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Nonconstant periodic solutions for a class of ordinary p-Laplacian systems

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Abstract

In this paper, we study the existence of periodic solutions for a class of ordinary p-Laplacian systems. Our technique is based on the generalized mountain pass theorem of Rabinowitz.

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Keywords: periodic solutions; ordinary p-Laplacian systems; generalized mountain pass theorem

1 Introduction and main results

We consider the existence of periodic solutions for the following ordinary p-Laplacian system:

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + \nabla F(t,u(t)) = 0, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1}$$

where $p > 1$, $T > 0$, and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t for all $x \in \mathbb{R}^N$ and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$\bigl|F(t,x)\bigr| \le a\bigl(|x|\bigr)b(t), \qquad \bigl|\nabla F(t,x)\bigr| \le a\bigl(|x|\bigr)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x.

As we all know, for $p = 2$, system (1) reduces to the following second-order Hamiltonian system:

$$\begin{cases} u''(t) + \nabla F(t,u(t)) = 0, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{2}$$

In 1978, Rabinowitz [1] published his pioneer paper for the existence of periodic solutions for problem (2) under the following Ambrosetti-Rabinowitz superquadratic condition: there exist $\mu > 2$ and $L^* > 0$ such that

$$0 < \mu F(t,x) \leq \bigl(\nabla F(t,x), x\bigr) \quad \text{for all } |x| \geq L^* \text{ and a.e. } t \in [0,T]. \tag{3}$$

From then on, various conditions have been applied to study the existence and multiplicity of periodic solutions for Hamiltonian systems by using the critical point theory; see [2–17] and references therein.

Over the last few decades, many researchers tried to replace the Ambrosetti-Rabinowitz superquadratic condition (3) by other superquadratic conditions. Some new superquadratic conditions are discovered. Especially, by using linking methods Schechter [13] obtained the following theorems.

Theorem 1.1 ([13], Theorem 1.1) *Suppose that $F(t,x)$ satisfies* (A) *and the following conditions*:

(V_0) *$F(t,x) \geq 0$ for all $t \in [0,T]$ and $x \in \mathbb{R}^N$;*
(V_1) *There are constants $m > 0$ and $\alpha \leq \frac{6m^2}{T^2}$ such that*

$$F(t,x) \leq \alpha \quad \textit{for all } |x| < m, x \in \mathbb{R}^N \textit{ and a.e. } t \in [0,T];$$

(V_2) *There are constants $\beta > \frac{2\pi^2}{T^2}$ and $C > 0$ such that*

$$F(t,x) \geq \beta |x|^2 \quad \textit{for all } |x| > C, x \in \mathbb{R}^N \textit{ and a.e. } t \in [0,T];$$

(V_3) *There exist a constant $\xi > 2$ and a function $W(t) \in L^1(0,T;\mathbb{R})$ such that*

$$\xi F(t,x) - \bigl(\nabla F(t,x), x\bigr) \leq W(t)|x|^2 \quad \textit{for all } |x| > C, x \in \mathbb{R}^N \textit{ and a.e. } t \in [0,T]$$

and

$$\limsup_{|x| \to \infty} \frac{\xi F(t,x) - (\nabla F(t,x), x)}{|x|^2} \leq 0$$

uniformly for a.e. $t \in [0,T]$.

Then system (1) *possesses a nonconstant T-periodic solution.*

Theorem 1.2 ([13], Theorem 1.2) *Suppose that $F(t,x)$ satisfies* (A), (V_0), (V_2), (V_3), *and the following condition*:

(V_1') *There is a constant $q > 2$ such that*

$$F(t,x) \leq C\bigl(|x|^q + 1\bigr) \quad \textit{for all } t \in [0,T] \textit{ and } x \in \mathbb{R}^N,$$

and there are constants $m > 0$ and $\alpha < \frac{2\pi^2}{T^2}$ such that

$$F(t,x) \leq \alpha |x|^2 \quad \textit{for all } |x| \leq m, x \in \mathbb{R}^N, \textit{ and a.e. } t \in [0,T].$$

Then system (1) *possesses a nonconstant T-periodic solution.*

Moreover, Schechter [14] proved the existence of a periodic solution for system (2) if condition (V_2) is replaced by the following local superquadratic condition: there is a subset $E \subset [0,T]$ with $\operatorname{meas}(E) > 0$ such that

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^2} > 0 \quad \text{uniformly for a.e. } t \in E. \tag{4}$$

Wang, Zhang, and Zhang [17] established the existence of a nonconstant T-periodic solution of system (2) under condition (4). They obtained the following theorem.

Theorem 1.3 ([17], Theorem 1.1) *Suppose that* $F(t,x)$ *satisfies* (A), (V_0), (V_1), (4), *and the following conditions*:

(V_4) *There exist constants* $\xi > 2, 1 \le \gamma < 2, L > 0$ *and the function* $d(t) \in L^1(0,T;\mathbb{R}^+)$ *such that*

$$\xi F(t,x) \le \bigl(\nabla F(t,x),x\bigr) + d(t)|x|^\gamma$$

for all $|x| \ge L, x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$;

(V_5) *There exists a constant* $M^* > 0$ *such that* $d(t) \le M^*$ *for a.e.* $t \in E$.

Then system (1) *possesses a nonconstant* T*-periodic solution*.

Recently, there are many results concerning the existence of periodic and subharmonic solutions for system (1); see [18–25] and references therein. Manasevich and Mawhin [21] generalized the Hartman-Knobloch results to perturbations of a vector p-Laplacian ordinary operator. Xu and Tang [23] proved the existence of periodic solutions for problem (1) by using the saddle point theorem. With the aid of the generalized mountain pass theorem, Ma and Zhang [20] extended the results of [16] to systems (1).

In this paper, motivated by the works [13, 14, 17], we consider the existence of periodic solutions for ordinary p-Laplacian systems (1). The main result is the following theorem.

Theorem 1.4 *Suppose that* $F(t,x)$ *satisfies the following conditions*:

(H_0) $F(t,x) \ge 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^N$;

(H_1) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^p} = 0$ *uniformly for a.e.* $t \in [0,T]$;

(H_2) *There exist constants* $\mu > p$ *and* $L_0 > 0$ *and* $W(t) \in L^1(0,T;\mathbb{R})$ *such that*

$$\mu F(t,x) - \bigl(\nabla F(t,x),x\bigr) \le W(t)|x|^p$$

for all $|x| \ge L_0, x \in \mathbb{R}^N$, *and a.e.* $t \in [0,T]$, *and*

$$\limsup_{|x|\to\infty} \frac{\mu F(t,x) - (\nabla F(t,x),x)}{|x|^p} \le 0$$

uniformly for a.e. $t \in [0,T]$;

(H_3) *There exists* $\Omega \subset [0,T]$ *with* $\operatorname{meas}(\Omega) > 0$ *such that*

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > 0$$

uniformly for a.e. $t \in \Omega$.

Then system (1) *possesses a nonconstant* T*-periodic solution.*

Remark 1.5 For $p = 2$, it is easy to see that the conclusion in Theorem 1.4 is the same if condition (H_1) is replaced by (V_1) or (V_1'). Thus, Theorem 1.4 generalizes Theorems 1.1 and 1.2 in [13] and Theorems 1.1 and 1.2 in [14]. Furthermore, Theorem 1.4 extends Theorem 1.1 in [17]. There are functions F satisfying our Theorem 1.4 but not satisfying the results mentioned before. For example, let

$$F(t,x) = \frac{\psi(t)}{T^2}\bigl(|x|^4 + |x|^2\bigr) \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N,$$

where

$$\psi(t) = \begin{cases} \sin(2\pi t/T), & t \in [0, T/2], \\ 0, & t \in [T/2, T]. \end{cases}$$

Taking $\Omega = [T/6, T/4]$, a straightforward computation implies that F does not satisfy the results in [13, 14, 17].

2 Proof of the main results

Let us consider the functional φ on $W_T^{1,p}$ given by

$$\varphi(u) = \frac{1}{p}\int_0^T |u'|^p\,dt - \int_0^T F(t,u)\,dt$$

for each $u \in W_T^{1,p}$, where

$$\begin{aligned} W_T^{1,p} = \bigl\{u : [0,T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous}, u(0) = u(T), \\ \text{and } u' \in L^p\bigl(0,T;\mathbb{R}^N\bigr)\bigr\} \end{aligned}$$

is a reflexive Banach space with norm

$$\|u\| = \left(\int_0^T |u(t)|^p\,dt + \int_0^T |u'(t)|^p\,dt\right)^{1/p} \quad \text{for all } u \in W_T^{1,p}.$$

For $u \in W_T^{1,p}$, let

$$\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt, \qquad \widetilde{u} = u(t) - \overline{u}$$

and

$$\widetilde{W}_T^{1,p} = \bigl\{u \in W_T^{1,p} \mid \overline{u} = 0\bigr\}.$$

Then we have

$$W_T^{1,p} = \widetilde{W}_T^{1,p} \oplus \mathbb{R}^N$$

and

$$\|u\|_{L^p} \le C_0 \|u'\|_{L^p} \quad \text{(Wirtinger's inequality)},$$
$$\|u\|_{\infty} \le C_0 \|u'\|_{L^p} \quad \text{(Sobolev inequality)}$$

for all $u \in \widetilde{W}_T^{1,p}$, where C_0 is a positive constant.

It follows from assumption (A) that the functional φ is continuously differentiable on $W_T^{1,p}$. Moreover, we have

$$\langle \varphi'(u), v\rangle = \int_0^T |u'|^{p-2}(u', v')\,dt - \int_0^T (\nabla F(t,u), v)\,dt$$

for all $u, v \in W_T^{1,p}$. It is well known that the problem of finding a T-periodic solution of problem (1) is equal to that of finding the critical points of φ.

Now, we can state the proof of our result.

Proof of Theorem 1.4 Firstly, we will show that φ satisfies (P.-S.) condition, i.e., for every sequence $\{u_n\} \subset W_T^{1,p}$, $\{u_n\}$ has a convergent subsequence if

$$\{\varphi(u_n)\} \text{ is bounded} \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{5}$$

According to a standard argument, we only need to show that $\{u_n\}$ is a bounded sequence in $W_T^{1,p}$. Otherwise, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, so that $\|w_n\| = 1$. If necessary, taking a subsequence, still denoted by $\{w_n\}$, we suppose that

$$w_n \rightharpoonup w_0 \quad \text{weakly in } W_T^{1,p},$$
$$w_n \to w_0 \quad \text{strongly in } C(0,T;\mathbb{R}^N)$$

as $n \to \infty$, and we have

$$\overline{w}_n \to \overline{w}_0 \quad \text{as } n \to \infty. \tag{6}$$

By (5) there exists $M_1 > 0$ such that

$$\begin{aligned}\left(\frac{\mu}{p} - 1\right)\|u_n'\|_{L^p}^p &= \mu\varphi(u_n) - \langle\varphi'(u_n), u_n\rangle + \int_0^T \big(\mu F(t,u_n) - (\nabla F(t,u_n), u_n)\big)\,dt \\ &\le M_1\big(1 + \|u_n\|\big) + \int_0^T \big(\mu F(t,u_n) - (\nabla F(t,u_n), u_n)\big)\,dt.\end{aligned}$$

So, we obtain

$$\left(\frac{\mu}{p} - 1\right)\|w_n'\|_{L^p}^p \le \frac{M_1(1 + \|u_n\|)}{\|u_n\|^p} + \frac{\int_0^T (\mu F(t,u_n) - (\nabla F(t,u_n), u_n))\,dt}{\|u_n\|^p}. \tag{7}$$

In view of (A) and (H_2), let $\Omega_0 \subset \Omega$ with $|\Omega_0| = 0$ be such that

$$\big|F(t,x)\big| \le a\big(|x|\big)b(t) \quad \text{and} \quad \big|\nabla F(t,x)\big| \le a\big(|x|\big)b(t) \tag{8}$$

for all $x \in \mathbb{R}^N$ and $t \in [0,T] \setminus \Omega_0$ and

$$\limsup_{|x|\to\infty} \frac{\mu F(t,x) - (\nabla F(t,x),x)}{|x|^p} \le 0$$

uniformly for $t \in [0,T] \setminus \Omega_0$.

In fact, we have

$$\limsup_{n\to\infty} \frac{\mu F(t,u_n) - (\nabla F(t,u_n),u_n)}{\|u_n\|^p} \le 0 \tag{9}$$

for $t \in [0,T] \setminus \Omega_0$. Otherwise, there exist $t_0 \in [0,T] \setminus \Omega_0$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{\|u_n\|^p} > 0. \tag{10}$$

If $\{u_n(t_0)\}$ is bounded, then there exists a positive constant M_2 such that $|u_n(t_0)| \le M_2$ for all $n \in \mathbb{N}$. By (8) we find

$$\begin{aligned} &\frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{\|u_n\|^p} \\ &\quad \le \frac{(\mu + M_2)\max_{0\le s\le M_2} a(s)b(t)}{\|u_n\|^p} \to 0 \end{aligned}$$

as $n \to \infty$, which contradicts (10). So, there is a subsequence of $\{u_n(t_0)\}$, still denoted by $\{u_n(t_0)\}$, such that $|u_n(t_0)| \to \infty$ as $n \to \infty$. From (H_2) we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{\|u_n\|^p} \\ &\quad = \limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{|u_n(t_0)|^p} \big|w_n(t_0)\big|^p \\ &\quad = \limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{|u_n(t_0)|^p} \lim_{n\to\infty} \big|w_n(t_0)\big|^p \\ &\quad \le 0. \end{aligned}$$

This contradicts (10). Thus, (9) holds. From (7) and (9) we obtain

$$\limsup_{n\to\infty} \left(\frac{\mu}{p} - 1\right) \big\|w_n'\big\|_{L^p}^p \le 0.$$

Since $\mu > p$, we get

$$\big\|w_n'\big\|_{L^p}^p \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Combining with (6), this yields

$$w_n \to \overline{w}_0 \quad \text{as } n \to \infty,$$

which means that

$$w_0 = \overline{w}_0 \quad \text{and} \quad T|\overline{w}_0| = \|\overline{w}_0\| = 1.$$

Then we have

$$\left|u_n(t)\right| \to \infty \quad \text{as } n \to \infty$$

uniformly for a.e. $t \in [0, T]$. We deduce from (H_0), (H_3), and Fatou's lemma that

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^T \frac{F(t,u_n)}{\|u_n\|^p}\,dt &\geq \int_0^T \liminf_{n\to\infty} \frac{F(t,u_n)}{\|u_n\|^p}\,dt \\
&= \int_0^T \left[\liminf_{n\to\infty} \frac{F(t,u_n)}{|u_n|^p}|w_0|^p\right] dt \\
&\geq \int_\Omega \left[\liminf_{n\to\infty} \frac{F(t,u_n)}{|u_n|^p}|w_0|^p\right] dt \\
&> 0.
\end{aligned} \tag{12}$$

Since

$$\frac{\varphi(u_n)}{\|u_n\|^p} = \frac{\frac{1}{p}\int_0^T |u_n'|^p\,dt}{\|u_n\|^p} - \frac{\int_0^T F(t,u_n)\,dt}{\|u_n\|^p}$$

and $\varphi(u_n)$ is bounded, we obtain from (11) that

$$\liminf_{n\to\infty} \int_0^T \frac{F(t,u_n)}{\|u_n\|^p}\,dt = 0,$$

which contradicts (12). Hence, $\{u_n\}$ is a bounded sequence in $W_T^{1,p}$, and we conclude that φ satisfies (P.-S.) condition.

Now, by the generalized mountain pass theorem [12], Theorem 5.3, we only need to show that

(G_1) $\inf_{u\in S}\varphi(u) > 0$,
(G_2) $\sup_{u\in Q}\varphi(u) < +\infty, \sup_{u\in\partial Q}\varphi(u) \leq 0$,

where $S = \widetilde{W}_T^{1,p} \cap \partial B_\rho$, $Q = \{x + se | x \in \mathbb{R}^N \cap B_{r_1}, s \in [0, r_2]\}$, $r_1 > 0$, $\rho < r_2$, $e \in \widetilde{W}_T^{1,p}$, and $B_r = \{u \in W_T^{1,p} : \|u\| \leq r\}$.

By (H_2) and (H_3) there exist constants $M_3 > L_0$ and $\eta > 0$ and a subset of Ω, still denoted by Ω, with $|\Omega| > 0$ such that

$$F(t,x) > \frac{2\eta}{\mu - p}|x|^p \tag{13}$$

and

$$\mu F(t,x) - \big(\nabla F(t,x),x\big) \leq \eta |x|^p \tag{14}$$

for all $|x| \geq M_3$ and $t \in \Omega$. For $x \in \mathbb{R}^N \setminus \{0\}$ and $t \in [0,T]$, let

$$f(s) = F(t,sx) \quad \text{for all } s \geq \frac{M_3}{|x|}.$$

We deduce from (14) that

$$\begin{aligned} f'(s) &= \frac{1}{s}\big(\nabla F(t,sx),sx\big) \\ &\geq \frac{\mu}{s}F(t,sx) - \eta s^{p-1}|x|^p \\ &= \frac{\mu}{s}f(s) - \eta s^{p-1}|x|^p, \end{aligned}$$

which yields

$$g(s) = f'(s) - \frac{\mu}{s}f(s) + \eta s^{p-1}|x|^p \geq 0$$

for all $s \geq \frac{M_3}{|x|}$. From the above expression we have

$$f(s) = \left(\int_{\frac{M_3}{|x|}}^{s} \frac{g(r) - \eta r^{p-1}|x|^p}{r^\mu}\,dr + M_4\right)s^\mu \tag{15}$$

for all $s \geq \frac{M_3}{|x|}$, where

$$M_4 = \left(\frac{|x|}{M_3}\right)^\mu f\left(\frac{M_3}{|x|}\right).$$

It follows from (15) that

$$\begin{aligned} f(s) &= \left(\int_{\frac{M_3}{|x|}}^{s} \frac{g(r) - \eta r^{p-1}|x|^p}{r^\mu}\,dr + M_4\right)s^\mu \\ &= \left(\int_{\frac{M_3}{|x|}}^{s} \frac{g(r)}{r^\mu}\,dr - \eta|x|^p \int_{\frac{M_3}{|x|}}^{s} r^{p-1-\mu}\,dr + M_4\right)s^\mu \\ &\geq M_4 s^\mu + \left(\frac{\eta|x|^p}{\mu - p}s^{p-\mu} - \frac{\eta|x|^\mu}{(\mu-p)M_3^{\mu-p}}\right)s^\mu \\ &\geq \left(M_3^{-\mu}F\big(t,M_3|x|^{-1}x\big) - \frac{\eta}{(\mu-p)M_3^{\mu-p}}\right)|x|^\mu s^\mu. \end{aligned}$$

Combining this with (13) yields

$$\begin{aligned} F(t,x) &= f(1) \\ &\geq \left(M_3^{-\mu}F\big(t,M_3|x|^{-1}x\big) - \frac{\eta}{(\mu-p)M_3^{\mu-p}}\right)|x|^\mu \end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{2\eta M_3^{p-\mu}}{\mu - p} - \frac{\eta M_3^{p-\mu}}{\mu - p}\right)|x|^\mu \\
&\geq \frac{\eta M_3^{p-\mu}}{\mu - p}|x|^\mu \\
&= M_5|x|^\mu
\end{aligned}$$

for all $|x| \geq M_3$ and $t \in \Omega$, where $M_5 = \eta M_3^{p-\mu}/(\mu - p)$. So, we get

$$F(t,x) \geq M_5|x|^\mu - M_5 M_3^\mu \tag{16}$$

for all $x \in \mathbb{R}^N$ and $t \in \Omega$.

Choose

$$z(t) = \bigl(\sin(\omega t), 0, \ldots, 0\bigr) \in \widetilde{W}_T^{1,p},$$

where $\omega = 2\pi/T$. Let

$$\overline{W}_T^{1,p} = \mathbb{R}^N \oplus \operatorname{span}\bigl\{z(t)\bigr\}$$

and

$$Q = \bigl\{x \in \mathbb{R}^N \,|\, |x| \leq r_1\bigr\} \oplus \{sz \,|\, 0 \leq s \leq r_2\}.$$

Since $\dim(\overline{W}_T^{1,p}) < \infty$, all the norms are equivalent. For any $u \in \overline{W}_T^{1,p}$, there exists a positive constant K such that

$$\|u\|_{L^p(\Omega)} \geq K\|u\|_{L^2(\Omega)}. \tag{17}$$

According to (16), we have

$$F(t,x) \geq M_6|x|^p - M_7 \tag{18}$$

for all $x \in \mathbb{R}^N$ and $t \in \Omega$, where

$$M_6 = \frac{2\omega^p T}{pK^p}\left(\int_\Omega \bigl|z(t)\bigr|^2\,dt\right)^{-p/2} \quad \text{and} \quad M_7 = M_5 M_3^\mu + M_6\left(\frac{M_6}{M_5}\right)^{p/(\mu-p)}.$$

Now, it follows from (17) and (18) that

$$\begin{aligned}
\varphi(x + sz) &= \frac{1}{p}\int_0^T \bigl|sz'(t)\bigr|^p\,dt - \int_0^T F(t, x + sz)\,dt \\
&\leq \frac{1}{p}\int_0^T \bigl|sz'(t)\bigr|^p\,dt - \int_\Omega F(t, x + sz)\,dt \\
&\leq \frac{1}{p}\omega^p|s|^p\int_0^T |\cos\omega t|^p\,dt - M_6\int_\Omega |x + sz|^p\,dt + M_7|\Omega| \\
&\leq \frac{1}{p}\omega^p|s|^p T - M_6\int_\Omega |x + sz|^p\,dt + M_7|\Omega|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{p}\omega^p|s|^p T - M_6K^p\left(\int_\Omega |x+sz|^2\,dt\right)^{p/2} + M_7|\Omega| \\
&= \frac{1}{p}\omega^p|s|^p T - M_6K^p\left(\int_\Omega \left(|x|^2+|sz|^2\right)dt\right)^{p/2} + M_7|\Omega| \\
&= \frac{1}{p}\omega^p|s|^p T - M_6K^p\left(|\Omega||x|^2 + s^2\int_\Omega |z|^2\,dt\right)^{p/2} + M_7|\Omega|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varphi(x+sz) &\leq \frac{1}{p}\omega^p|s|^p T - M_6K^p\left(s^2\int_\Omega |z|^2\,dt\right)^{p/2} + M_7|\Omega| \\
&\leq -\frac{1}{p}\omega^p|s|^p T + M_7|\Omega| \qquad (19)
\end{aligned}$$

and

$$\varphi(x+sz) \leq \frac{1}{p}\omega^p|s|^p T - M_6K^p|\Omega|^{p/2}|x|^p + M_7|\Omega|. \qquad (20)$$

Let

$$r_1 = \left(\frac{pM_7|\Omega|}{\omega^p T}\right)^{1/p} \quad \text{and} \quad r_2 = \left(\frac{2M_7|\Omega|^{1-p/2}}{M_6K^p}\right)^{1/p}.$$

For $x + r_1 z \in \partial Q$, we get from (19) that

$$\varphi(x + r_1 z) \leq 0, \qquad (21)$$

and, for $x + sz \in \partial Q$ with $0 \leq s \leq r_1$, $|x| = r_2$, we obtain from (20) that

$$\varphi(x+sz) \leq 0. \qquad (22)$$

If $s = 0$, then by (H_1) we get

$$\varphi(x) = -\int_0^T F(t,x)\,dt \leq 0 \qquad (23)$$

for all $x \in \mathbb{R}^N$. By (21), (22), and (23) condition (G_2) holds.

On the other hand, it follows from (H_1) that there exist two positive constants $\varepsilon < 1/(pC_0)$ and $\delta < C_0$ such that

$$F(t,x) \leq \varepsilon|x|^p \qquad (24)$$

for all $|x| \leq \delta$ and a.e. $t \in [0,T]$.

For $u \in \widetilde{W}_T^{1,p}$ with $\|u\| \leq \frac{1}{C_0}\delta$, we have $\|u\|_\infty \leq \delta$. We obtain from (24) and Wirtinger's inequality that

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\int_0^T \left|u'\right|^p dt - \int_0^T F(t,u)\,dt \\
&\geq \frac{1}{p}\int_0^T \left|u'\right|^p dt - \varepsilon\int_0^T |u|^p\,dt
\end{aligned}$$

$$\geq \left(\frac{1}{p} - \varepsilon C_0\right) \|u'\|_{L^p}^p$$
$$\geq \left(\frac{1}{p} - \varepsilon C_0\right)(1 + C_0)^{-1} \|u\|^p.$$

Choose $\rho \in (0, \delta/C_0)$ to obtain

$$\inf_{u \in S} \varphi(u) > 0,$$

where $S = \widetilde{W}_T^{1,p} \cap \partial B_\rho$. So, condition (G_1) holds.

Hence, there is a nonconstant T-periodic solution of system (1). □

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors typed, read, and approved the final manuscript.

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