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Infinitely many solutions for a class of fourth-order partially sublinear elliptic problem

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Abstract

In this paper, we study the existence of infinitely many solutions for a class of fourth-order partially sublinear elliptic problem with Navier boundary value condition by using an extension of Clark's theorem.

Keywords: multiple solution; fourth-order elliptic problem; critical point; variational method

1 Introduction

Consider the following fourth-order boundary value problem:

$$\begin{cases} \triangle^2 u + c\triangle u = g(x,u) & \text{in } \Omega, \\ u = \triangle u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\triangle^2$ denotes the biharmonic operator, $\Omega \subset \mathbb{R}^N$ $(N > 4)$ is a bounded domain with smooth boundary, $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and $c < \lambda_1$ is a parameter, where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$.

The fourth-order elliptic equations can describe the static form change of beam or the motion of rigid body, so they are widely applied in physics and engineering. In the 1990s, Lazer and Mckenna (see [1, 2]) investigated the problem (1) as $g(x,u) = d[(u+1)^+ - 1]$. In [1], they pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. In [2], they got $2k-1$ solutions when $N = 1$ and $d > \lambda_k(\lambda_k - c)$ (λ_k is the sequence of the eigenvalues of $-\triangle$ in $H_0^1(\Omega)$) by the global bifurcation method. In [3], Micheletti and Pistoia used a variational linking theorem to investigate the existence of two solutions for a more general nonlinearity $g(\cdot,u)$. In 2001, Micheletti and Saccon (see [4]) obtained two results about the existence of two nontrivial solutions and four nontrivial solutions by a similar variational approach, depending on the position of a suitable parameter with respect to the eigenvalues of the linear part.

In recent years, many researchers have used variational approach to investigate the fourth-order elliptic equations. In [5], Pu *et al.* used the least action principle, the Ekeland variational principle and the mountain pass theorem to prove the existence and multiplicity of solutions of (1) when $g(x,u) = a(x)|u|^{s-2}u + f(x,u)$ $(a \in L^\infty(\Omega)$, $s \in (1,2)$ and

$f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$). In [6], Hu and Wang obtained the existence of nontrivial solutions to problem (1) under suitable assumptions of $g(x,u)$ by a variant version of the mountain pass theorem. In [7], we have studied the existence of multiple solutions for problem (1) by using the variant fountain theorem without the condition $c < \lambda_1$ (λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$). In [8], Wang and Shen, under an improved Hardy-Rellich inequality, studied the existence of multiple and sign-changing solutions for a class of biharmonic equations in an unbounded domain by the minimax method and linking theorem. In 2015, Liu and Chen (see [9]) investigated the existence of ground-state solution and nonexistence of nontrivial solution for a similar biharmonic equation in [8] by using variational methods, also they explored the phenomenon of concentration of solutions. For other related results, see [10–15] and the references therein.

In critical point theory, Clark's theorem [16] asserts the existence of a sequence of negative critical values tending to 0 for even coercive functionals. It is constantly and effectively applied to sublinear differential equations with symmetry. In 2001, Wang (see [17]) explored a variant of the Clark theorem given by Heinz in [18] to investigate a variety of nonlinear boundary value problems. Then, in 2015, Liu and Wang improved Clark's theorem and gave an extension of Clark's theorem in [19]. Their new results gave a more detailed structure of the set of critical points near the origin and are powerful in applications. In this paper, inspired by [17], we will use the new result of Liu and Wang in [19] to investigate the existence of infinitely many solutions for partially sublinear problems (1).

Our main result is the following theorem.

Theorem 1 *Assume that there exists a constant $\delta > 0$ such that $g \in C(\overline{\Omega} \times [-\delta, \delta], \mathbb{R})$ is odd and bounded. The primitive $G(x,u) := \int_0^u g(x,s)\,ds$ of the nonlinearity g satisfies*

$$\liminf_{|u|\to 0} \frac{G(x,u)}{|u|^2} = \infty \quad \text{uniformly for } x \in \overline{\Omega}. \tag{2}$$

Then problem (1) possesses infinitely many solutions u_k such that $\|u_k\|_\infty \to 0$ as $k \to \infty$, where $\|\cdot\|_\infty$ is the usual norm of $L^\infty \equiv L^\infty(\Omega)$.

Remark The advantage of this theorem is that it not only obtained the existence of infinite solutions, but it also pointed out their positions.

2 Preliminaries

Let $E = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$(u,v)_E = \int_\Omega \bigl(\triangle u \triangle v - c\langle \nabla u, \nabla v\rangle\bigr)\,dx$$

and the norm

$$\|u\| = (u,u)_E^{\frac{1}{2}}.$$

A weak solution of problem (1) is a $u \in E$ such that

$$\int_\Omega \bigl(\triangle u \triangle v - c\langle \nabla u, \nabla v\rangle\bigr)\,dx - \int_\Omega g(x,u)v\,dx = 0$$

for any $v \in E$. Here and in the sequel, $\langle\cdot,\cdot\rangle$ always denotes the standard inner product in $\mathbb{R}^N$.

Let $\Phi : E \to R$ be the functional defined by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega \left(|\triangle u|^2 - c|\nabla u|^2\right) dx - \int_\Omega G(x,u)\,dx \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)\,dx. \end{aligned} \tag{3}$$

It is well known that a critical point of the functional Φ in E corresponds to a weak solution of problem (1).

Direct computation shows that

$$\Phi'(u)v = (u,v)_E - \int_\Omega g(x,u)v\,dx \tag{4}$$

for all $u, v \in E$. It is well known that $\Phi' : E \to E$ is compact.

Denote by $\|\cdot\|_p$ the usual norm of $L^p \equiv L^p(\Omega)$ as

$$\|u\|_p = \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}}, \quad \text{for } u \in L^p, 1 \le p < +\infty.$$

Lemma 1 *The norm $\|u\|$ is equivalent to the norm $\|\triangle u\|_2$ in E.*

Proof This result can be found in [6] (Lemma 2.3), so we omit the proof here. □

Then by Lemma 1 and Sobolev embedding theorem, there exists a $\tau_p > 0$ such that

$$\|u\|_p \le \tau_p\|u\|, \quad \forall u \in E, \tag{5}$$

for all $1 \le p \le \frac{2N}{N-4}$.

Let λ_i $(i = 1, 2, \ldots)$ be the eigenvalues of $-\triangle$ in $H_0^1(\Omega)$. Then the eigenvalue problem

$$\begin{cases} \triangle^2 u + c\triangle u = \mu u & \text{in } \Omega, \\ u = \triangle u = 0 & \text{on } \partial\Omega, \end{cases}$$

has infinitely many eigenvalues $\mu_i = \lambda_i(\lambda_i - c)$, $i = 1, 2, \ldots$.

Define a selfadjoint linear operator $\mathscr{A} : L^2 \to L^2$ by

$$(\mathscr{A}u, v)_2 = \int_\Omega \left(\triangle u \triangle v - c\langle\nabla u, \nabla v\rangle\right) dx \tag{6}$$

with domain $D(\mathscr{A}) = E$. Here, $(\cdot,\cdot)_2$ denotes the inner product in L^2. Then the sequence of eigenvalues of $\mathscr{A}$ is just $\{\mu_i\}$ $(i = 1, 2, \ldots)$. Denote the corresponding system of eigenfunctions by $\{e_n\}$, it forms an orthogonal basis in L^2.

Denote

$$n^- = \sharp\{i \mid \mu_i < 0\}, \qquad n^0 = \sharp\{i \mid \mu_i = 0\}, \qquad \overline{n} = n^- + n^0. \tag{7}$$

Here, $\sharp\{\cdot\}$ denotes the cardinality of a set and $\overline{n}$ can be 0. Let

$$L^- = \operatorname{span}\{e_1, \dots, e_{n^-}\}, \quad L^0 = \operatorname{span}\{e_{n^-+1}, \dots, e_{\overline{n}}\}, \quad L^+ = \left(L^- \oplus L^0\right)^\perp = \overline{\operatorname{span}\{e_{\overline{n}+1}, \dots\}}.$$

Decompose L^2 as

$$L^2 = L^- \oplus L^0 \oplus L^+.$$

Then E also possesses the orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+ \tag{8}$$

with

$$E^- = L^-, \qquad E^0 = L^0 \quad \text{and} \quad E^+ = E \cap L^+ = \overline{\operatorname{span}\{e_{\overline{n}+1}, \dots\}}. \tag{9}$$

To prove our main result, Theorem 1, we give the improved Clark theorem in [19].

Theorem 2 (See [19], Theorem 1.1) *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.*

(ii) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $\Phi(u) = 0$.*

In order to apply this theorem to prove our main result, we set our working space $E = H^2(\Omega) \cap H_0^1(\Omega)$ as the space X in Theorem 2.

3 Proof of Theorem 1

Now we prove our main result, Theorem 1.

Proof of Theorem 1 Choose $\widehat{g} \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ so that $\widehat{g}$ is odd in $u \in \mathbb{R}$,

$$\widehat{g}(x,u) = \begin{cases} g(x,u), & x \in \overline{\Omega}, |u| \le \delta/2, \\ \text{odd}, & x \in \overline{\Omega}, \delta/2 < |u| \le \delta, \\ 0, & x \in \overline{\Omega}, |u| > \delta. \end{cases} \tag{10}$$

Since g is bounded, together with (10), we see that there exists a constant c_1 such that

$$\left|\widehat{g}(x,u)\right| \le c_1 \quad \text{for all } x \in \overline{\Omega} \text{ and } u \in \mathbb{R}. \tag{11}$$

Here and in the sequel, we denote $c_i > 0$ $(i = 1, 2, \dots)$ for different positive constants.

In order to obtain solutions of (1), we study the system

$$\begin{cases} \triangle^2 u + c\triangle u = \widehat{g}(x,u) & \text{in } \Omega, \\ u = \triangle u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

with the functional defined by

$$\widehat{\Phi}(u) = \frac{1}{2}\int_{\Omega}\left(|\triangle u|^2 - c|\nabla u|^2\right)dx - \int_{\Omega}\widehat{G}(x,u)\,dx = \frac{1}{2}\|u\|^2 - \int_{\Omega}\widehat{G}(x,u)\,dx, \tag{13}$$

for all $u \in E$. Here, $\widehat{G}(x,u) := \int_0^u \widehat{g}(x,s)\,ds$ is the primitive of the nonlinearity $\widehat{g}$. It is easy to find that $\widehat{\Phi} \in C^1(E,\mathbb{R})$, $\widehat{\Phi}(0) = 0$ and $\widehat{\Phi}$ is even. In view of the boundedness of $\widehat{G}(x,u)$, $\widehat{\Phi}$ is bounded from below.

At the same time

$$\widehat{\Phi}'(u)v = (u,v)_E - \int_{\Omega}\widehat{g}(x,u)v\,dx \tag{14}$$

for all $u,v \in E$. It is well known that $\widehat{\Phi}' : E \to E$ is compact.

Now the proof of conclusion of Theorem 1 is divided into three steps.

Step 1. $\widehat{\Phi}$ satisfies the (PS) condition.

We assume the sequence $\{u_n\} \subset E$ satisfies the requirement that $\{\widehat{\Phi}(u_n)\}$ is bounded and $\widehat{\Phi}'(u_n) \to 0$ as $n \to \infty$. Now we claim that the sequence $\{u_n\}$ is bounded in E and possesses a strong convergent subsequence.

First, we claim that $\{u_n\}$ is bounded in E. By (8), we set $u_n = u_n^- + u_n^0 + u_n^+ \in E^- \oplus E^0 \oplus E^+$.

Considering $\widehat{\Phi}'(u_n) \to 0$ as $n \to \infty$, we have $\|\widehat{\Phi}'(u_n)\| < 1$ as n is large enough. Then

$$\left|\widehat{\Phi}'(u_n)u_n^+\right| \le \left\|\widehat{\Phi}'(u_n)\right\|\left\|u_n^+\right\| < \left\|u_n^+\right\|. \tag{15}$$

By (4), (5), (11), (14), and (15), we have

$$\begin{aligned}
\left\|u_n^+\right\| &> \left|\widehat{\Phi}'(u_n)u_n^+\right| \\
&= \left(u_n, u_n^+\right)_E - \int_{\Omega}\widehat{g}(x,u_n)u_n^+\,dx \\
&\ge \left(u_n^+, u_n^+\right)_E - \int_{\Omega}\left|\widehat{g}(x,u_n)\right|\left|u_n^+\right|dx \\
&\ge \left\|u_n^+\right\|^2 - c_1\left\|u_n^+\right\|_1 \\
&\ge \left\|u_n^+\right\|^2 - c_2\left\|u_n^+\right\|.
\end{aligned} \tag{16}$$

From (16), we learn that $\{u_n^+\}$ is bounded in E. Similarly, we see that $\{u_n^-\}$ is bounded in E, too.

Now we consider $\{u_n^0\}$. By the assumption that $\{\widehat{\Phi}(u_n)\}$ is bounded, there exists a constant $K > 0$ such that $|\widehat{\Phi}(u_n)| \le K$. Thus, by (13), we have

$$\begin{aligned}
K &\ge \frac{1}{2}\|u_n\|^2 - \int_{\Omega}\left|\widehat{G}(x,u_n)\right|dx \\
&\ge \frac{1}{2}\left(\left\|u_n^-\right\|^2 + \left\|u_n^0\right\|^2 + \left\|u_n^+\right\|^2\right) - \int_{\Omega}\left|\widehat{G}(x,u_n)\right|dx \\
&\ge \frac{1}{2}\left\|u_n^0\right\|^2 - \int_{\Omega}\left|\widehat{G}(x,u_n) - \widehat{G}\left(x,u_n^0\right)\right|dx - \int_{\Omega}\left|\widehat{G}\left(x,u_n^0\right)\right|dx.
\end{aligned} \tag{17}$$

Here,

$$\begin{aligned} \left|\widehat{G}(x,u_n)-\widehat{G}\left(x,u_n^0\right)\right| &= \left|\widehat{g}(x,\xi_1)\right|\left|\left(u_n^+ + u_n^-\right)\right|, \\ &\le \left|\widehat{g}(x,\xi_1)\right|\left|u_n^+\right| + \left|\widehat{g}(x,\xi_1)\right|\left|u_n^-\right|, \quad \xi_1 \text{ is between } u_n \text{ and } u_n^0, \end{aligned} \tag{18}$$

and

$$\left|\widehat{G}\left(x,u_n^0\right)\right| = \left|\int_0^{u_n^0}\widehat{g}(x,s)\,ds\right| = \left|\widehat{g}(x,\xi_2)\right|\left|u_n^0\right|, \quad \xi_2 \text{ is between } 0 \text{ and } u_n^0. \tag{19}$$

By (5), (11), (17)-(19), together with the boundedness of $\{u_n^+\}$ and $\{u_n^-\}$, we have

$$\begin{aligned} K &\ge \frac{1}{2}\left\|u_n^0\right\|^2 - \int_\Omega \left(\left|\widehat{g}(x,\xi_1)\right|\left|u_n^+\right| + \left|\widehat{g}(x,\xi_1)\right|\left|u_n^-\right|\right)dx - \int_\Omega \left|\widehat{g}(x,\xi_2)\right|\left|u_n^0\right|dx \\ &\ge \frac{1}{2}\left\|u_n^0\right\|^2 - c_3\left(\left\|u_n^+\right\|_1 + \left\|u_n^-\right\|_1\right) - c_4\left\|u_n^0\right\|_1 \\ &\ge \frac{1}{2}\left\|u_n^0\right\|^2 - c_5 - c_6\left\|u_n^0\right\|. \end{aligned} \tag{20}$$

From (20), we can see that $\{u_n^0\}$ is bounded in E. Thus, $\{u_n\}$ is bounded in E.

Second, we claim that $\{u_n\}$ possesses a strong convergent subsequence in E.

Now, define an operator $T: E \to E$ as

$$(Tu,v)_E = \int_\Omega \widehat{g}(x,u)v\,dx.$$

It is obvious that T is a compact operator. Then (14) changes into

$$\widehat{\Phi}'(u)v = (u,v)_E - (Tu,v)_E = (u-Tu,v)_E.$$

So, $\widehat{\Phi}'(u) = u - Tu$. Set $u = u_n$, we have

$$\widehat{\Phi}'(u_n) = u_n - Tu_n. \tag{21}$$

Considering the boundedness of $\{u_n\}$ and the compactness of the operator T, $\{Tu_n\}$ has a strong convergent subsequence. Combining $\widehat{\Phi}'(u_n) \to 0$ as $n \to \infty$, we see that $\{u_n\}$ possesses a strong convergent subsequence in E.

Step 2. Construction of space $X^k \cap S_{\rho_k}$ and proof of $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$.

In view of (2), we learn that for any $M > 0$ there exists a constant $\beta > 0$ such that

$$\frac{\widehat{G}(x,u)}{|u|^2} > M, \quad |u| < \beta, \text{ uniformly for } x \in \overline{\Omega}. \tag{22}$$

For any $k \in \mathbb{N}$, we choose k-dimensional subspace of E as X^k (*e.g.* the subspace spanned by eigenfunctions $\{e_n\}$). By the equivalence of any two norms on finite-dimensional space X^k, we can choose a sufficient small constant $\rho_k > 0$ such that, for any $u \in X^k$, $\|u\|_\infty < \beta$, as $\|u\| = \rho_k$ uniformly for $x \in \overline{\Omega}$. That is for any $u \in X^k$,

$$|u| < \beta, \quad \text{as } \|u\| = \rho_k \text{ uniformly for } x \in \overline{\Omega}. \tag{23}$$

Now we can construct the space $X^k \cap S_{\rho_k}$ where $S_{\rho_k} = \{u \in X \mid \|u\| = \rho_k\}$. For any $u \in X^k \cap S_{\rho_k}$, by the equivalence of any two norms on finite-dimensional space X^k, together with (13) and (22), we have

$$\begin{aligned}\widehat{\Phi}(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega \frac{\widehat{G}(x,u)}{|u|^2}|u|^2\,dx \\ &< \frac{1}{2}\|u\|^2 - M\|u\|_2^2 \\ &\leq \frac{1}{2}\|u\|^2 - Mc_7\|u\|^2. \end{aligned} \tag{24}$$

For any M large enough, we can choose a sufficiently small ρ_k such that $\widehat{\Phi}(u) < 0$. Thus,

$$\sup_{X^k \cap S_{\rho_k}} \widehat{\Phi} < 0. \tag{25}$$

Now we appeal to Theorem 2 to obtain infinitely many solutions u_k for (12) such that $\|u_k\| \to 0$ as $k \to \infty$.

Step 3. Solutions of (12) are solutions of (1).

By virtue of the boundedness of $\widehat{g}(x,u)$, we have the solutions of (12) $u_k \in W^{2,q}(\Omega)$ for any $q \in [1,+\infty)$. Then we have

$$u_k \in W^{2,q}(\Omega) \cap H_0^1(\Omega) \hookrightarrow C^{1,\alpha}(\Omega) \subset C^0(\Omega) \tag{26}$$

(the embedding theorem is given in [20], Theorem 7.26).

Since $\|u_k\| \to 0$, then $\|u_k\|_\infty \to 0$. When k is large enough, there exists a constant $\delta > 0$ such that $|u_k(x)| < \delta/2$. Then $g(x,u_k(x)) = \widehat{g}(x,u_k(x))$. Therefore, u_k are the solutions of (1) with k sufficiently large and $\|u_k\|_\infty \to 0$ as $k \to \infty$. □

Competing interests
The authors declare that they have no financial or non-financial competing interests.

Authors' contributions
HG wrote the first draft and TA corrected and improved the final version. All authors read and approved the final draft.

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