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Study on a kind of fourth-order p-Laplacian Rayleigh equation with linear autonomous difference operator

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Abstract

In this paper, we consider the following fourth-order Rayleigh type p-Laplacian generalized neutral differential equation with linear autonomous difference operator:

$$(\varphi_p(x(t)-c(t)x(t-\delta(t)))'')''+f(t,x'(t))+g(t,x(t-\tau(t)))=e(t).$$

By applications of coincidence degree theory and some analysis skills, sufficient conditions for the existence of periodic solutions are established.

MSC: 34C25; 34K13; 34K40

Keywords: periodic solution; p-Laplacian; fourth-order; linear autonomous difference operator; Rayleigh type

1 Introduction

In this paper, we consider the following fourth-order Rayleigh type p-Laplacian neutral differential equation with linear autonomous difference operator:

$$\left(\varphi_p\left(x(t)-c(t)x\left(t-\delta(t)\right)\right)''\right)''+f\left(t,x'(t)\right)+g\left(t,x\left(t-\tau(t)\right)\right)=e(t), \tag{1.1}$$

where $p\geq 2$, $\varphi_p(x)=|x|^{p-2}x$ for $x\neq 0$ and $\varphi_p(0)=0$; $|c(t)|\neq 1$, $c,\delta\in C^2(\mathbb{R},\mathbb{R})$ and c, δ are T-periodic functions for some $T>0$; f and g are continuous functions defined on $\mathbb{R}^2$ and periodic in t with $f(t,\cdot)=f(t+T,\cdot)$, $g(t,\cdot)=g(t+T,\cdot)$ and $f(t,0)=0$, $e,\tau:\mathbb{R}\to\mathbb{R}$ are continuous periodic functions with $e(t+T)\equiv e(t)$ and $\tau(t+T)\equiv\tau(t)$.

In recent years, there has been a good amount of work on periodic solutions for fourth-order differential equations (see [1–17] and the references cited therein). For example, in [12], applying Mawhin's continuation theorem, Shan and Lu studied the existence of periodic solution for a kind of fourth-order p-Laplacian functional differential equation with a deviating argument as follows:

$$\left[\varphi_p\left(u''(t)\right)\right]''+f\left(u(t)\right)u'(t)+g\left(t,u(t),u\left(t-\tau(t)\right)\right)=e(t). \tag{1.2}$$

Afterwards, Lu and Shan [8] observed a fourth-order p-Laplacian differential equation

$$\left[\varphi_p\left(u''(t)\right)\right]'' + f\left(u''(t)\right) + g\left(u\left(t-\tau(t)\right)\right) = e(t) \tag{1.3}$$

and presented sufficient conditions for the existence of periodic solutions for (1.3). Recently, by means of Mawhin's continuation theorem, Wang and Zhu [14] studied a kind of fourth-order p-Laplacian neutral functional differential equation

$$\left[\varphi_p\left(x(t)-cx(t-\delta)\right)''\right]'' + f\left(x(t)\right)x'(t) + g\left(t, x\left(t-\tau\left(t, |x|_\infty\right)\right)\right) = e(t). \tag{1.4}$$

Some sufficient criteria to guarantee the existence of periodic solutions were obtained.

However, the fourth-order p-Laplacian neutral differential equation (1.1), which includes the p-Laplacian neutral differential equation, has not attracted much attention in the literature. In this paper, we try to fill the gap and establish the existence of periodic solution of (1.1) using Mawhin's continuation theory. Our new results generalize some recent results contained in [2, 8, 12, 14] in several aspects.

2 Preparation

Lemma 1 (See [18]) *If $|c(t)| \neq 1$, then the operator $(Au)(t) := x(t) - c(t)x(t-\delta(t))$ has a continuous inverse A^{-1} on the space*

$$C_T := \left\{u | u \in (\mathbb{R}, \mathbb{R}), u(t+T) \equiv u(t), \forall t \in \mathbb{R}\right\},$$

and satisfies

(1) $\int_0^T |(A^{-1}u)(t)|\,dt \leq \frac{\int_0^T |u(t)|\,dt}{1-c_\infty}$ *for* $c_\infty := \max_{t\in[0,T]} |c(t)| < 1$ $\forall u \in C_T$;

(2) $\int_0^T |(A^{-1})(t)|\,dt \leq \frac{\int_0^T |u(t)|\,dt}{c_0-1}$ *for* $c_0 := \min_{t\in[0,T]} |c(t)| > 1$ $\forall u \in C_T$.

Lemma 2 (Gaines and Mawhin [19]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to apply Mawhin's continuation degree theorem to study the existence of periodic solution for (1.1), we rewrite (1.1) in the form:

$$\begin{cases} (Ax_1)''(t) = \varphi_q(x_2(t)), \\ x_2''(t) = -f(t, x_1'(t)) - g(t, x_1(t-\tau(t))) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to (2.1), then $x_1(t)$ must be a T-periodic solution to (1.1). Thus, the problem of finding a T-periodic solution for (1.1) reduces to finding one for (2.1).

Now, set $X = \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $|x|_\infty = \max\{|x_1|_\infty, |x_2|_\infty\}$; $Y = \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \{x \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t+T) = x(t), t \in \mathbb{R}\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)''(t) \\ x_2''(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(t, x_1'(t)) - g(t, x_1(t - \tau(t))) + e(t) \end{pmatrix}. \tag{2.2}$$

Then (2.1) can be converted to the abstract equation $Lx = Nx$.

From $\forall x \in \operatorname{Ker} L$, $x = \binom{x_1}{x_2} \in \operatorname{Ker} L$, *i.e.*, $\begin{cases} (x_1(t) - c(t)x_1(t-\delta(t)))'' = 0, \\ x_2''(t) = 0, \end{cases}$ we have

$$\begin{cases} x_1(t) - c(t)x_1(t - \delta(t)) = a_2 t + a_1, \\ x_2(t) = b_2 t + b_1, \end{cases}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ are constant. Let $\phi(t) \not\equiv 0$ be a solution of $x(t) - c(t)x(t - \delta(t)) = 1$, then $\operatorname{Ker} L = u = \binom{a_1\phi(t),}{b_1}$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \quad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}; \qquad Qy = \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Let K denote the inverse of $L|_{\operatorname{Ker} p \cap D(L)}$. It is easy to see that $\operatorname{Ker} L = \operatorname{Im} Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s) y(s)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{-s(T-t)}{T}, & 0 \le s \le t \le T, \\ \frac{-t(T-s)}{T}, & 0 \le t < s \le T. \end{cases} \tag{2.3}$$

From (2.2) and (2.3), it is clear that QN and $K(I - Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I - Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\bar{\Omega}$.

3 Main results

Theorem 1 *Assume that the following conditions hold*:

(H_1) *There exists a positive constant K such that $|f(t,u)| \leq K$ for $(t,u) \in \mathbb{R} \times \mathbb{R}$*;
(H_2) *There exists a positive constant D such that $xg(t,x) > 0$, and $|g(t,x)| > K + |e|_\infty$, here $|e|_\infty = \max_{t\in[0,T]} |e(t)|$ for $|x| > D$ and $t \in \mathbb{R}$*;
(H_3) *There exists a positive constant M and $M > |e|_\infty$ such that $g(t,x) \geq -M$ for $x \geq D$ and $t \in \mathbb{R}$.*

Then (1.1) *has at least non-constant T-periodic solution if one of the following conditions is satisfied*:
(i) *If $c_\infty < 1$ and $1 - c_\infty - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1) > 0$*;
(ii) *If $c_0 > 1$ and $c_0 - 1 - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1) > 0$*;
where $\delta_i = \max_{t\in[0,\omega]} |\delta^{(i)}(t)|$, $c_i = \max_{t\in[0,\omega]} |c^{(i)}(t)|$, $i = 1,2$.

Proof Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} (Ax_1)''(t) = \lambda\varphi_q(x_2(t)), \\ x_2''(t) = -\lambda f(t, x_1'(t)) - \lambda g(t, x_1(t-\tau(t))) + \lambda e(t). \end{cases} \tag{3.1}$$

Then the second equation of (3.1) and $x_2(t) = \lambda^{1-p}\varphi_p[(Ax_1)''(t)]$ imply

$$\left(\varphi_p(Ax_1)''(t)\right)'' + \lambda^p f\left(t, x_1'(t)\right) + \lambda^p g\left(t, x_1\left(t-\tau(t)\right)\right) = \lambda^p e(t). \tag{3.2}$$

We first claim that there is a constant $t_0 \in \mathbb{R}$ such that

$$\left|x_1(t_0)\right| \leq D. \tag{3.3}$$

Integrating both sides of (3.2) on the interval $[0, T]$, we arrive at

$$\int_0^T \left\{f\left(t, x_1(t)\right) + g\left(t, x_1\left(t-\tau(t)\right)\right) - e(t)\right\} dt = 0, \tag{3.4}$$

which yields that there exists at least a point t_1^* such that

$$f\left(t_1^*, x_1'\left(t_1^*\right)\right) + g\left(t_1^*, x_1\left(t_1^* - \tau\left(t_1^*\right)\right)\right) = e\left(t_1^*\right),$$

and we get

$$g\left(t_1^*, x_1\left(t_1^* - \tau\left(t_1^*\right)\right)\right) = e\left(t_1^*\right) - f\left(t_1^*, x_1'\left(t_1^*\right)\right),$$

and then by (H_1) we have

$$\left|g\left(t_1^*, x_1\left(t_1^* - \tau\left(t_1^*\right)\right)\right)\right| \leq \left|e\left(t_1^*\right)\right| + \left|f\left(t_1^*, x_1'\left(t_1^*\right)\right)\right| \leq |e|_\infty + K_1,$$

and from (H_2) we can get $|x_1(t_1^* - \tau(t_1^*))| \le D_1$. Since $x_1(t)$ is periodic with period T, take $t_1^* - \tau(t_1^*) = nT + t_0$, $t_0 \in [0,T]$, where n is some integer; then $|x_1(t_0)| \le D$, (3.3) is proved. Then we have

$$\begin{aligned}
|x_1|_\infty &= \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[\xi,\xi+T]} |x_1(t)| \\
&= \frac{1}{2}\max_{t\in[\xi,\xi+T]} \bigl(|x_1(t)| + |x_1(t-T)|\bigr) \\
&= \frac{1}{2}\max_{t\in[\xi,\xi+T]} \left(\left|x_1(t_0) + \int_{t_0}^{T} x'(s)\,ds\right| + \left|x_1(t_0) - \int_{t-T}^{t_0} x'(s)\,ds\right|\right) \\
&\le D + \frac{1}{2}\left(\int_{\xi}^{t} |x_1'(s)|\,ds + \int_{t-T}^{\xi} |x_1'(s)|\,ds\right) \\
&\le D + \frac{1}{2}\int_0^T |x_1'(s)|\,ds. \qquad (3.5)
\end{aligned}$$

Multiplying both sides of (3.2) by $(Ax_1)(t)$ and integrating over $[0,T]$, we get

$$\begin{aligned}
\int_0^T \bigl(\varphi_p(Ax_1)''(t)\bigr)''\bigl(Ax_1(t)\bigr)\,dt = {}&-\lambda^p \int_0^T f\bigl(t,x_1'(t)\bigr)(Ax_1)(t)\,dt \\
&- \lambda^p \int_0^T g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)(Ax_1)(t)\,dt \\
&+ \lambda^p \int_0^T e(t)(Ax_1)(t)\,dt. \qquad (3.6)
\end{aligned}$$

Substituting $\int_0^T (\varphi_p(Ax_1)''(t))''(Ax_1(t))\,dt = \int_0^T |(Ax_1)''(t)|^p\,dt$ into (3.6), in view of (H_2), we have

$$\begin{aligned}
\int_0^T \bigl|(Ax_1)''(t)\bigr|^p\,dt \le{}& \int_0^T \bigl|f\bigl(t,x_1'(t)\bigr)\bigr|\bigl|\bigl[x_1(t) - c(t)x_1\bigl(t-\delta(t)\bigr)\bigr]\bigr|\,dt \\
&+ \int_0^T \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\bigl|\bigl[x_1(t) - c(t)x_1\bigl(t-\delta(t)\bigr)\bigr]\bigr|\,dt \\
&+ \int_0^T |e(t)|\bigl|\bigl[x_1(t) - c(t)x_1\bigl(t-\delta(t)\bigr)\bigr]\bigr|\,dt \\
\le{}& (1+c_\infty)|x_1|_\infty\left[\int_0^T \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt + T\bigl(K + |e|_\infty\bigr)\right]. \qquad (3.7)
\end{aligned}$$

Besides, we can assert that there exists some positive constant N_1 such that

$$\int_0^T \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt \le 2TN_1 + T\bigl(K + |e|_\infty\bigr). \qquad (3.8)$$

In fact, from (H_1) and (3.4), we have

$$\begin{aligned}
\int_0^T \bigl\{g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr) - K - |e|_\infty\bigr\}\,dt &\le \int_0^T \bigl\{g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr) - \bigl|f\bigl(t,x_1'(t)\bigr)\bigr| - |e|_\infty\bigr\}\,dt \\
&\le \int_0^T \bigl\{g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr) + f\bigl(t,x_1'(t)\bigr) - e(t)\bigr\}\,dt \\
&= 0.
\end{aligned}$$

Define

$$\begin{aligned}
&E_1 = \{t \in [0,T] : x_1(t-\tau(t)) < -D\};\\
&E_2 = \{t \in [0,T] : |x_1(t-\tau(t))| \le D\} \cup \{t \in [0,T] : x_1(t-\tau(t)) > D\}.
\end{aligned}$$

With these sets we get

$$\begin{aligned}
&\int_{E_2} |g(t,x_1(t-\tau(t)))|\,dt \le T\max\Big\{M, \sup_{t\in[0,T],|x_1(t-\tau(t))|\le D} |g(t,x_1)|\Big\},\\
&\int_{E_1} \{|g(t,x_1(t-\tau(t)))| - K - |e|_\infty\}\,dt = \int_{E_1} \{g(t,x_1(t-\tau(t))) - K - |e|_\infty\}\,dt\\
&\qquad \le -\int_{E_2} \{g(t,x_1(t-\tau(t))) - K - |e|_\infty\}\,dt\\
&\qquad \le \int_{E_2} \{|g(t,x_1(t-\tau(t)))| + K + |e|_\infty\}\,dt,
\end{aligned}$$

which yields

$$\begin{aligned}
\int_{E_1} |g(t,x_1(t-\tau(t)))|\,dt &\le \int_{E_2} |g(t,x_1(t-\tau(t)))|\,dt + \int_{E_1\cup E_2} (K+|e|_\infty)\,dt\\
&= \int_{E_2} |g(t,x_1(t-\tau(t)))|\,dt + T(K+|e|_\infty).
\end{aligned}$$

That is,

$$\begin{aligned}
\int_0^T |g(t,x_1(t-\tau(t)))|\,dt &= \int_{E_1} |g(t,x_1(t-\tau(t)))|\,dt + \int_{E_2} |g(x_1(t-\tau(t)))|\,dt\\
&\le 2\int_{E_2} |g(t,x_1(t-\tau(t)))|\,dt + T(K+|e|_\infty)\\
&\le 2T\max\Big\{M, \sup_{t\in[0,T],|x_1(t-\tau(t))|<D} |g(t,x_1)|\Big\} + T(K+|e|_\infty)\\
&= 2TN_1 + T(K_1+|e|_\infty),
\end{aligned}$$

where $N_1 = \max\{M, \sup_{t\in[0,T],|x_1(t-\tau(t))|<D} |g(t,x_1)|\}$, proving (3.8).

Substituting (3.8) into (3.7) and recalling (3.5), we get

$$\begin{aligned}
\int_0^T |(Ax_1)''(t)|^p\,dt &\le 2T(1+c_\infty)|x_1|_\infty(K+|e|_\infty+N_1)\\
&\le (1+c_\infty)\left(D_1 + \frac{1}{2}\int_0^T |x_1'(t)|\,dt\right)2T(K+|e|_\infty+N_1)\\
&= \frac{(1+c_\infty)N_2}{2}\int_0^T |x_1'(t)|\,dt + (1+c_\infty)N_2D_1,
\end{aligned} \tag{3.9}$$

where $N_2 = 2T(K+|e|_\infty+N_1)$.

On the other hand, since $(Ax_1)(t) = x_1(t) - c(t)x_1(t-\delta(t))$, we have

$$\begin{aligned}
(Ax_1)'(t) &= \big(x_1(t) - c(t)x_1\big(t-\delta(t)\big)\big)' \\
&= x_1'(t) - c'(t)x_1\big(t-\delta(t)\big) - c(t)x_1'\big(t-\delta(t)\big) + c(t)x_1'\big(t-\delta(t)\big)\delta'(t), \\
(Ax_1)''(t) &= \big(x_1'(t) - c'(t)x_1\big(t-\delta(t)\big) - c(t)x_1'\big(t-\delta(t)\big) + c(t)x_1'\big(t-\delta(t)\big)\delta'(t)\big)' \\
&= x_1''(t) - \big[c''(t)x\big(t-\delta(t)\big) + c'(t)x'\big(t-\delta(t)\big)\big(1-\delta'(t)\big) + c'(t)x'\big(t-\delta(t)\big) \\
&\quad + c(t)x''\big(t-\delta(t)\big)\big(1-\delta'(t)\big) - c'(t)x'\big(t-\delta(t)\big)\delta'(t) \\
&\quad - c(t)x''\big(t-\delta(t)\big)\big(1-\delta'(t)\big)\delta'(t) - c(t)x'\big(t-\delta(t)\big)\delta''(t)\big] \\
&= x_1''(t) - c(t)x_1''\big(t-\delta(t)\big) - \big[c''(t)x\big(t-\delta(t)\big) + \big(2c'(t) - 2c'(t)\delta'(t) \\
&\quad - c(t)\delta''(t)\big)x_1'\big(t-\delta(t)\big) + \big(c(t)\big(\delta'(t)\big)^2 - 2c(t)\delta'(t)\big)x_1''\big(t-\delta(t)\big)\big]
\end{aligned}$$

and

$$\begin{aligned}
\big(Ax_1''\big)(t) &= (Ax_1)''(t) + c''(t)x\big(t-\delta(t)\big) + \big(2c'(t) - 2c'(t)\delta'(t) - c(t)\delta''(t)\big)x_1'\big(t-\delta(t)\big) \\
&\quad + \big(c(t)\big(\delta'(t)\big)^2 - 2c(t)\delta'(t)\big)x_1''\big(t-\delta(t)\big).
\end{aligned}$$

Case (I): If $|c(t)| \leq c_\infty < 1$, by applying Lemma 1, we have

$$\begin{aligned}
\int_0^T \big|x_1''(t)\big|\,dt &= \int_0^T \big|A^{-1}Ax_1''(t)\big|\,dt \\
&\leq \frac{\int_0^T |Ax_1''(t)|\,dt}{1-c_\infty} \\
&\leq \bigg(\int_0^T \big|(Ax_1)''(t)\big|\,dt + c_2T|x_1|_\infty + T(2c_1 + 2c_1\delta_1 + c_\infty\delta_2)\big|x_1'\big|_\infty \\
&\quad + \big(c_\infty\delta_1^2 + 2c_\infty\delta_1\big)\int_0^T \big|x''(t)\big|\,dt\bigg) \Big/ (1-c_\infty),
\end{aligned}$$

where $c_i = \max_{t\in[0,T]} |c^{(i)}(t)|$ and $\delta_i = \max_{t\in[0,T]} |\delta^{(i)}(t)|$, $i = 1, 2$. From (3.5), we have

$$\begin{aligned}
|x_1|_\infty &\leq D + \frac{1}{2}\int_0^T \big|x_1'(t)\big|\,dt \\
&\leq D + \frac{T}{2}\big|x_1'\big|_\infty \\
&\leq D + \frac{T}{4}\int_0^T \big|x_1''(t)\big|\,dt.
\end{aligned}\tag{3.10}$$

From $x_1(0) = x_1(T)$, there exists a point $t^* \in [0, T]$ such that $x_1'(t^*) = 0$, then we have

$$\big|x_1'\big|_\infty \leq x_1'\big(t^*\big) + \frac{1}{2}\int_0^T \big|x_1''(t)\big|\,dt = \frac{1}{2}\int_0^T \big|x_1''(t)\big|\,dt.\tag{3.11}$$

Therefore, we have

$$\begin{aligned}\int_0^T \left|x_1''(t)\right| dt &\le \frac{\int_0^T |(Ax_1)''(t)|\,dt + c_2 T(D + \frac{T}{4}\int_0^T |x_1''(t)|\,dt)}{1-c_\infty} \\ &\quad + \frac{\frac{T}{2}(2c_1 + 2c_1\delta_1 + c_\infty\delta_2)\int_0^T |x_1''(t)|\,dt + (c_\infty\delta_1^2 + 2c_\infty\delta_1)\int_0^T |x''(t)|\,dt}{1-c_\infty} \\ &= \left(\int_0^T \left|(Ax_1)''(t)\right| dt + \left(\frac{T^2}{4}c_2 + T\left(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2\right) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right. \\ &\quad \left.\times \int_0^T \left|x_1''(t)\right| dt + Tc_2D\right) \Big/ (1-c_\infty).\end{aligned}$$

Since $1 - c_\infty - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1) > 0$, we have

$$\begin{aligned}\int_0^T \left|x_1''(t)\right| dt &\le \frac{\int_0^T |(Ax_1)''(t)|\,dt + Tc_2D}{1 - c_\infty - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1)} \\ &\le \frac{T^{\frac{1}{q}}(\int_0^T |(Ax_1)''(t)|^p\,dt)^{\frac{1}{p}} + Tc_2D}{1 - c_\infty - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1)}.\end{aligned} \tag{3.12}$$

Applying the inequality $(a+b)^k \le a^k + b^k$ for $a, b > 0$, $0 < k < 1$, from (3.9) and (3.12) we obtain

$$\begin{aligned}\int_0^T \left|x_1''(t)\right| dt &\le \frac{T^{\frac{1}{q}}(\frac{(1+c_\infty)N_2^*}{2})^{\frac{1}{p}}(\int_0^T |x_1'(t)|\,dt)^{\frac{1}{p}} + ((1+c_\infty)N_2^*D_1)^{\frac{1}{p}} + Tc_2D_1}{1 - c_\infty - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1)} \\ &\le \frac{T(\frac{(1+c_\infty)N_2^*}{2})^{\frac{1}{p}}(\frac{1}{2}\int_0^T |x_1''(t)|\,dt)^{\frac{1}{p}} + ((1+c_\infty)N_2^*D_1)^{\frac{1}{p}} + Tc_2D_1}{1 - c_\infty - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1)}.\end{aligned}$$

It is easy to see that there exists a positive constant M^* (independent of λ) such that

$$\int_0^T \left|x_1''(t)\right| dt \le M^*.$$

Case (ii): If $c_0 > 1$, we have

$$\begin{aligned}\int_0^T \left|x_1''(t)\right| dt &= \int_0^T \left|A^{-1}Ax_1''(t)\right| dt \\ &\le \frac{\int_0^T |Ax_1''(t)|\,dt}{c_0 - 1} \\ &\le \left(\int_0^T \left|(Ax_1)''(t)\right| dt + \left(\frac{T^2}{4}c_2 + T\left(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2\right) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right. \\ &\quad \left.\times \int_0^T \left|x_1''(t)\right| dt + Tc_2D\right) \Big/ (c_0 - 1).\end{aligned}$$

Since $c_0 - 1 - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1) > 0$, we have

$$\int_0^T |x_1''(t)|\,dt \le \frac{T^{\frac{1}{q}}(\int_0^T |(Ax_1)''(t)|^p\,dt)^{\frac{1}{p}} + Tc_2D}{c_0 - 1 - (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1)}.$$

Similarly, we can get $\int_0^T |x_1''(t)|\,dt \le M^*$.

It follows from (3.10) that

$$|x_1|_\infty \le D + \frac{T}{4}\int_0^T |x_1''(t)|\,dt \le D + \frac{T}{4}M^* := M_{11}.$$

By (3.11)

$$|x_1'|_\infty \le \frac{1}{2}\int_0^T |x_1''(t)|\,dt \le \frac{1}{2}M^* := M_{12}.$$

On the other hand, from $x_2(0) = x_2(T)$, we know that there is a point $t_2 \in [0,T]$ such that $x_2'(t_2) = 0$; then by the second equation of (3.1) we get

$$\begin{aligned} |x_2'|_\infty &\le \frac{1}{2}\int_0^T |x_2''(t)|\,dt \\ &\le \int_0^T \left(\left|f\left(t,x_1'(t)\right)\right| + \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| + \left|e(t)\right|\right)dt \\ &\le TK + T\left(b + |e|_\infty\right) + 2TN_1 := M_{21}. \end{aligned}$$

Integrating the first equation of (3.1) over $[0,T]$, we have $\int_0^T |x_2(t)|^{q-2}x_2(t)\,dt = 0$, which implies that there is a point $t_3 \in [0,T]$ such that $x_2(t_3) = 0$, so

$$|x_2|_\infty \le \frac{1}{2}\int_0^T |x_2'(t)|\,dt \le T|x_2'|_\infty \le TM_{21} := M_{22}.$$

Let $M = \max\{M_{11}, M_{12}, M_{21}, M_{22}\} + 1$, $\Omega = \{x = (x_1,x_2)^\top : \|x\| < M\}$ and $\Omega_2 = \{x : x \in \partial\Omega \cap \operatorname{Ker} L\}$, then $\forall x \in \partial\Omega \cap \operatorname{Ker} L$

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(t,x_1'(t)) - g(t,x_1(t-\tau(t))) + e(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0$, $x_1 = M$ or $-M$. But if $x_1(t) = M$, we know

$$0 = \int_0^T \left\{g(t,M) - e(t)\right\} dt.$$

From assumption (H_2), we have $x_1(t) \le D \le M$, which yields a contradiction. Similarly, if $x_1 = -M$, we also have $QNx \ne 0$, *i.e.*, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so conditions (1) and (2) of Lemma 2 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x_1,x_2)^\top = (x_2,x_1)^\top.$$

Let $H(\mu,x) = \mu x + (1-\mu)JQNx$, $(\mu,x) \in [0,1] \times \Omega$, then $\forall(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu,x) = \begin{pmatrix} \mu x_1 - \frac{1-\mu}{T}\int_0^T [f(t,0) + g(t,x_1) - e(t)]\,dt \\ \mu x_2 + (1-\mu)\varphi_q(x_2) \end{pmatrix}$$
$$\forall(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L).$$

From $f(t,0) = 0$ and (H_2), it is obvious that $x^\top H(\mu,x) > 0$, $\forall(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(0,x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(1,x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

So condition (3) of Lemma 2 is satisfied. By applying Lemma 2, we conclude that equation $Lx = Nx$ has a solution $x = (x_1,x_2)^\top$ on $\bar{\Omega} \cap D(L)$, *i.e.*, (2.1) has a T-periodic solution $x_1(t)$.

Finally, observe that $y_1^*(t)$ is not constant. If $y_1^* \equiv a$ (constant), then from (1.1) we have $g(t,a,a,0,0) - e(t) \equiv 0$, which contradicts the assumption that $g(t,a,a,0,0) - e(t) \not\equiv 0$. The proof is complete. □

If $c(t) \equiv c$ and $|c| \neq 1$, $\delta(t) \equiv \delta$, then (1.1) translates into the following form:

$$\bigl(\varphi_p\bigl(x(t) - cx(t-\delta)\bigr)''\bigr)'' + f\bigl(t,x'(t)\bigr) + g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr) = e(t). \tag{3.13}$$

Similarly, we can get the following result.

Theorem 2 *Assume that conditions* (H_1)-(H_3) *hold. Then* (3.13) *has at least non-constant* T*-periodic solution.*

We illustrate our results with some examples.

Example 1 Consider the following fourth-order p-Laplacian generalized neutral functional differential:

$$\begin{aligned} &\left(\varphi_p\left(x(t) - \frac{1}{32}\sin(4t)x\left(t - \frac{1}{32}\cos(4t)\right)\right)''\right)'' - \frac{\cos^2(2t)}{64}\sin x'(t) \\ &\quad - \arctan\left(\frac{x(t-\sin(4t))}{1+\cos^2(2t)}\right) = \frac{1}{3}e^{\cos 4t}, \end{aligned} \tag{3.14}$$

where p is a constant.

It is clear that $T = \frac{\pi}{2}, c(t) = \frac{1}{32}\sin 4t,\ \delta(t) = \frac{1}{32}\cos 4t,\ \tau(t) = \sin 4t,\ e(t) = \frac{1}{3}e^{\cos 4t},\ c_1 = \max_{t\in[0,T]} |\frac{1}{16}\cos 4t| = \frac{1}{16},\ c_2 = \max_{t\in[0,T]} |-\frac{1}{2}\sin 4t| = \frac{1}{2},\ \delta_1 = \max_{t\in[0,T]} |-\frac{1}{8}\sin 4t| = \frac{1}{8},\ \delta_2 = \max_{t\in[0,T]} |-\frac{1}{2}\cos 4t| = \frac{1}{2}. f(t,u) = -\frac{1}{64}\cos^2(2t)\sin u, g(t,x) = -\arctan(\frac{x}{1+\cos^2(2t)})$ and $g(t,a) - e(t) = -\arctan(\frac{a}{1+\cos^2(2t)}) - \frac{1}{3}e^{\cos(4t)} \not\equiv 0$. Choose $K = \frac{1}{64}$, $b = 0$, $D > \frac{\pi}{2}$ and $M = \frac{\pi}{2}$; it is obvious that (H_1)-(H_3) hold. Next, we consider

$$\begin{aligned}
&1 - c_\infty - \left(\frac{T^2}{4}c_2 + T\left(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2\right) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\\
&\quad = 1 - \frac{1}{32} - \left(\frac{1}{4}\times\left(\frac{\pi}{2}\right)^2 \times \frac{1}{2} + \frac{\pi}{2}\left(\frac{1}{8} + \frac{1}{8}\times\frac{1}{8} + \frac{1}{2}\times\frac{1}{32}\times\frac{1}{2}\right)\right.\\
&\qquad \left. + \frac{1}{32}\times\frac{1}{64} + \frac{1}{16}\times\frac{1}{8}\right)\\
&\quad > 0.
\end{aligned}$$

Therefore, by Theorem 1, (3.14) has at least one non-constant $\frac{\pi}{2}$-periodic solution.

Example 2 Consider a kind of fourth-order p-Laplacian neutral functional differential as follows:

$$\left(\varphi_p\left(x(t) - 5x(t-\delta)\right)''\right)'' + \sin t\cos x'(t) + \arctan\left(\frac{x(t-\cos t)}{1+\sin^3(t)}\right) = \frac{1}{5}e^{\sin t}. \tag{3.15}$$

Here p is some positive integer and δ is a constant. It is clear that $T = 2\pi$, $c = 5$, $\tau(t) = \cos t$, $e(t) = \frac{1}{3}e^{\sin t}$, $f(t,u) = \sin t\cos u$, $g(t,x) = \arctan(\frac{x}{1+\sin^3(t)})$ and $g(t,a) - e(t) = \arctan(\frac{a}{1+\sin^3(t)}) - \frac{1}{3}e^{\cos 4t} \not\equiv 0$. Choose $K = 1$, $D > \frac{\pi}{2}$ and $M = \frac{\pi}{2}$; it is obvious that (H_1)-(H_3) hold. So (3.15) has at least one non-constant 2π-periodic solution by application of Theorem 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YX, XFH and ZBC worked together on the derivation of mathematical results. Both authors read and approved the final manuscript.

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