

and then

$$\begin{aligned}
\bigl|Tx(t)\bigr| &\le |c_0| + \bigl|h(x)\bigr| + \bigl|g(x)\bigr| + \frac{1}{\Gamma(\alpha+1)}\bigl(L_1\|x\|_{PC}+M_1\bigr) + \frac{m}{\Gamma(\alpha)}\bigl(L_1\|x\|_{PC}+M_1\bigr) \\
&\quad + \frac{m}{\Gamma(\alpha+1)}\bigl(L_1\|x\|_{PC}+M_1\bigr) + m\bigl(L_2\|x\|_{PC}+M_2\bigr) + m\bigl(L_3\|x\|_{PC}+M_3\bigr) \\
&\le \frac{4m+2}{\Gamma(\alpha)}\bigl(L_1\|x\|_{PC}+M_1\bigr) + 2m\bigl(L_2\|x\|_{PC}+M_2\bigr) + 2m\bigl(L_3\|x\|_{PC}+M_3\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(L_4\|x\|_{PC} + M_4\right) + \left(L_5\|x\|_{PC} + M_5\right) \\
&\leq R_0.
\end{aligned}
$$

Consequently T maps U_0 into itself.

Now, it remains to show that T is a contraction. Let $x, y \in PC(J, \mathbb{R})$, then, for any $t \in J_k, k = 0, 1, \ldots, m$, we can easily to see from (7) and (8) that

$$
\begin{aligned}
\left|Tx(t) - Ty(t)\right| &\leq \left[\frac{2(m+1)L_1}{\Gamma(\alpha+1)} + \frac{2m}{\Gamma(\alpha)}L_1 + 2mL_2 + 2mL_3 + L_4 + L_5\right]\|x-y\|_{PC} \\
&\leq \left(\frac{4m+2}{\Gamma(\alpha)}L_1 + 2m(L_2 + L_3) + L_4 + L_5\right)\|x-y\|_{PC} \\
&\leq \|x-y\|_{PC}.
\end{aligned}
$$

Hence, T is a contraction and the Banach fixed point theorem implies that T has a unique fixed point on $PC(J, \mathbb{R})$ which is a unique solution to (1). □

The second result is based on the Schauder fixed point theorem. We introduce the following assumptions.

(H3) There exists a nonnegative function $a(t) \in L^1(J, \mathbb{R})$ such that $|f(x,t)| \leq a(t) + l_1|x|^\rho$, and $|I_k(x)| \leq l_2|x|^\mu$, $|\bar{I}_k(x)| \leq l_3|x|^\nu$, $|h(x)| \leq l_4|x|^\theta$, $|g(x)| \leq l_5|x|^\gamma$, for any $x \in \mathbb{R}, k = 1, 2, \ldots, m$, where $l_1 \geq 0, l_i > 0, i = 2, \ldots, 5$, and $0 < \rho, \mu, \nu, \theta, \gamma \leq 1$. Furthermore

$$
\frac{2(2m+1)l_1}{\Gamma(\alpha)} + 2ml_2 + 2ml_3 + l_4 + l_5 < 1.
$$

(H4) $|f(x,t)| \leq l_1|x|^\rho$, and $|I_k(x)| \leq l_2|x|^\mu$, $|\bar{I}_k(x)| \leq l_3|x|^\nu$, $|h(x)| \leq l_4|x|^\theta$, $|g(x)| \leq l_5|x|^\gamma$, for any $x \in \mathbb{R}, k = 1, 2, \ldots, m$, where $l_i > 0, i = 1, 2, \ldots, 5$, and $\rho, \mu, \nu, \theta, \gamma > 1$.

Theorem 2 *Assume that* (H3) *or* (H4) *is satisfied, then the impulsive boundary value problem* (1) *has at least one solution*.

Proof We will use the Schauder fixed point theorem to prove this result. The proof will be given in several steps.

Step 1. T is a continuous operator.

It is very easy to find that T is continuous since $f, g, h, I_k, \bar{I}_k, k = 0, 1, \ldots, m$, are continuous functions. We omit the details.

Step 2. T maps bounded sets into uniformly bounded sets in $PC(J, \mathbb{R})$.

At first, we let the condition (H3) be valid. We denote

$$
U := \left\{x(t) \in PC(J, \mathbb{R}) : \|x\|_{PC} \leq R\right\},
$$

where

$$
R \geq \max\left\{1, \frac{\frac{4}{\Gamma(\alpha)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-2}a(s)\,\mathrm{d}s}{1 - [\frac{2(2m+1)l_1}{\Gamma(\alpha)} + 2ml_2 + 2ml_3 + l_4 + l_5]}\right\}.
$$

Now, we will illustrate that T maps U into itself in the following part. Indeed, for any $x \in U$, we have

$$\begin{aligned}
|c_0| \leq{} & \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} a(s)\,\mathrm{d}s + \frac{l_1 R^\rho}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\,\mathrm{d}s \\
& + ml_2 R^\mu + ml_3 R^\nu + \frac{1}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s \\
& + \frac{l_1 R^\rho}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\,\mathrm{d}s \\
\leq{} & \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} a(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s \\
& + \frac{(m+1) l_1 R^\rho}{\Gamma(\alpha+1)} + \frac{m l_1 R^\rho}{\Gamma(\alpha)} + ml_2 R^\mu + ml_3 R^\nu \\
\leq{} & \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \bigl[(t_k - s)^{\alpha-1} + (t_k - s)^{\alpha-2}\bigr] a(s)\,\mathrm{d}s + \frac{(2m+1) l_1 R^\rho}{\Gamma(\alpha)} + ml_2 R^\mu + ml_3 R^\nu \\
\leq{} & \frac{2}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s + \frac{(2m+1) l_1 R^\rho}{\Gamma(\alpha)} + ml_2 R^\mu + ml_3 R^\nu,
\end{aligned}$$

and then

$$\begin{aligned}
\bigl|Tx(t)\bigr| \leq{} & |c_0| + \bigl|g(x)\bigr| + \bigl|h(x)\bigr| + \frac{l_1 R^\rho}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} a(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} a(s)\,\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} a(s)\,\mathrm{d}s + \frac{l_1 R^\rho}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\,\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s + \frac{l_1 R^\rho}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2}\,\mathrm{d}s \\
& + ml_2 R^\mu + ml_3 R^\nu \\
\leq{} & |c_0| + \bigl|g(x)\bigr| + \bigl|h(x)\bigr| + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} a(s)\,\mathrm{d}s \\
& + \frac{l_1 R^\rho}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\,\mathrm{d}s + \frac{1}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s \\
& + \frac{l_1 R^\rho}{\Gamma(\alpha-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2}\,\mathrm{d}s + ml_2 R^\mu + ml_3 R^\nu \\
\leq{} & |c_0| + \frac{2}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s + \frac{(m+1) l_1 R^\rho}{\Gamma(\alpha+1)} + \frac{m l_1 R^\rho}{\Gamma(\alpha)} \\
& + ml_2 R^\mu + ml_3 R^\nu + l_4 R^\theta + l_5 R^\gamma \\
\leq{} & |c_0| + \frac{2}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s + \frac{(2m+1) l_1 R^\rho}{\Gamma(\alpha)}
\end{aligned}$$

$$\begin{aligned}
&+ml_2R^{\mu}+ml_3R^{\nu}+l_4R^{\theta}+l_5R^{\gamma}\\
&\le \frac{4}{\Gamma(\alpha)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}a(s)\,\mathrm{d}s+\frac{2(2m+1)l_1R^{\rho}}{\Gamma(\alpha)}+2ml_2R^{\mu}+2ml_3R^{\nu}\\
&+l_4R^{\theta}+l_5R^{\gamma}\\
&\le \frac{4}{\Gamma(\alpha)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}a(s)\,\mathrm{d}s+\frac{2(2m+1)l_1R}{\Gamma(\alpha)}+2ml_2R+2ml_3R+l_4R+l_5R\\
&\le R.
\end{aligned}$$

Second, we are in the position to let (H4) be satisfied. Choose

$$0<R\le\min\left\{\left[\frac{\Gamma(\alpha)}{10(2m+1)l_1}\right]^{\frac{1}{\rho-1}},\left(\frac{1}{10ml_2}\right)^{\frac{1}{\mu-1}},\left(\frac{1}{10ml_3}\right)^{\frac{1}{\nu-1}},\left(\frac{1}{5l_4}\right)^{\frac{1}{\theta-1}},\left(\frac{1}{5l_5}\right)^{\frac{1}{\gamma-1}}\right\}.$$

Repeating arguments similar to that above we can arrive at

$$|c_0|\le\frac{(2m+1)l_1R^{\rho}}{\Gamma(\alpha)}+ml_2R^{\mu}+ml_3R^{\nu}$$

and

$$\bigl|Tx(t)\bigr|\le\frac{2(2m+1)l_1R}{\Gamma(\alpha)}+2ml_2R+2ml_3R+l_4R+l_5R\le R.$$

Consequently we have shown that T maps U into itself, it means that T maps bounded sets into uniformly bounded sets.

Step 3. T maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.

U be a bounded set of $PC(J,\mathbb{R})$ as in Step 2. For any $x\in U$, take $N_f=\max_{t\in J}|f(x,t)|+1$, let $t,\tau\in J_k$ with $t<\tau$, we have

$$\begin{aligned}
|c_0|&\le\frac{N_f}{\Gamma(\alpha)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\,\mathrm{d}s+\frac{N_f}{\Gamma(\alpha-1)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\,\mathrm{d}s+ml_2R^{\mu}+ml_3R^{\nu}\\
&\le\frac{(m+1)N_f}{\Gamma(\alpha+1)}+\frac{mN_f}{\Gamma(\alpha)}+ml_2R^{\mu}+ml_3R^{\nu},
\end{aligned}$$

and then

$$\begin{aligned}
\bigl|Tx(\tau)-Tx(t)\bigr|&\le|c_0|(\tau-t)+\frac{1}{\Gamma(\alpha)}\left|\int_{t_k}^{\tau}(\tau-s)^{\alpha-1}f(x,s)\,\mathrm{d}s-\int_{t_k}^{t}(t-s)^{\alpha-1}f(x,s)\,\mathrm{d}s\right|\\
&\quad+\frac{\tau-t}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\bigl|f(x,s)\bigr|\,\mathrm{d}s+\sum_{k=1}^{m}(\tau-t)\bigl|\overline{I}_k(x)\bigr|\\
&\quad+l_4R^{\theta}(\tau-t)+l_5R^{\gamma}(\tau-t)\\
&\le|c_0|(\tau-t)+\frac{N_f}{\Gamma(\alpha)}\left|\int_{t_k}^{\tau}(\tau-s)^{\alpha-1}\,\mathrm{d}s-\int_{t_k}^{t}(t-s)^{\alpha-1}\,\mathrm{d}s\right|+ml_3R^{\nu}(\tau-t)\\
&\quad+\frac{N_f(\tau-t)}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\,\mathrm{d}s+l_4R^{\theta}(\tau-t)+l_5R^{\gamma}(\tau-t)
\end{aligned}$$

$$\begin{aligned}
&\leq \big(|c_0| + ml_3R^{\nu} + l_4R^{\theta} + l_5R^{\gamma}\big)(\tau - t) + \frac{N_f(\tau - t)}{\Gamma(\alpha)} \sum_{k=1}^{m} (t_{k-1} - t_k)^{\alpha-1} \\
&\quad + \frac{N_f}{\Gamma(\alpha)} \left| \int_{t_k}^{t} (\tau - s)^{\alpha-1}\,\mathrm{d}s - \int_{t_k}^{t} (t - s)^{\alpha-1} ds \right| \\
&\leq \big(|c_0| + ml_3R^{\nu} + l_4R^{\theta} + l_5R^{\gamma}\big)(\tau - t) + \frac{N_f(\tau - t)}{\Gamma(\alpha)} \sum_{k=1}^{m} (t_{k-1} - t_k)^{\alpha-1} \\
&\quad + \frac{N_f}{\Gamma(\alpha + 1)} \big[(\tau - t_k)^{\alpha} - (t - t_k)^{\alpha}\big].
\end{aligned}$$

As $\tau \to t$, the right-hand side of the above inequality tends to zero, hence we conclude that T maps bounded sets into equicontinuous sets of $PC(J, \mathbb{R})$.

As a consequence of Step 1 to 3 together with the Ascoli-Arzela theorem we assert that T is a completely continuous operator. From the Schauder fixed point theorem one deduces that T has at least one fixed point which is a solution of the impulsive boundary value problem (1) and the theorem is proved. □

Remark 1 The condition

$$\frac{2(2m + 1)l_1}{\Gamma(\alpha)} + 2ml_2 + 2ml_3 + l_4 + l_5 < 1$$

in the assumption (H3) can be removed, in the proof of this situation, we choose the bounded set $U := \{x(t) \in PC(J, \mathbb{R}) : \|x\|_{PC} \leq R\}$ with

$$R \geq \max\left\{6K, (12ml_3)^{\frac{1}{1-\nu}}, (6l_4)^{\frac{1}{1-\theta}}, (6l_5)^{\frac{1}{1-\gamma}}, \left[\frac{12(2m + 1)l_1}{\Gamma(\alpha)}\right]^{\frac{1}{1-\rho}}\right\},$$

where $K = \frac{4}{\Gamma(\alpha)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} a(s)\,\mathrm{d}s$.

Remark 2 We state that the coupled system likes (2) can be investigated in the same way as the whole arguments we made above. We omit the details.

In the end of this section, we will give some examples to illustrate our results.

Example 1 Consider the following problem:

$$\begin{cases}
{}^cD_{0^+}^{\frac{7}{4}} x(t) = \frac{1}{10} t(t - \frac{1}{3}) \frac{x}{1+x^2}, & t \in J = [0, 1], t \neq \frac{1}{3}, \\
\Delta x(\frac{1}{3}) = \frac{1}{10} \ln(1 + x(\frac{1}{3})^2), & \Delta x'(\frac{1}{3}) = \frac{1}{10+|x(\frac{1}{3})|}, \\
x(0) = \min_j \frac{|x(\zeta_j)|}{15+|x(\zeta_j)|}, & x(1) = \max_j \frac{1}{15+|x(\xi_j)|},
\end{cases}$$

where $\xi_j, \zeta_j \neq \frac{1}{3}, j = 1, 2, \ldots, 10$.

A simple computation shows that $L_1 = L_4 + L_5 = \frac{1}{15}, L_2 = L_3 = \frac{1}{10}$, then

$$\frac{4m + 2}{\Gamma(\alpha)} L_1 + 2m(L_2 + L_3) + L_4 + L_5 \approx 0.969 < 1,$$

Hence, by Theorem 1, this impulsive boundary value problem has a unique solution.

Example 2 Consider the problem

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{3}}x(t) = (t-\frac{1}{2})^4\frac{|x|^{\frac{1}{4}}}{1+|x|}, & t \in J = [0,1], t \neq \frac{1}{2}, \\ \Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2})|^{\frac{1}{4}}}{12+|x(\frac{1}{2})|}, & \Delta x'(\frac{1}{2}) = \frac{|x(\frac{1}{2})|^{\frac{1}{5}}}{10+|x(\frac{1}{2})|}, \\ x(0) = \min_j \frac{|x(\zeta_j)|^{\frac{1}{3}}}{15+|x(\zeta_j)|}, & x(1) = \max_j \frac{|x(\xi_j)|^{\frac{1}{2}}}{15+|x(\xi_j)|}, \end{cases}$$

where $\xi_j, \zeta_j \neq \frac{1}{2}, j = 1, 2, \ldots, 10$.

Since we can get $l_1 = \frac{1}{16}, l_2 = l_5 = \frac{1}{12}, l_3 = l_4 = \frac{1}{15}$, we have

$$\frac{2(2m+1)l_1}{\Gamma(\alpha)} + 2ml_2 + 2ml_3 + l_4 + l_5 \approx 0.865 < 1.$$

Therefore, by Theorem 2, we know that the above problem has at least one solution.

Example 3 We can pay attention to the following class of problems:

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) = (t-\frac{1}{2})^4\frac{|x|^{\frac{1}{\rho}}}{1+|x|}, & t \in J = [0,1], t \neq \frac{1}{2}, \\ \Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2})|^{\frac{1}{\mu}}}{12+|x(\frac{1}{2})|}, & \Delta x'(\frac{1}{2}) = \frac{|x(\frac{1}{2})|^{\frac{1}{\nu}}}{10+|x(\frac{1}{2})|}, \\ x(0) = \min_j \frac{|x(\zeta_j)|^{\frac{1}{\theta}}}{15+|x(\zeta_j)|}, & x(1) = \max_j \frac{|x(\xi_j)|^{\frac{1}{\gamma}}}{15+|x(\xi_j)|}, \end{cases}$$

where $\xi_j, \zeta_j \neq \frac{1}{2}, j = 1, 2, \ldots, 10, 1 < \alpha \leq 2, 0 < \rho, \mu, \nu, \theta, \gamma \leq 1$, or $\rho, \mu, \nu, \theta, \gamma > 1$. By invoking Theorem 2 and Remark 1, we can imply that the above problem has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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