Kong and Ru *Boundary Value Problems* (2017) 2017:17
DOI 10.1186/s13661-017-0751-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global solutions for a new geometric flow with rotational invariance

De-Xing Kong[1] and Qiang Ru[2*]

*Correspondence:
ruqiang666@163.com
[2]Department of Mathematics, China University of Mining and Technology, Xuzhou, 221116, China
Full list of author information is available at the end of the article

Abstract

In this paper we introduce a new geometric flow with rotational invariance and prove the existence of a global solution.

MSC: 53C21; 53C44; 35M20

Keywords: system of hyperbolic-parabolic equations; rotational invariance; global smooth solution; hypersurface

1 Introduction

Since the last quarter of the 20th century, using partial differential equations to formulate and solve geometric problems has become a trend and a dominating force. A new area called geometric analysis was born. When looking back at the history of geometric analysis, one could see numerous success stories of utilizing differential equations to tackle important problems in geometry, topology, and physics. Typical and important examples would include Yau's solution to the Calabi conjecture using the complex Monge-Ampère equation (see Yau [1]), Schoen's solution of the Yamabe conjecture (see Schoen [2]), Schoen-Yau's proof of the positive mass conjecture (see Schoen-Yau [3]), Donaldson's work on 4-dimensional smooth manifolds using the Yang-Mills equation (see Donaldson [4]), and recently, Perelman's solution to the century-old Poincaré conjecture using Hamilton's beautiful theory on the Ricci flow, which is just a nonlinear version of the classical heat equation (see [5–7]). However, despite all these success, the equations studied and utilized in geometry so far are almost exclusively of elliptic or parabolic type. With few exceptions, hyperbolic equations have not yet found their way into the study of geometric or topological problems. More recently, Kong *et al.* introduced the hyperbolic geometric flow which is a fresh start of an attempt to introduce hyperbolic partial differential equations into the realm of geometry (see [8] or [9]). The kind of flow is a very natural tool to understand the wave character of metrics, the wave phenomenon of curvatures, the evolution of manifolds and their structures (see [10–12]).

In this paper, we introduce a new geometric flow with rotational invariance. This flow is described formally by a system of parabolic partial differential equations, essentially a coupled system of hyperbolic-parabolic partial differential equations with rotational invariance. More precisely, let $\mathcal{S}_t$ be a family of hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with coordinates $(x_1,\dots,x_{n+1})$, without loss of generality, we may assume that the family of hypersurfaces $\mathcal{S}_t$ is given by

$$x = x(t, \theta_1, \ldots, \theta_n), \tag{1.1}$$

where $x = (x_1, \ldots, x_{n+1})^T$ is a vector-valued smooth function of t and $\theta = (\theta_1, \ldots, \theta_n)$, the new flow considered here is given by the following evolution equation:

$$\frac{\partial x}{\partial t} + \sum_{i=1}^{n} \frac{\partial (f_i(|x|)x)}{\partial \theta_i} = \frac{x}{|x|} \Delta |x|, \tag{1.2}$$

where $f_i(\nu)$ $(i = 1, \ldots, n)$ are n given smooth functions, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial \theta_i^2}$ is the Laplacian operator, and $|\bullet|$ stands for the norm of the vector $\bullet$ in $\mathbb{R}^{n+1}$. It is easy to verify that equation (1.2) possesses the rotational invariance which plays an important role in the present paper.

We are interested in the deformation of a smooth closed hypersurface $x = x_0(\theta_1, \ldots, \theta_n)$ under the flow (1.2), that is, we consider how the hypersurface x_0 is smoothly deformed, say, embedded into a smooth family of hypersurfaces depending on a time parameter. This can be reduced to solve the Cauchy problem for (1.2) with the initial data

$$t = 0 : x = x_0(\theta_1, \ldots, \theta_n). \tag{1.3}$$

Obviously, in the present situation, $x_0 = x_0(\theta_1, \ldots, \theta_n)$ is a vector-valued periodic function, say, defined on $[0, 1]^n$. In Section 2, we shall prove the following.

Theorem 1.1 *If $f \in C^1$, $x_0 \in L^\infty$ and $|x_0(\theta_1, \ldots, \theta_n)| > 0$, then the Cauchy problem* (1.2), (1.3) *admits a unique global smooth solution on* $[0, \infty) \times \mathbb{R}^n$.

The paper is organized as follows. In Section 2 we prove the global existence and uniqueness of smooth solutions for the Cauchy problem (1.2), (1.3); in Section 3, we state conclusions obtained in the present paper and give some open problems.

2 Global existence and uniqueness of smooth solutions

This section is devoted to the global existence and uniqueness of smooth solution of the following equation:

$$\frac{\partial x}{\partial t} + \sum_{i=1}^{m} \frac{\partial (f_i(|x|)x)}{\partial \theta_i} = \frac{x}{|x|} \Delta |x|, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)^T$ is the unknown vector-valued function, $f(\nu) = (f_1(\nu), \ldots, f_m(\nu))^T$ is a given smooth vector-valued function, $\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial \theta_i^2}$ is the Laplacian operator, and $|\bullet|$ stands for the norm of the vector $\bullet$ in $\mathbb{R}^n$.

Let

$$x = rP, \quad r = |x|, P = (p_1, \ldots, p_n) \in \mathbb{S}^{n-1}. \tag{2.2}$$

Then it is easy to verify that equation (2.1) can be rewritten as

$$\frac{\partial r}{\partial t} + \sum_{i=1}^{m} \frac{\partial (f_i(r)r)}{\partial \theta_i} = \Delta r \tag{2.1a}$$

and

$$\frac{\partial P}{\partial t}+\sum_{i=1}^{m}\frac{\partial P}{\partial \theta_i}f_i(r)=0 \tag{2.1b}$$

for smooth solutions.

We now consider the Cauchy problem for equation (2.1), equivalently, the system (2.1a)-(2.1b) with initial data

$$t=0: r=r_0(\theta), \qquad P=P_0(\theta), \tag{2.3}$$

where $r_0(\theta)$ is a given scalar function of θ, and $P_0(\theta)$ is a given vector-valued function of θ. In the following, we first investigate the local existence of smooth solution of the above Cauchy problem.

As the standard way, let $K(t,\theta)$ be the fundamental solution associated with the operator $\frac{\partial}{\partial t}-\triangle$. That is to say,

$$K(t,\theta)=(4\pi t)^{-\frac{n}{2}}\exp\left\{-\frac{|\theta|^2}{4t}\right\}. \tag{2.4}$$

Then the solution $r=r(t,\theta)$ of the Cauchy problem

$$\begin{cases} \frac{\partial r}{\partial t}+\sum_{i=1}^{m}\frac{\partial (f_i(r)r)}{\partial \theta_i}=\Delta r, \\ t=0: r=r_0(\theta), \end{cases} \tag{2.5}$$

has the following integral representation:

$$r(t,\theta)=K(t,\theta)*r_0(\theta)+\sum_{j=1}^{m}\int_0^t K_{\theta_j}(t-s,\theta)*\big(f_j\big(r(s,\theta)\big)r(s,\theta)\big)\,ds, \tag{2.6}$$

where $*$ denotes the convolution with the space variables. We have the following.

Lemma 2.1 *Assume that*

$$f\in C^1, \quad r_0\in L^\infty, \tag{2.7}$$

then there exists a positive constant T *such that the Cauchy problem* (2.5) *admits a unique smooth solution* $=r(t,\theta)$ *on the strip*

$$\Pi_T=\big\{(t,\theta)\big|t\in[0,T],\theta\in\mathbb{R}^m\big\}, \tag{2.8}$$

where

$$T=\min\left\{\left(\frac{M\pi^{\frac{1}{2}}}{2H}\right)^2,\left(\frac{\pi^{\frac{1}{2}}}{4Hm}\right)^2\right\}, \tag{2.9}$$

in which

$$M = \|r_0(\theta)\|_{L^\infty},$$
$$H = \max_{i,j=1,\dots,m}\left\{\sup_{|r|\le(m+1)M}\left|g_i\bigl(r(t,\theta)\bigr)\right|, \sup_{|r|\le(m+1)M}\left|\frac{\partial}{\partial\theta_j}g_i\bigl(r(t,\theta)\bigr)\right|\right\}, \tag{2.10}$$

where $g_j = rf_j(r)$ $(j = 1,\dots,m)$.

Proof Set

$$G_T = \bigl\{r : [0,T]\times\mathbb{R}^m \to L^\infty(\mathbb{R}^m) \mid \|r(t,\cdot)\|_{L^\infty} \le (m+1)M \text{ for } t\in[0,T]\bigr\} \tag{2.11}$$

and let $\mathcal{T}$ be the following integral operator:

$$\mathcal{T}r(t,\theta) = K(t,\theta) * r_0(\theta) + \sum_{j=1}^m \int_0^t K_{\theta_j}(t-s,\theta) * \bigl(f_j\bigl(r(s,\theta)\bigr)r(s,\theta)\bigr)\,ds. \tag{2.12}$$

The solution $r = r(t,\theta)$ can be obtained as the L^∞-limit of the sequence $\{r^k\}$ defined by

$$r^0(t,\theta) = K(t,\theta) * r_0(\theta), \qquad r^{k+1} = \mathcal{T}r^k \quad (n = 0,1,\dots). \tag{2.13}$$

To prove the above statement, we first claim that, for any $t\in[0,T]$, we have

$$\|r^k(t,\theta)\|_{L^\infty} \le (m+1)M, \quad \forall k\in\{0,1,2,\dots\}. \tag{2.14}$$

In the following, we prove (2.14) by the method of induction.

When $k = 0$, we have

$$\|r^0(t,\theta)\|_{L^\infty} = \|K(t,\theta) * r_0(\theta)\|_{L^\infty}. \tag{2.15}$$

By Young's inequality, we obtain

$$\|r^0(t,\theta)\|_{L^\infty} \le \|K(t,\theta)\|_{L^1}\|r_0(\theta)\|_{L^\infty} = \|r_0(\theta)\|_{L^\infty} = M \le (m+1)M. \tag{2.16}$$

Now we assume that $\|r^k(t,\theta)\|_{L^\infty} \le (m+1)M$ $(k\in\mathbb{N})$ holds. We next prove

$$\|r^{k+1}(t,\theta)\|_{L^\infty} \le (m+1)M. \tag{2.17}$$

In fact,

$$\begin{aligned}\|r^{k+1}(t,\theta)\|_{L^\infty} &= \|\mathcal{T}r^k(t,\theta)\|_{L^\infty}\\ &\le \|K(t,x) * r_0(\theta)\|_{L^\infty} + \sum_{j=1}^m\int_0^t \left\|K_{x_j}(t-s,\theta) * \bigl(f_j\bigl(r^k(s,\theta)\bigr)r^k(s,\theta)\bigr)\right\|_{L^\infty} ds\\ &\le M + \sum_{j=1}^m\int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1}\left\|\bigl(f_j\bigl(r^k(s,\theta)\bigr)r^k(s,\theta)\bigr)\right\|_{L^\infty} ds.\end{aligned} \tag{2.18}$$

Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \left|K_{\theta_j}(t-s,\theta)\right| d\theta \\
&= \int_{\mathbb{R}^m} \left[4\pi(t-s)\right]^{-\frac{m}{2}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{|\theta|^2}{4(t-s)}\right\} d\theta \\
&= \left[4\pi(t-s)\right]^{-\frac{m}{2}} \int_{\mathbb{R}^{m-1}} \left\{\exp\left\{-\frac{\theta_1^2}{4(t-s)}\right\} + \cdots + \exp\left\{-\frac{\theta_{j-1}^2}{4(t-s)}\right\}\right. \\
&\quad \left. + \exp\left\{-\frac{\theta_{j+1}^2}{4(t-s)}\right\} + \cdots + \exp\left\{-\frac{\theta_m^2}{4(t-s)}\right\}\right\} d\theta_1 \cdots d\theta_{j-1}\, d\theta_{j+1} \cdots d\theta_m \\
&\quad \times \int_{\mathbb{R}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\theta_j \\
&= \left[4\pi(t-s)\right]^{-\frac{m}{2}} \left\{\left[4(t-s)\right]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{\theta_1^2}{4(t-s)}\right\} d\left(-\frac{\theta_1}{[4(t-s)]^{\frac{1}{2}}}\right)\right\}^{m-1} \\
&\quad \times \int_{\mathbb{R}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\theta_j \\
&= \left[4\pi(t-s)\right]^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\theta_j \\
&= \left[4\pi(t-s)\right]^{-\frac{1}{2}} \left[-2\int_0^\infty \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\left(-\frac{\theta_j^2}{4(t-s)}\right)\right] \\
&= \left[\pi(t-s)\right]^{-\frac{1}{2}}.
\end{aligned} \tag{2.19}$$

It follows from (2.18) that

$$\begin{aligned}
\left\|r^{k+1}(t,\theta)\right\|_{L^\infty} &\le M + \sum_{j=1}^m \int_0^t \left\|K_{\theta_j}(t-s,\theta)\right\|_{L^1} \left\|\left(f_j\left(r^k(s,\theta)\right)r^k(s,\theta)\right)\right\|_{L^\infty} ds \\
&\le M + \sum_{j=1}^m \pi^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \left\|\left(f_j\left(r^k(s,\theta)\right)r^k(s,\theta)\right)\right\|_{L^\infty} ds \\
&\le M + m\pi^{-\frac{1}{2}} H \int_0^t (t-s)^{-\frac{1}{2}} ds \\
&= M + 2m\pi^{-\frac{1}{2}} H t^{\frac{1}{2}} \\
&\le M + 2m\pi^{-\frac{1}{2}} H T^{\frac{1}{2}} \le (m+1)M.
\end{aligned} \tag{2.20}$$

This is the desired estimate (2.14). Thus, the proof of (2.14) is completed.

In the following, we prove that $\{r^k(t,\theta)\}$ is uniformly convergent in the strip $(0,T] \times \mathbb{R}^m$. To do so, it suffices to show that

$$\sum_{k=1}^\infty \left[r^{k+1}(t,\theta) - r^k(t,\theta)\right]$$

is uniformly convergent in the strip $(0,T] \times \mathbb{R}^m$.

In fact, we have

$$
\begin{aligned}
&\left\|r^{k+1}-r^k\right\|_{L^\infty} \\
&\le \sum_{j=1}^{m}\int_0^t \left\|K_{\theta_j}(t-s,\theta)*\left[\left(f_j\left(r^k(s,\theta)\right)r^k(s,\theta)\right)-\left(f_j\left(r^{k-1}(s,\theta)\right)r^{k-1}(s,\theta)\right)\right]\right\|_{L^\infty}ds \\
&\le \sum_{j=1}^{m}\int_0^t \left\|K_{\theta_j}(t-s,\theta)\right\|_{L^1}\left\|\left[\left(f_j\left(r^k(s,\theta)\right)r^k(s,\theta)\right)-\left(f_j\left(r^{k-1}(s,\theta)\right)r^{k-1}(s,\theta)\right)\right]\right\|_{L^\infty}ds \\
&\le \sum_{j=1}^{m}\int_0^t \left\|K_{\theta_j}(t-s,\theta)\right\|_{L^1}\left\|g_j\left(r^k(s,\theta)\right)-g_j\left(r^{k-1}(s,\theta)\right)\right\|_{L^\infty}ds \\
&\le \sum_{j=1}^{m}\int_0^t \left\|K_{\theta_j}(t-s,\theta)\right\|_{L^1}\left\|\nabla g_j(\beta_k)\right\|_{L^\infty}\left\|r^k(s,\theta)-r^{k-1}(s,\theta)\right\|_{L^\infty}ds \\
&\le mH\max\left\{\left|r^k(s,\theta)-r^{k-1}(s,\theta)\right|\right\}\int_0^t \left\|K_{\theta_j}(t-s,\theta)\right\|_{L^1}ds \\
&\le 2\pi^{-\frac{1}{2}}mHt^{\frac{1}{2}}\max\left\{\left|r^k(s,\theta)-r^{k-1}(s,\theta)\right|\right\} \\
&\le 2\pi^{-\frac{1}{2}}mHT^{\frac{1}{2}}\max\left\{\left|r^k(s,\theta)-r^{k-1}(s,\theta)\right|\right\} \\
&\le \left(2\pi^{-\frac{1}{2}}mHT^{\frac{1}{2}}\right)^2\max\left\{\left|r^{k-1}(s,\theta)-r^{k-2}(s,\theta)\right|\right\} \\
&\le \cdots \\
&\le \left(2\pi^{-\frac{1}{2}}mHT^{\frac{1}{2}}\right)^k\max\left\{\left|r^1(s,\theta)-r^0(s,\theta)\right|\right\},
\end{aligned}
\tag{2.21}
$$

where

$$\beta_k\in\left[\min\left\{r^k(s,x),r^{k-1}(s,x)\right\},\max\left\{r^k(s,x),r^{k-1}(s,x)\right\}\right].$$

Noting

$$\left\|r^1(s,\theta)-r^0(s,\theta)\right\|_{L^\infty}\le 2\pi^{-\frac{1}{2}}mHT^{\frac{1}{2}}, \tag{2.22}$$

we obtain from (2.21)

$$\left\|r^{k+1}-r^k\right\|_{L^\infty}\le\left(2\pi^{-\frac{1}{2}}mHT^{\frac{1}{2}}\right)^{k+1}. \tag{2.23}$$

By (2.9), we have

$$\left\|r^{k+1}-r^k\right\|_{L^\infty}\le\left(\frac{1}{2}\right)^{k+1}, \tag{2.24}$$

which implies that $\sum_{k=1}^{\infty}[r^{k+1}(t,\theta)-r^k(t,\theta)]$ is uniformly convergent in the strip $(0,T]\times\mathbb{R}^m$. Therefore, $\lim_{k\to\infty}r^k(t,\theta)$ gives the unique local solution of the Cauchy problem (2.5). Thus, the proof of Lemma 2.1 is completed. □

Lemma 2.2 *Suppose that*

$$f \in C^1, \qquad r_0 \in L^\infty$$

and let $M \triangleq \|r_0\|_{L^\infty}$. *Suppose furthermore that* $r(t,\theta)$ *is the solution of Cauchy problem* (2.5) *on the strip* Π_T, *then we have*

$$\big\|r(t,\theta)\big\|_{L^\infty(\Pi_T)} \le M. \tag{2.25}$$

Proof It follows from the proof of Lemma 2.1 that

$$\big\|r(t,\theta)\big\|_{L^\infty(\Pi_T)} \le (m+1)M \triangleq K. \tag{2.26}$$

Introduce

$$w(t,\theta) = r(t,\theta) - M - \frac{K}{L^2}\big(|\theta|^2 + CLe^t\big), \tag{2.27}$$

where C and L are positive constants to be determined. By (2.27),

$$r_t = w_t + \frac{CK}{L}e^t, \qquad \triangle r = \triangle w + \frac{2Km}{L^2}. \tag{2.28}$$

On the other hand,

$$\sum_{j=1}^m \big(f_j(r)r\big)_{\theta_j} = \sum_{j=1}^m \big(g_j(r)\big)_{\theta_j} = \sum_{j=1}^m g_j'(r)r_{\theta_j} = \sum_{j=1}^m g_j'(r)\left(w_{\theta_j} + \frac{2K}{L^2}\theta_j\right). \tag{2.29}$$

Thus,

$$w_t + \sum_{j=1}^m g_j'(r)w_{\theta_j} + \frac{2K}{L^2}\sum_{j=1}^m g_j'(r)\theta_j + \frac{CK}{L}e^t - \frac{2Km}{L^2} = \triangle w. \tag{2.30}$$

Choose sufficiently large C such that

$$w(0,\theta) = r_0(\theta) - M - \frac{K}{L^2}\big(|\theta|^2 + CL\big) < 0, \quad \forall\theta \in \mathbb{R}^m, \tag{2.31}$$

and

$$\begin{cases} w(t,\pm L,\theta_2,\ldots,\theta_m) = r(t,\pm L,\theta_2,\ldots,\theta_m)M \\ \quad - \frac{K}{L^2}[(L^2 + \theta_2^2 + \cdots + \theta_m^2) + CLe^t] < 0, \\ w(t,\theta_1,\pm L,\ldots,\theta_m) = r(t,\theta_1,\pm L,\ldots,\theta_m) - M \\ \quad - \frac{K}{L^2}[(\theta_1^2 + L^2 + \cdots + \theta_m^2) + CLe^t] < 0, \\ \cdots \\ w(t,\theta_1,\ldots,\theta_{m-1},\pm L) = r(t,\theta_1,\ldots,\theta_{m-1},\pm L) - M \\ \quad - \frac{K}{L^2}[(\theta_1^2 + \cdots + \theta_{m-1}^2 + |L|^2) + CLe^t] < 0 \end{cases} \tag{2.32}$$

for all $t \in [0,T]$.

In the following, we prove that, for any $(t,\theta) \in (0,T) \times (-L,L)^m$, that

$$w(t,\theta) < 0. \tag{2.33}$$

In fact, if (2.33) is not true, then we can define $\bar{t}$ by

$$\bar{t} = \inf_{t\in(0,T]} \{ t \,|\, w(t,\theta) = 0 \text{ for some } \theta \in (-L,L)^m \}. \tag{2.34}$$

It is easy to see that there exists a point, denoted by $\bar{\theta} \in (-L,L)^m$, such that

$$w(\bar{t},\bar{\theta}) = 0, \qquad w_{\theta_1}(\bar{t},\bar{\theta}) = 0, \qquad \ldots, \qquad w_{\theta_m}(\bar{t},\bar{\theta}) = 0 \tag{2.35}$$

and

$$w_{\theta_i\theta_i}(\bar{t},\bar{\theta}) \le 0, \quad \forall i \in \{1,\ldots,m\}. \tag{2.36}$$

By (2.35)-(2.36), it follows from (2.30) that

$$w_t(\bar{t},\bar{\theta}) + \frac{2K}{L^2}\sum_{j=1}^{m} g_j'\bigl(r(\bar{t},\bar{\theta})\bigr)\bar{\theta}_j + \frac{CK}{L}e^{\bar{t}} - \frac{2Km}{L^2} \le 0. \tag{2.37}$$

Noting

$$\bigl\|g_j'(\bullet)\bigr\|_{L^\infty} < \infty \quad \text{and} \quad (\bar{t},\theta_j) \in (0,T] \times (-L,L), \tag{2.38}$$

we can choose a sufficiently large C such that

$$\frac{2K}{L^2}\sum_{j=1}^{m} g_j'\bigl(r(\bar{t},\bar{\theta})\bigr)\bar{\theta}_j + \frac{CK}{L}e^{\bar{t}} - \frac{2Km}{L^2} > 0. \tag{2.39}$$

Combining (2.37) and (2.39)

$$w_t(\bar{t},\bar{\theta}) < 0. \tag{2.40}$$

On the other hand, by the definition of $(\bar{t},\bar{\theta})$ we have

$$w_t(\bar{t},\bar{\theta}) = \lim_{\triangle t\to 0} \frac{w(\bar{t},\bar{\theta}) - w(\bar{t}-\triangle t,\bar{\theta})}{\triangle t} \ge 0, \tag{2.41}$$

which is a contradiction. This proves (2.33).

Noting (2.27) and (2.33) and letting $L \to \infty$ gives

$$r(t,\theta) \le M, \quad \forall (t,\theta) \in \Pi_T. \tag{2.42}$$

Similarly, letting

$$w(t,\theta) = r(t,\theta) + M + \frac{K}{L^2}\bigl(|\theta|^2 + CLe^t\bigr), \tag{2.43}$$

we can prove

$$r(t,\theta) \geq -M, \quad \forall (t,\theta) \in \Pi_T. \tag{2.44}$$

Combining (2.42) and (2.44) leads to

$$\left\| r(t,\theta) \right\|_{L^\infty(\Pi_T)} \leq M. \tag{2.45}$$

Thus, the proof of Lemma 2.2 is completed. □

By Lemma 2.1 and Lemma 2.2, we have the following.

Theorem 2.1 *If* $f \in C^1$ *and* $r_0 \in L^\infty$, *then the Cauchy problem* (2.5) *admits a unique global smooth solution on* $[0,\infty) \times \mathbb{R}^m$.

Now we turn to a consideration of the Cauchy problem (2.1) (*i.e.*, (2.1a)-(2.1b)), (2.3). We have the following.

Theorem 2.2 *Under the assumptions of Theorem* 1.1, *the Cauchy problem* (2.1), (2.3) *admits a unique global smooth solution on* $[0,\infty) \times \mathbb{R}^m$.

Proof Noting (1.4), by the maximum principle we obtain the result that, on the existence domain of smooth solution, we have

$$r(t,\theta) > 0. \tag{2.46}$$

On the one hand, we observe that, under the condition (2.46), equation (2.1) can be reduced to the system (2.1a)-(2.1b); on the other hand, we notice that, once $r = r(t,\theta)$ is solved from the Cauchy problem (2.5), equation (2.1b) becomes linear. Therefore, Theorem 2.2 follows from Theorem 2.1 directly. □

Obviously, Theorem 1.1 follows from Theorem 2.2 directly.

3 Conclusions and open problems

In the present paper, we introduce a new geometric flow with rotational invariance. This flow is described formally by a system of hyperbolic partial differential equations with viscosity, essentially a coupled system of hyperbolic-parabolic partial differential equations with rotational invariance, which possesses very interesting geometric properties and dynamical behavior. We only investigate the global solutions for the flow equation (1.2) in the Euclidean space $\mathbb{R}^n$ ($n \geq 2$), there are some fundamental and interesting problems. In particular, the following open problems seem to us to be more interesting and important: (i) use the flow equation (2.1) to investigate the deformation of a closed m-dimensional sub-manifold $x_0 = x_0(\theta_1, \ldots, \theta_m)$; (ii) find a suitable way to extend the results presented in this paper to the case of Riemannian manifolds instead of the Euclidean space $\mathbb{R}^n$; (iii) introduce the theory of viscous shock waves to investigate geometric problems. These problems are worthy of study in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Zhejiang University, Hangzhou, 310027, China. [2]Department of Mathematics, China University of Mining and Technology, Xuzhou, 221116, China.

Acknowledgements
This research was supported by the Fundamental Research Funds for the Central Universities (2014QNA63), the Natural Science Foundation of Jiangsu Province (BK20150172) and the TianYuan Special Funds of the National Natural Science Foundation of China (11526191).

Received: 30 October 2016 Accepted: 16 January 2017 Published online: 25 January 2017

References

1. Yau, S-T: Calabi's conjecture and some new results in algebraic geometry. Proc. Natl. Acad. Sci. USA **74**, 1798-1799 (1977)
2. Schoen, R: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ. Geom. **20**, 479-495 (1984)
3. Schoen, R, Yau, S-T: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. **65**, 45-76 (1979)
4. Donaldson, S: An application of gauge theory to four-dimensional topology. J. Differ. Geom. **18**, 279-315 (1983)
5. Perelman, G: The entropy formula for the Ricci flow and its geometric applications. arXiv:math.DG/0211159
6. Perelman, G: Ricci flow with surgery on three-manifolds. arXiv:math.DG/0303109
7. Perelman, G: Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arXiv:math.DG/0307245
8. Kong, D-X: Hyperbolic geometric flow. In: Proceedings of ICCM 2007, vol. II, pp. 95-110. Higher Educational Press, Beijing (2007)
9. Kong, D-X, Liu, K-F: Wave character of metrics and hyperbolic geometric flow. J. Math. Phys. **48**, 103508 (2007)
10. Dai, W-R, Kong, D-X, Liu, K-F: Hyperbolic geometric flow (I): short-time existence and nonlinear stability. Pure Appl. Math. Q. (Special Issue: In honor of Michael Atiyah and Isadore Singer) **6**, 331-359 (2010)
11. Kong, D-X, Liu, K-F, Wang, Y-Z: Life-span of classical solutions to hyperbolic geometric flow in two space variables with slow decay initial data. Commun. Partial Differ. Equ. **36**, 162-184 (2011)
12. Kong, D-X, Liu, K-F, Xu, D-L: The hyperbolic geometric flow on Riemann surfaces. Commun. Partial Differ. Equ. **34**, 553-580 (2009)