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Weak solutions nonlinear fractional integrodifferential equations in nonreflexive Banach spaces

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Abstract
The aim of this paper is to discuss the existence of weak solutions for a nonlinear two-point boundary value problem of integrodifferential equations of fractional order $\alpha \in (1,2]$. Our analysis relies on the Krasnoselskii fixed point theorem combined with the technique of measure of weak noncompactness.

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1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and they have been emerging as an important area of investigation in the last few decades; see [1–8]. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

In [9], Zhou discusses the existence of solutions for a nonlinear multi-point boundary value problem of integrodifferential equations of fractional order as follows:

$$\begin{cases} {}^cD_{0+}^{\alpha}x(t) = f(t, x(t), (Hx)(t), (Kx)(t)), & t \in [0,1], \alpha \in (1,2], \\ a_1x(0) - b_1x'(0) = d_1x(\xi_1), & a_2x(1) + b_2x'(1) = d_2x(\xi_2), \end{cases}$$

where ${}^cD_{0+}^{\alpha}$ denotes the fractional Caputo derivative and

$$(Hx)(s) = \int_0^t g(t,s)u(s)\,ds, \qquad (Kx)(s) = \int_0^t h(t,s)\,ds,$$

with respect to the strong topology. In [10], Bouffak investigates the existence of weak solutions for a class of boundary value problem of fractional differential equations involving nonlinear integral conditions of the form

$$\begin{cases} {}^cD_{0+}^{\alpha}x(t) = f(t, x(t)), & t \in [0,T], \alpha \in (0,1], \\ x(0) + \mu \int_0^T x(s)\,ds = x(T), \end{cases}$$

by used of the measure of weak noncompactness and Pettis integrals.

In recent years, the theory for boundary value problem of integrodifferential equations of fractional order in Banach spaces endowed with its weak topology has been few studied until now and, in [11], Li and Gou discussed the existence theorem of weak solutions for a class of nonlinear integral equations and obtain a new results by using the techniques of measure of weak noncompactness and Henstock-Kurzweil-Pettis integrals, motivated by this work, in this paper, we use the techniques of the measure of weak noncompactness combined with the fixed point theorem to discuss the existence theorem of weak solutions for a class of nonlinear fractional integrodifferential equations of the form

$$\begin{cases} {}^cD^\alpha_{0+}x(t) = f(t,x(t),(Tx)(t),(Sx)(t)), & t\in[0,1], \alpha\in(1,2], \\ a_1x(0)-a_2x'(0)=\gamma_1, & b_1x(1)+b_2x'(1)=\gamma_2, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha_{0+}$ denotes the fractional Caputo derivative and

$$(Tx)(s) = \int_0^s k_1(s,\tau)g\bigl(\tau,x(\tau)\bigr)\,d\tau, \qquad (Sx)(s) = \int_0^1 k_2(s,\tau)h\bigl(\tau,x(\tau)\bigr)\,d\tau,$$

$f: I\times E^3 \to E$ is a given function satisfying some assumptions that will be specified later, E is a nonreflexive Banach space and the integrals are taken in the sense of Henstock-Kurzweil-Pettis. Also, it is assumed that $a_i, b_i \geq 0$, γ_i, $i=1,2$ are real numbers.

The paper is organized as follows: In Section 2 we recall some basic known results. In Section 3 we discuss the existence theorem of weak solutions for the problem (1.1).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary results which will be used throughout this paper.

Let $I=[0,1]$ be the real interval, let E be a real Banach space with norm $\|\cdot\|$ and its dual space E^*, and also $E_w=(E,w)=(E,\sigma(E,E^*))$ denotes the space E with its weak topology. Denote by $C(I,E_\omega)=(C(I,E),\omega)$ the space of all continuous functions from I to E endowed with the weak topology and the usual supremum norm $\|x\|_\infty = \sup_{t\in I}\|x(t)\|$.

The fundamental tool in this paper is the measure of weak noncompactness developed by De Blasi, for more details see [12].

Now, for the convenience of the reader, we recall some useful definitions of integrals.

Definition 2.1 ([13]) A function $u: I\to E$ is said to be Henstock-Kurzweil integrable on I if there exists an $J\in E$ such that, for every $\varepsilon>0$, there exists $\delta(\xi): I\to\mathbb{R}^+$ such that, for every δ-fine partition $D=\{(I_i,\xi_i)\}_{i=1}^n$, we have

$$\left\| \sum_{i=1}^n u(\xi_i)\mu(I_i) - J \right\| < \varepsilon,$$

we denote the Henstock-Kurzweil integral J by $(HK)\int_a^b u(s)\,ds$.

Definition 2.2 ([13]) A function $f: I\to E$ is said to be Henstock-Kurzweil-Pettis integrable, or simply HKP-integrable on I, if there exists a function $g: I\to E$ with the following properties:

(i) $\forall x^* \in E^*$, x^*f is Henstock-Kurzweil integrable on I;

(ii) $\forall t \in I$, $\forall x^* \in E^*$, $x^*g(t) = (HK)\int_0^t x^*f(s)\,ds$.

This function g will be called a primitive of f and we will denote by $g(t) = \int_0^t f(t)\,dt$ the Henstock-Kurzweil-Pettis integral of f on the interval I.

Theorem 2.1 ([13]; mean value theorem for the *HKP* integral) *If the function $f : I_\alpha \to E$ is HKP integrable, then*

$$\int_I f(t)\,dt \in |I| \cdot \overline{\text{conv}} f(I),$$

where $\overline{\text{co}} f(I)$ is the closure of the convex of $f(I)$, I is an arbitrary subinterval of I_α and $|I|$ is the length of I.

Theorem 2.2 ([14]) *Let $f : I \to E$ and assume that $f_n : I \to E$, $n \in N$, are HKP integrable on I. For each $n \in N$, let F_n be a primitive of f_n. If we assume that:*

(i) $\forall x^* \in E^*$, $x^*(f_n(t)) \to x^*(f(t))$ *a.e. on I,*

(ii) *for each $x^* \in E^*$, the family $G = \{x^*F_n : n = 1, 2, \ldots\}$ is uniformly ACG_* on I (i.e. weakly uniformly ACG_* on I),*

(iii) *for each $x^* \in E^*$, the set G is equicontinuous on I, then f is HKP integrable on I and $\int_0^t f_n(s)\,ds$ tends weakly in E to $\int_0^t f(s)\,ds$ for each $t \in I$.*

Lemma 2.1 ([15]) *If $B \subset C(I,E)$ is equicontinuous, $u_0 \in C(I,E)$, then $\overline{\text{co}}\{B, u_0\}$ is also equicontinuous in $C(I,E)$.*

Lemma 2.2 ([15]) *If $B \subset C(I,E)$ is equicontinuous and bounded, then $\beta(B) = \max_{t\in I} \beta(B(t))$.*

Lemma 2.3 ([15]) *If $B \subset C(I,E)$ is equicontinuous and bounded, then $\beta(B(t)) \in C(I,\mathbb{R}+)$ and*

$$\beta\bigl(B(s)\,ds\bigr) \le \beta\bigl(B(s)\bigr)\,ds, \quad \forall t \in I. \tag{2.1}$$

We give some fixed point theorem, which play a key role in the proofs of our main results.

Theorem 2.3 ([15]) *Let M be a nonempty bounded closed convex subset of a Banach space E. Suppose that $T : M \to M$ is weakly sequentially continuous and there exist an integer n_0 and a vector $x_0 \in M$ such that T is power-convex condensing about x_0 and n_0. Then T has at least one fixed point in M.*

For completeness we recall the definition of the Caputo derivative of fractional order.

Definition 2.3 ([16]) Let $x : I \to E$ be a function. The fractional HKP-integral of the function x of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_{0+}^{\alpha} x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\,ds.$$

In the above definition the sign '$\int$' denotes the HKP-integral.

Lemma 2.4 *For a function $f : I \to E$, the Caputo fractional order derivative of f is defined by*

$$ {}^cD^\alpha_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \quad n-1 < \alpha < n, $$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

3 Main results

In this section, we prove the existence of solutions to the problem (1.1) in the space $C(I, E_\omega)$. Let us start by defining what we mean by a solution of the problem (1.1).

Definition 3.1 A function $x \in C(I, E_w)$ is said to be a solution of the problem (1.1) if x satisfies the equation ${}^cD^\alpha_{0+}x(t) = f(t, x(t), (Tx)(t), (Sx)(t))$ on I and satisfies the conditions $a_1x(0) - a_2x'(0) = \gamma_1$, $b_1x(1) + b_2x'(1) = \gamma_2$.

Lemma 3.1 *Let $\alpha > 0$, then the differential equation*

$$ {}^cD^\alpha_{0+}u(t) = 0 $$

has a solution $u(t) = c_0 + c_1t + c_2t^2 + \cdots + c_nt^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

From the lemma above, we deduce the following statement.

Lemma 3.2 *Let $\alpha > 0$, then*

$$ I^\alpha_{0+}\big({}^cD^\alpha_{0+}u(t)\big) = u(t) + c_0 + c_1t + c_2t^2 + \cdots + c_nt^{n-1} $$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

We derive the corresponding Green's function for the boundary value problem (1.1) which will play major role in our next analysis.

Lemma 3.3 *Let $\rho \in C(I, E_w)$ and $\alpha \in (1, 2]$, then the unique solution of*

$$ \begin{cases} {}^cD^\alpha_{0+}x(t) = \rho(t), & t \in I, \\ a_1x(0) - a_2x'(0) = \gamma_1, & b_1x(1) + b_2x'(1) = \gamma_2, \end{cases} \tag{3.1} $$

is given by

$$ x(t) = \int_0^1 G(t,s)\rho(s)\,ds + \frac{(b_1 + b_2)\gamma_1 + a_2\gamma_2}{l} + \frac{a_1\gamma_2 - b_1\gamma_1}{l}t, \tag{3.2} $$

where the Green's function G is given by

$$ G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}[(t-s)^{\alpha-1} - \frac{a_2b_1}{l}(1-s)^{\alpha-1} - \frac{a_1b_1}{l}(1-s)^{\alpha-1}t] \\ \quad + \frac{1}{\Gamma(\alpha-1)}[-\frac{a_2b_2}{l}(1-s)^{\alpha-2} - \frac{a_1b_2}{l}(1-s)^{\alpha-2}t], & 0 \le s \le t, \\ \frac{1}{\Gamma(\alpha)}[-\frac{a_2b_1}{l}(1-s)^{\alpha-1} - \frac{a_1b_1}{l}(1-s)^{\alpha-1}t] \\ \quad + \frac{1}{\Gamma(\alpha-1)}[-\frac{a_2b_2}{l}(1-s)^{\alpha-2} - \frac{a_1b_2}{l}(1-s)^{\alpha-2}t], & t \le s < 1, \end{cases} \tag{3.3} $$

and $p(t) = \frac{(b_1+b_2)\gamma_1+a_2\gamma_2}{l} + \frac{a_1\gamma_2-b_1\gamma_1}{l}t$, $l = a_1b_1 + a_1b_2 + a_2b_1 \neq 0$, $a_i, b_i \ge 0$, $i = 1, 2$.

Proof Assume that $x(t)$ satisfies (3.1), then Lemma 3.2 implies that

$$x(t) = I_{0+}^{\alpha}\rho(t) - c_1 - c_2 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(t)\,ds - c_1 - c_2 t \tag{3.4}$$

for some constants $c_1, c_2 \in \mathbb{R}$.

On the other hand, by the relations $D_{0+}^{\alpha} I_{0+}^{\alpha} x(t) = x(t)$ and $I_{0+}^{\alpha} I_{0+}^{\beta} x(t) = I_{0+}^{\alpha+\beta} x(t)$, for $\alpha, \beta > 0$, $x \in C(I, E_w)$, we have

$$x'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\rho(s)\,ds - c_2.$$

By the boundary conditions of (3.1), we obtain

$$\begin{cases} -a_1 c_1 + a_2 c_2 = \gamma_1 - a_1 I_{0+}^{\alpha}\rho(0) + a_2 I_{0+}^{\alpha-1}\rho(0), \\ -b_1 c_1 - (b_1 + b_2) c_2 = \gamma_2 - b_2 I_{0+}^{\alpha-1}\rho(1) - b_1 I_{0+}^{\alpha}\rho(1), \end{cases}$$

that is,

$$\begin{aligned} c_1 &= \frac{1}{l}\begin{vmatrix} \gamma_1 - a_1 I_{0+}^{\alpha}\rho(0) + a_2 I_{0+}^{\alpha-1}\rho(0) & a_2 \\ \gamma_2 - b_2 I_{0+}^{\alpha-1}\rho(1) - b_1 I_{0+}^{\alpha}\rho(1) & -(b_1+b_2) \end{vmatrix} \\ &= \frac{a_2 b_1}{l}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\rho(s)\,ds \\ &\quad + \frac{a_2 b_2}{l}\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\rho(s)\,ds - \frac{(b_1+b_2)\gamma_1 + a_2\gamma_2}{l}, \\ c_2 &= \frac{1}{l}\begin{vmatrix} -a_1 & \gamma_1 - a_1 I_{0+}^{\alpha}\rho(0) + a_2 I_{0+}^{\alpha-1}\rho(0) \\ -b_1 & \gamma_2 - b_2 I_{0+}^{\alpha-1}\rho(1) - b_1 I_{0+}^{\alpha}\rho(1) \end{vmatrix} \\ &= \frac{a_1 b_1}{l}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\rho(s)\,ds \\ &\quad + \frac{a_1 b_2}{l}\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\rho(s)\,ds - \frac{a_1\gamma_2 - b_1\gamma_1}{l}, \end{aligned}$$

where $l = a_1 b_1 + a_1 b_2 + a_2 b_1 \neq 0$. Substituting the values of c_1 and c_2 in (3.4), we get the solution given by (3.2), which completes the proof. □

Remark 3.1 From the expression of the function $G(t,s)$, it is obvious that $G(t,s)$ is continuous on I, and is bounded. Let

$$G^* = \sup\left\{\int_0^1 \left|G(t,s)\right| ds : t \in I\right\}.$$

To facilitate our discussion, let $B_r = \{x \in E : \|x\| < b\}$, $D_r = \{z \in C(I, E_w), \|z\| \leq r\}$, $\mathrm{BV}(I,\mathbb{R})$ represents the space of real bounded variation functions with its classical norm $\|\cdot\|_{\mathrm{BV}}$, $p, x : I \to E$, $f : I \times E^3 \to E$, $g, h : I \times E \to E$ and $G, k_1, k_2 : I \times I \to \mathbb{R}$ satisfy the following assumptions:

(1) p is weakly continuous function from I to E.
(2) For each uniformly ACG_* function $x : I \to E$, the functions $k_1(t,\cdot)g(\cdot, x(\cdot))$, $k_2(t,\cdot)h(\cdot, x(\cdot))$, $f(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot))$ are HKP integrable, f, g, h are weakly-weakly continuous functions and $\int_0^t g(s, x(s))\,ds$, $\int_0^1 h(s, x(s))\,ds$ are bounded on I.

(3) For any $r > 0$, there exist a HK-integrable function $M_r : I \to \mathbb{R}^+$ and a nondecreasing continuous function $\Omega : [0,\infty) \to (0,\infty)$ such that $\|f(t,x,y,z)\| \le M_r(t)\Omega(r)$ for all $t \in I$, $(x,y,z) \in D_r \times D_r \times D_r$.

(4) For each $t \in I$, $G(t,\cdot)$, $k_i(t,\cdot) \in \mathrm{BV}(I,\mathbb{R})$, $i = 1,2$ are continuous, *i.e.* the applications $t \mapsto G(t,\cdot)$ and $t \mapsto k(t,\cdot)$ are $\|\cdot\|_{\mathrm{BV}}$-continuous.

(5) The family

$$\bigl\{x^* f\bigl(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot)\bigr) : x^* \in E^*, \bigl\|x^*\bigr\| \le 1\bigr\}$$

is uniformly HK-integrable over I for every $x \in D_r$.

(6) For each bounded set $X, Y, Z \subset D_r$, and each for each closed interval $J \subset I$, $t \in I$, there exists a positive constant $L_1, L_2 \in (0,1)$ such that

$$\beta\bigl(k_1(J,J)g(J,Y)\bigr) \le L_1\beta\bigl(Y(J)\bigr), \qquad \beta\bigl(k_2(J,J)h(J,Z)\bigr) \le L_2\beta\bigl(Z(J)\bigr),$$
$$\beta\bigl(f(t,X,Y,Z)\bigr) \le M_r(t)\max\bigl\{\beta(X),\beta(Y),\beta(Z)\bigr\}.$$

(7) There exists a constant $r_0 > 0$ such that

$$\frac{r_0}{\|p\|_\infty + \|M_{r_0}\|_\infty \Omega(r_0) G^*} > 1.$$

Now, we present the existence theorem for the problem (1.1).

Theorem 3.1 *Assume that the conditions* (1)-(7) *and the families*

$$\biggl\{x^* \int_0^{(\cdot)} k_1(t,x) g\bigl(s, x_n(s)\bigr)\,ds\biggr\}_{n=1}^\infty, \qquad \biggl\{x^* \int_0^{(\cdot)} k_2(t,x) h\bigl(s, x_n(s)\bigr)\,ds\biggr\}_{n=1}^\infty, \tag{3.5}$$

$$\biggl\{x^* \int_0^t G(t,s) f\biggl(t, x_n(s), \int_0^{(\cdot)} k_1(t,s) g\bigl(s,x_n(s)\bigr), \int_0^{(\cdot)} k_2(t,s) h\bigl(s,x_n(s)\bigr)\biggr)\,ds\biggr\}_{n=1}^\infty, \tag{3.6}$$

are uniformly ACG$_*$ and equicontinuous on I for every $t \in I$ be satisfied, and let $r(K)$ be the spectral radius of the integral operator K defined by

$$(K\varphi)(t) = \int_0^t G(t,s) M_r(s)\varphi(s)\,ds, \quad \varphi \in D_r.$$

If $r(K) < 1$, then the problem (1.1) *has at least one solution $x \in C(I, E_w)$.*

Proof To simplify, we denote $m = \sup_{t\in I}\|k_1(t,\cdot)\|_{\mathrm{BV}}$, $c = \sup_{t\in I}\|p(t)\|$, and $k_0 = \max\{\sup_{t\in I}\int_0^t g(s,x(s))\,ds, \sup_{t\in I}\int_0^1 h(s,x(s))\,ds\}$. Let $c < k_0 < \min(r_0, \frac{r_0}{m})$. For $x \in D_{r_0}$ and $x^* \in E^*$ such that $\|x\|^* \le 1$, we have

$$\begin{aligned}\bigl|x^*\bigl(Tx(s)\bigr)\bigr| &= \biggl|(HK)\int_0^t x^*\bigl(k_1(t,s)g\bigl(s,x(s)\bigr)\bigr)\,ds\biggr| \\ &\le \bigl\|x^*\bigr\| \sup_{t\in I}\bigl\|k_1(t,\cdot)\bigr\|_{\mathrm{BV}} \int_0^1 \bigl\|g\bigl(s,x(s)\bigr)\bigr\|\,ds \le m\cdot k_0 \le r_0,\end{aligned}$$

and also

$$\sup\{|x^* Tx| : x \in E^*, \|x^*\| \leq 1\} \leq r_0.$$

So $Tx \in D_{r_0}$. Similarly, we prove $Sx \in D_{r_0}$.

Define the operator $F : C(I, E_w) \to C(I, E_w)$ by

$$Fx(t) = p(t) + \int_0^1 G(t,s)f\big(s, x(s), (Tx)(s), (Sx)(s)\big)\,ds, \quad t \in I,$$

where $G(\cdot,\cdot)$ is the Green's function defined by (3.3). Clearly the fixed points of the operator F are solutions of the problem (1.1). According to the assumptions (2), (3), and (5), let $G(t,\cdot) \in \mathrm{BV}(I,\mathbb{R})$ for each I, $f(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot))$ is HKP-integrable over I for every $x \in D_{r_0}$ and the family

$$\{x^* f\big(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot)\big) : x^* \in E^*, \|x^*\| \leq 1\}$$

be uniformly HK-integrable over I for every $x \in D_{r_0}$. Now, let $\mathcal{F}$ be a family of functions which are uniformly HK-integrable over an interval I. Then it is easy to see (the proof is in the spirit of [17], Th. 4.28) that $\mathcal{F}$ satisfies uniformly the Cauchy criterion over any closed subinterval $J \subset I$. Analogously to [17], Th. 4.27, the condition

$$\forall \varepsilon > 0\ \exists \text{ gauge } \gamma \text{ on } I\ \forall P_1, P_2\ \gamma \text{ fine-partitions } \forall f \in \mathcal{F}\ \big|\mathcal{S}(f, P_1) - \mathcal{S}(f, P_2)\big| < \varepsilon,$$

implies

$$\forall \varepsilon > 0\ \exists \text{ gauge } \gamma \text{ on } I \forall P_1, P_2\ \gamma \text{ fine-partitions } \forall f \in \mathcal{F}\ \left|\mathcal{S}(f, P) - (HK)\int_I f(t)\,dt\right| < \varepsilon.$$

Therefore, if $\mathcal{F}$ is family uniformly HK-integrable over interval $[0,b]$, then family $\mathcal{F}$ is uniformly HK-integrable over $[0,\tau]$ for every $\tau < b$. Consequently, in view of assumption (5) the family

$$\{x^* f\big(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot)\big) : x^* \in E^*, \|x^*\| \leq 1\}$$

will be uniformly HK-integrable over any subinterval $[0,\tau] \subset I$, for every $x \in D_{r_0}$. This entails the *weak**-continuity of the linear functional

$$x^* \in E^* \mapsto (HK)\int_0^\tau G(t,s)x^* f\big(s, x(s), T(x)(s), S(x)(s)\big)\,ds$$

for all $\tau \in I$. The latter in turn means that there is $x_{t,\tau} \in E$ such that

$$x^* x_{t,\tau} = (HK)\int_0^\tau x^* G(t,s)f\big(s, x(s), T(x)(s), S(x)(s)\big)\,ds, \quad \forall x^* \in E^*,$$

i.e. the function $G(t,\cdot)f(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot))$ is HKP-integrable on I and thus the operator F makes sense.

Let $r_0 > 0$, and consider the set

$$Q = \left\{ x \in D_{r_0} : \|x\|_\infty \le r_0, \forall t_1, t_2 \in I, \right.$$
$$\left. \big\|x(t_2) - x(t_1)\big\| \le \big\|p(t_2) - p(t_1)\big\| + \|M_{r_0}\|_\infty \Omega(r_0) \int_0^1 \big|G(t_2, s) - G(t_1, s)\big| \, ds \right\},$$

it is clear that the convex closed and equicontinuous subset $Q \subset D_{r_0} \subset C(I, E_w)$. We will show that F satisfies the assumptions of Theorem 2.3; the proof will be given in three steps.

Step 1. We shall show that the operator F maps into itself. First of all, we begin to show that $F : Q \to Q$. To see this, let $x \in Q$, $t \in I$. Without loss of generality, assume that $Fx(t) \neq 0$. By the Hahn-Banach theorem, there exists $x^* \in E^*$ with $\|x^*\| = 1$ and $\|Fx(t)\| = |x^*(Fx(t))|$. Thus

$$\begin{aligned} \big\|Fx(t)\big\| &= \big|x^*\big(Fx(t)\big)\big| \le x^*\big(p(t)\big) + x^*\left(\int_0^1 G(t,s) f\big(s, x(s), (Tx)(s), (Sx)(s)\big)\right) ds \\ &\le \big\|p(t)\big\| + \Omega(r_0) \sup_{t \in I} \int_0^1 G(t,s) M_{r_0}(s) \, ds \\ &\le \|p\|_\infty + \|M_{r_0}\|_\infty \Omega(r_0) G^* \le r_0, \end{aligned}$$

then $\|Fx\|_\infty = \sup_{t \in I} \|Fx(t)\| \le r_0$. Hence $F : Q \to Q$.

Let $0 < t_1 < t_2$, without loss of generality, assume that $Fx(t_2) - Fx(t_1) \neq 0$. By the Hahn-Banach theorem, there exists $x^* \in E^*$ with $\|x^*\| = 1$ and

$$\begin{aligned} &\big\|Fx(t_2) - Fx(t_1)\big\| \\ &\quad = x^*\big(Fx(t_2) - Fx(t_1)\big) \\ &\quad \le x^*\big|p(t_2) - p(t_1)\big| + \int_0^1 \big|G(t_2, s) - G(t_1, s)\big| \cdot \big|x^*\big(f\big(s, x(s), (Tx)(s), (Sx)(s)\big)\big)\big| \, ds \\ &\quad \le x^*\big|p(t_2) - p(t_1)\big| + \|M_{r_0}\|_\infty \Omega(r_0) \int_0^1 \big|G(t_2, s) - G(t_1, s)\big| \, ds \\ &\quad \le \big\|p(t_2) - p(t_1)\big\| + \|M_{r_0}\|_\infty \Omega(r_0) \int_0^1 \big|G(t_2, s) - G(t_1, s)\big| \, ds. \end{aligned}$$

This estimation shows that F maps Q into itself.

Step 2. We will show that the operator F is weakly sequentially continuous. To see this, by Lemma 9 of [18], a sequence $x_n(\cdot)$ weakly convergent to $x(\cdot) \in Q$ if and only if $x_n(\cdot)$ tends weakly to $x(t)$ for each $t \in I$. Because $g(s, \cdot)$ is weakly-weakly sequentially continuous, so if $x_n \to x$ in $C(I, E_w)$, then $g(s, x_n(s)) \to g(s, x(x))$ and $h(s, x_n(s)) \to h(s, x(x))$ in $C(I, E_w)$, and by Theorem 2.2 (see our assumptions (3.5), (3.6)), we have

$$\lim_{n \to \infty} \int_0^1 k_1(t,s) g\big(s, x_n(s)\big) \, ds = \int_0^1 k_1(t,s) g\big(s, x(s)\big) \, ds$$

weakly in E for each $t \in I$ and $Tx_n(t) \to Tx(t)$.

Similarly, we have

$$\lim_{n \to \infty} \int_0^1 k_2(t,s) h\big(s, x_n(s)\big) \, ds = \int_0^1 k_2(t,s) h\big(s, x(s)\big) \, ds$$

weakly in E for each $t \in I$ and $Sx_n(t) \to Sx(t)$. Therefore, the operator T, S are weakly sequentially continuous in Q.

Moreover, because f is weakly-weakly sequentially continuous,

$$f\big(s, x_n(s), (Tx_n)(s), (Sx_n)(s)\big) \to f\big(s, x(s), (Tx)(s), (Sx)(s)\big)$$

weakly in E, for each I. Now, applying assumption (5), Theorem 5 in [19] and Lemma 25 in [20], then the function $G(t,\cdot)a(\cdot)f(\cdot, x_n(\cdot), (Tx_n)(\cdot), (Sx_n)(\cdot))$ is HKP-integrable on I for every $n \geq 1$, by Theorem 2.2 and assumption (3.6), we have

$$\begin{aligned} &\lim_{n\to\infty} \int_0^1 G(t,s)a(s)f\big(s, x_n(s), (Tx_n)(s), (Sx_n)(s)\big)\,ds \\ &\quad= \int_0^1 G(t,s)a(s)f\big(s, x(s), (Tx)(s), (Sx)(s)\big)\,ds, \end{aligned}$$

then $F(x_n) \to F(x)$ in $C(I, E_w)$.

Step 3. We show that the operator $F : Q \to Q$ is power-convex condensing.

Let $B = \overline{\text{co}}F(Q) \subset Q$. Obviously, B is bounded, convex, and closed, and $F(\overline{\text{co}}F(Q)) \subset F(B) \subset \overline{\text{co}}F(Q)$, *i.e.*, $F : B \to B$. By Lemma 2.1, B is equicontinuous in $C(I, E_w)$. Obviously, F is bounded and continuous. Set $x_0 \in F$, we will prove that there exists n_0, such that, for any bounded $V \subset B$,

$$\beta\big(F^{(n_0,x_0)}(V)\big) \leq \beta(V).$$

By $V \subset B \subset Q$, $F(V)$ is equicontinuous. Then $F^{(2,x_0)}(V)$ is equicontinuous from $F^{(2,x_0)}(V) = F(\overline{\text{co}}F(V), x_0) \subset F(Q)$. Generally, $\forall n \in N$, $F^{(n,x_0)}(V)$ is equicontinuous. Since $F^{(n,x_0)}(V)$ is bounded, By Lemma 2.2,

$$\beta\big(F^{(n,x_0)}(V)\big) = \max_{t\in I}\big(F^{(n,x_0)}(V)(t)\big), \quad n = 2, 3, \ldots. \tag{3.7}$$

Now fix $t \in I$ and divide the interval I into n parts $0 = t_0 < t_1 < \cdots < t_n = 1$, for $s_1, s_2, s_3, r_1, r_2, r_3 \in T_i = [t_{i-1}, t_i]$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\big|M_r(s_1)G(t, s_2)\nu(s_3) - M_r(r_1)G(t, r_2)\nu(r_3)\big| < \epsilon,$$

if $|s_1 - r_1| < \delta$, $|s_2 - r_2| < \delta$, $|s_2 - r_2| < \delta$.

Let $\gamma_i = \sup_{s\in T_i} |G(t,s)| = |G(t,s_i)|$, $|M_r(\tau_i)| = \sup_{s\in T_i} |M_r(s)|$, $s_i, \tau_i \in T_i$ and $V_i = \{x(s) : x \in V, s \in T_i\}$. By the Ambrosetti lemma there exists $q_i \in T_i$ such that $\beta(V_i) = \nu(q_i)$, then

$$\begin{aligned} \beta\big(F^{1,x_0}(V)(t)\big) &= \beta\big(F(V)(t)\big) \\ &= \beta\left(p(t) + \int_0^1 G(t,s)f\big(s, V(s), (TV)(s), (SV)(s)\big)\,ds\right) \\ &= \beta\left(\int_0^1 G(t,s)f\big(s, V(s), (TV)(s), (SV)(s)\big)\,ds\right). \end{aligned}$$

By Theorem 2.1, we obtain

$$\begin{aligned}
&\int_0^1 G(t,s)f\bigl(s,V(s),(TV)(s),(SV)(s)\bigr)\,ds \\
&\quad=\sum_{i=1}^n \int_{T_i} G(t,s)f\bigl(s,V(s),(TV)(s),(SV)(s)\bigr)\,ds \\
&\quad\in \sum_{i=1}^n \mu(T_i)\overline{\mathrm{co}}\bigl\{G(t,s)f\bigl(s,x(s),(Tx)(s),(Sx)(s)\bigr):s\in T_i,x\in V\bigr\} \\
&\quad\subset \sum_{i=1}^n \mu(T_i)\overline{\mathrm{co}}\biggl(\bigcup_{|\gamma|\le\gamma_i}\gamma f\bigl(T_i\times V(T_i)\times T(V)(T_i)\times S(V)(T_i)\bigr)\biggr).
\end{aligned}$$

Furthermore, by the properties of β, we have

$$\begin{aligned}
&\beta\biggl(\int_0^1 G(t,s)f\bigl(s,V(s),T(V)(s),S(V)(s)\bigr)\,ds\biggr) \\
&\quad\le \beta\Biggl(\sum_{i=1}^n \mu(T_i)\overline{\mathrm{co}}\bigl\{G(t,s)f\bigl(s,x(s),(Tx)(s),(Sx)(s)\bigr):s\in T_i,x\in V\bigr\}\Biggr) \\
&\quad\le \sum_{i=1}^n \mu(T_i)\beta\biggl(\bigcup_{|\gamma|\le\gamma_i}\gamma f\bigl(T_i\times V(T_i)\times T(V)(T_i)\times S(V)(T_i)\bigr)\biggr) \\
&\quad= \sum_{i=1}^n \mu(T_i)\sup_{s\in T_i}\bigl|G(t,s)\bigr|\beta\bigl(f\bigl(T_i\times V(T_i)\times T(V)(T_i)\times S(V)(T_i)\bigr)\bigr) \\
&\quad\le \sum_{i=1}^n \mu(T_i)\bigl|G(t,s_i)\bigr|\sup_{s\in T_i}M_r(s)\max\bigl\{\beta\bigl(V(T_i)\bigr),\beta\bigl(T(V)(T_i)\bigr),\beta\bigl(S(V)(T_i)\bigr)\bigr\} \\
&\quad\le \sum_{i=1}^n \mu(T_i)\gamma_i M_r(\tau_i)\max\bigl\{\beta\bigl(V(T_i)\bigr),L_1\beta\bigl(V(T_i)\bigr),L_2\beta\bigl(V(T_i)\bigr)\bigr\} \\
&\quad\le \sum_{i=1}^n \mu(T_i)\gamma_i M_r(\tau_i)\max\{1,L_1,L_2\}\cdot\beta\bigl(V(T_i)\bigr) \\
&\quad\le \sum_{i=1}^n \mu(T_i)\gamma_i M_r(\tau_i)\beta\bigl(V(T_i)\bigr).
\end{aligned}$$

So

$$\begin{aligned}
\beta\biggl(\int_0^1 G(t,s)f\bigl(s,V(s),T(V)(s),S(V)(s)\bigr)\,ds\biggr) &\le \Biggl|\sum_{i=1}^n \mu(T_i)\gamma_i\sup_{s\in T_i}M_r(s)\beta\bigl(V(T_i)\bigr)\Biggr| \\
&\le \sum_{i=1}^n \mu(T_i)\bigl|G(t,s_i)\bigr|M_r(\tau_i)\nu(q_i),
\end{aligned}$$

where $s_i,\tau_i,q_i\in T_i$, and

$$\bigl|M_r(s)G(t,s)\nu(s)-M_r(\tau_i)G(t,s_i)\nu(q_i)\bigr|<\epsilon,\quad \text{for } s\in T_i,$$

we have

$$M_r(\tau_i)G(t,s_i)v(q_i)\mu(T_i) \le \int_{T_i} M_r(s)G(t,s)v(s)\,ds + \epsilon\mu(T_i).$$

Thus

$$\begin{aligned}&\beta\left(\int_0^1 G(t,s)f\big(s,V(s),T(V)(s),S(V)(s)\big)\,ds\right)\\&\quad\le \int_0^1 G(t,s)M_r(s)v(s)\,ds + \epsilon\sum_{i=1}^n \mu(T_i).\end{aligned}$$

Because ϵ is arbitrarily small, we get

$$\begin{aligned}&\beta\left(\left\{p(t)+\int_0^1 G(t,s)f\big(s,x(s),(Tx)(s),(Sx)(s)\big)\,ds : x\in V\right\}\right)\\&\quad\le \beta(V)\int_0^1 G(t,s)M_r(s)\,ds,\end{aligned}$$

i.e.

$$\beta\big(F^{(1,x_0)}(V)(t)\big) \le \beta(V)\int_0^1 G(t,s)M_r(s)\,ds = \beta(V)\cdot K\varphi_0(t),$$

where $\varphi_0(t)\equiv 1$, $\forall t\in I$.

By the equicontinuity of $F^{(1,x_0)}(V) = F(V)$ and $G(t,s)f(s,(\overline{\mathrm{co}}\{F^{(1,x_0)}(V)(s),x_0\}),(T\overline{\mathrm{co}}\{F^{(1,x_0)}(V)(s),x_0\}),(S\overline{\mathrm{co}}\{F^{(1,x_0)}(V)(s),x_0\}))$ is equicontinuous. Therefore,

$$\begin{aligned}&\beta\big(\big(F^{(2,x_0)}(V)\big)(t)\big)\\&\quad=\beta\big(F\overline{\mathrm{co}}\big\{\big(F^{(1,x_0)}(V)\big)(t),x_0\big\}\big)\\&\quad=\beta\bigg(\int_0^1 G(t,s)f\big(s,\big(\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big),\big(T\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big),\\&\qquad\big(S\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big)\big)\,ds\bigg)\\&\quad\le\int_0^1 \beta\big(\big(G(t,s)f\big(s,\big(\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big),\big(T\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big),\\&\qquad\big(S\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big)\big)\big)\big)\,ds\\&\quad\le\int_0^1 G(t,s)M_r(s)\beta\big(\overline{\mathrm{co}}\big\{F^{(1,x_0)}(V)(s),x_0\big\}\big)\,ds\\&\quad=\int_0^1 G(t,s)M_r(s)\beta\big(F^{(1,x_0)}(V)(s)\big)\,ds\\&\quad\le\beta(V)K^2\varphi_0(t).\end{aligned}$$

Generally,

$$\beta\big(\big(F^{(n,x_0)}(V)\big)(t)\big)\le\beta(V)K^n\varphi_0(t).$$

Since $r(K) < 1$, let $\varepsilon = \frac{1-r(K)}{2}$, then $\exists m_0 > 0$, when $n > m_0$,

$$\max_{t\in I}\left|K^n\varphi_0(t)\right| = \left\|K^n\varphi_0\right\| \leq \left\|K^n\right\|\|\varphi_0\| = \left\|K^n\right\| \\ \leq \left(r(K)+\varepsilon\right)^n = \left(\frac{1+r(K)}{2}\right)^n < 1.$$

Set $n_0 > m_0$, then $\forall t \in I$,

$$\beta\left(\left(F^{(n,x_0)}(V)\right)(t)\right) \leq \beta(V)\cdot K^{n_0}\varphi_0(t) \leq \left\|K^{n_0}\varphi_0\right\|\beta(V) \\ \leq \left(\frac{1+r(K)}{2}\right)^{n_0}\beta(V) \leq \beta(V).$$

By (3.7), $\beta(F^{(n,x_0)}(V)) \leq \beta(V)$. Therefore, $F: V \to V$ is convex-power condensing. By Theorem 2.3, F has one fixed point in $C(I,E_w)$, *i.e.*, the problem (1) has at least one solution in $C(I,E_w)$. □

Remark 3.2 The assumption (3) should instead agree with the following condition:

The function f is weakly-weakly sequentially continuous (for each convergent sequence $\{t_n\} \subset [0,1]$) and for all weakly convergent sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset E$, the sequence $\{f(t_n,x_n,y_n,z_n)\}$ is weakly convergent in E) such that, for all $r > 0$ and $\varepsilon > 0$, there exists $\delta_{\varepsilon,r} > 0$ such that

$$\left\|\int_\tau^t f\big(s,x(s),(Tx)(s),(Sx)(s)\big)\,ds\right\| < \varepsilon, \quad \forall|t-\tau| < \delta_{\varepsilon,r}, \forall x \in D_r.$$

Proof Let $r > 0$ and $x^* \in E^*$ such that $\|x^*\| \leq 1$. For $0 \leq t_1 < t_2 \leq 1$, we have

$$\left|x^*\int_{t_1}^{t_2} f\big(s,x(s),(Tx)(s),(Sx)(s)\big)\,ds\right| \leq (HK)\int_{t_1}^{t_2} x^*f\big(s,x(s)\big),(Tx)(s),(Sx)(s)\,ds.$$

Because $s \mapsto M_r(s)$ is Henstock-Kurzweil integrable and $|x^*f(s,x(s)),(Tx)(s),(Sx)(s)| \leq \|x^*\|\|f(s,x(s),(Tx)(s),(Sx)(s))\| \leq M_r$ for all $s \in I$, then by [21], Corollary 4.62, $s \mapsto x^*f(s,x(s),(Tx)(s),(Sx)(s))$ is absolutely Henstock-Kurzweil integrable on $[t_1,t_2] \subset I$ and

$$\left|(HK)\int_{t_1}^{t_2} x^*f\big(s,x(s),(Tx)(s),(Sx)(s)\big)\,ds\right| \leq (HK)\int_{t_1}^{t_2} M_r(s)\,ds.$$

Thus

$$\left\|\int_{t_1}^{t_2} f\big(s,x(s),(Tx)(s),(Sx)(s)\big)\,ds\right\| = \sup_{\|x^*\|\leq 1}\left|x^*\int_{t_1}^{t_2} f\big(s,x(s),Tx(s),Sx(s)\big)\,ds\right| \\ \leq (HK)\int_{t_1}^{t_2} M_r(s)\,ds.$$

Due to the continuity of the primitive function of the Henstock-Kurzweil integral, we have a t_2 less than ε and sufficiently close to t_1, and the proof is completed. □

4 Conclusions

In this paper, we use the techniques of the measure of weak noncompactness and Henstock-Kurzweil-Pettis integrals to discuss the existence theorem of weak solutions for a class of nonlinear fractional integrodifferential equations in a nonreflexive Banach space equipped with the weak topology.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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