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REVIEW

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Chebyshev spectral method for studying the viscoelastic slip flow due to a permeable stretching surface embedded in a porous medium with viscous dissipation and non-uniform heat generation

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Abstract
Herein, we study the numerical solution with the help of Chebyshev spectral collocation method for the ordinary differential equations which describe the flow of viscoelastic fluid over a stretching sheet embedded in a porous medium with viscous dissipation and slip velocity. The novel effects for the parameters which affect the flow and heat transfer, such as the Eckert number coupled with a porous medium and the velocity slip parameter, are studied. Also, the convergence analysis for the proposed method is addressed.

Keywords: viscoelastic fluid; Chebyshev spectral collocation method; porous medium; slip velocity

1 Introduction

Owing to the importance of the fluid flow over a stretching surface because of its practical applications such as hot rolling, fiber plating, and lubrication processes, Crane [1] was the first researcher to investigate an analytical solution to the problem of Newtonian boundary layer equations for the flow due to a stretching surface. In the same context, similar problems of Newtonian flow at different situations past a stretching surface have been extended by many authors [2–5].

The viscoelastic fluid belongs to a very important class of non-Newtonian fluids which is often found in many fields of engineering fluid mechanics because of its immense applications, such as inks, paints, and jet fuels. From this standpoint and because of this great importance to this type of fluid, many researchers turned to the study of this type under different conditions. From these researchers, for example, but not limited [6–14].

Motivated by the above mentioned studies, the effects of velocity slip is absence. However, this phenomenon is very interesting in fluid mechanics. The earliest slip boundary condition was proposed by Navier [15]. After the pioneering work of Navier [15], many papers dealing with this aspect are presented [16–18]. In the same context, velocity slip effect on the viscoelastic fluid flow and heat transfer was introduced by many authors [19–21].

The prediction of heat transfer characteristics for the non-Newtonian viscoelastic fluids in porous media is very important due to its practical engineering applications, such as oil recovery, flow through filtering media, and food processing. So, in this work a new visualization for the effects of viscous dissipation, non-uniform heat generation/absorption and velocity slip on the flow and heat transfer of a viscoelastic fluid over a stretching sheet embedded in porous medium is presented.

The Chebyshev collocation methods are used to solve many problems of ODEs and PDEs [22, 23]. We can solve ODEs or PDEs to high accuracy on a simple domain using these methods. They can often achieve 10 digits of accuracy where the finite difference method or finite element method would get two or three digits of accuracy [24]. At lower accuracies, they demand less computational time and computer memory than the alternatives. For a recent efficient use of spectral methods, in physical and engineering problems, see [25, 26]. So, in this work, we use the properties of the Chebyshev polynomials to derive an approximate formula of the integer derivative $D^{(n)}y(x)$ and estimate an error upper bound of this formula, then we use this formula to solve numerically the proposed problem.

2 Formulation of the problem

In this section, we will consider a two-dimensional boundary layer flow of an incompressible viscoelastic fluid over a stretching sheet embedded in a porous medium. The origin is located at a slit, through which the sheet (see Figure 1) is drawn through the fluid medium. The x-axis is chosen along the sheet and the y-axis is taken normal to it.

The sheet is assumed to have the velocity $U = cx$ where x is the coordinate measured along the stretching surface and $c(> 0)$ is a constant for a stretching sheet. Likewise, the temperature distribution for the sheet is assumed to be in the form $T_w = T_\infty + Ax^r$ where T_w is the temperature of the sheet, T_∞ is the temperature of the ambient, A and r are constants. Also, the sheet is assumed to be porous with the suction velocity $v_w > 0$.

Making the usual boundary layer approximations the boundary layer equations read

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{\mu_e}{\rho k}u - \frac{K_0}{\rho}\left(u\frac{\partial^3 u}{\partial x\,\partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho c_p}\left(\frac{\mu_e}{k}u^2 + \mu\left(\frac{\partial u}{\partial y}\right)^2\right) + \frac{q'''}{\rho c_p}, \tag{3}$$

where u and v are the velocity components in the x and y directions, respectively. ρ is the density of the fluid, κ is the fluid thermal conductivity and K_0 is a positive parameter associated with the viscoelastic fluid. T is the temperature of the fluid, μ is the fluid viscosity, μ_e is the dynamic viscosity of the fluid due to the flow in the porous medium, k is the permeability of the porous medium, q''' is the rate of internal heat generation, and c_p is the specific heat at constant pressure. We must observe that in the second term of the right hand side of equation (3), we follow [27–30].

The boundary conditions with the slip condition [19–21] can be written as

$$u = U + a\left(\frac{\partial u}{\partial y} - \frac{K_0}{\mu}\left(u\frac{\partial^2 u}{\partial x\,\partial y} + v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right)\right),$$
$$v = -v_w, \qquad T_w = T_\infty + Ax^r \quad \text{at } y = 0, \tag{4}$$

$$u \to 0, \qquad T \to T_\infty, \quad \text{as } y \to \infty, \tag{5}$$

where a is the velocity slip factor. The mathematical analysis of the problem is simplified by introducing the following dimensionless coordinates:

$$\eta = y\sqrt{\frac{c}{\nu}}, \qquad u = cxf'(\eta), \qquad v = -\sqrt{c\nu}f(\eta), \tag{6}$$

$$\theta(\eta) = \left(\frac{T - T_\infty}{T_w - T_\infty}\right), \tag{7}$$

where η is the similarity variable, $f(\eta)$ is the dimensionless stream function, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, and $\theta(\eta)$ is the dimensionless temperature. It can be seen that a similarity solution exists only when we take $r = 2$.

Likewise, the internal heat generation or absorption q''' is modeled according to the following formula [31]:

$$q''' = \left(\frac{\kappa U}{\nu x}\right)\left[a^*(T_w - T_\infty)e^{-\eta} + b^*(T - T_\infty)\right]. \tag{8}$$

Therefore, upon using these variables, the boundary layer governing equations (2)-(3) can be written in the following non-dimensional form:

$$f''' - f'^2 + ff'' - \beta f' + K\left(f''^2 - 2f'f''' + ff''''\right) = 0, \tag{9}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + f\theta' - 2f'\theta + \mathrm{Ec}\left(\beta f'^2 + f''^2\right) + \frac{1}{\mathrm{Pr}}\left(a^*e^{-\eta} + b^*\theta\right) = 0, \tag{10}$$

the boundary conditions are

$$f = f_w, \qquad f' = 1 + \lambda\left[\left(1 - 3Kf'\right)f'' + Kf_wf'''\right], \qquad \theta = 1, \quad \text{at } \eta = 0, \tag{11}$$

$$f' \to 0, \qquad f'' \to 0, \qquad \theta \to 0, \quad \text{at } \eta \to \infty, \tag{12}$$

where $\beta = \frac{\mu_e}{\rho c k}$ is the porous parameter, $K = \frac{cK_0}{\mu}$ is the viscoelastic parameter, $\mathrm{Pr} = \frac{\mu c_p}{\kappa}$ is the Prandtl number, $\mathrm{Ec} = \frac{c^2}{Ac_p}$ is the Eckert number, $f_w = \frac{v_w}{\sqrt{c\nu}} > 0$ is the suction velocity parameter, and $\lambda = a\sqrt{\frac{c}{\nu}}$ is the velocity slip parameter.

Finally, the local skin friction coefficient C_f and the local Nusselt number analysis Nu, which are very important, can be written as

$$C_f = -2R_{\mathrm{ex}}^{\frac{-1}{2}}\left[\left(1 - 3Kf'(0)\right)f''(0) + Kf_wf'''(0)\right], \tag{13}$$

$$\mathrm{Nu} = -R_{\mathrm{ex}}^{\frac{1}{2}}\theta'(0), \tag{14}$$

where R_{ex} is the local Reynolds number.

3 Procedure solution using Chebyshev spectral collocation method

3.1 An approximate formula of the derivative for Chebyshev polynomials expansion

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by

$$T_n(z) = n\sum_{i=0}^{[\frac{n}{2}]}(-1)^i 2^{n-2i-1}\frac{(n-i-1)!}{(i)!(n-2i)!}z^{n-2i}, \tag{15}$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

In order to use these polynomials on the interval $[0,1]$ for example, we define the so-called shifted Chebyshev polynomials by introducing the change of variable $z = 2x - 1$. The shifted Chebyshev polynomials are denoted by $ST_n(x) = T_n(2x - 1) = T_{2n}(\sqrt{x})$. We can generalize this formula in any interval $[0, \eta_\infty]$.

The function $y(x)$, which belongs to the space of square integrable functions in $[0,1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \sum_{i=0}^{\infty} c_i ST_i(x), \tag{16}$$

where the coefficients c_i are given by

$$c_0 = \frac{1}{\pi}\int_0^1 \frac{y(x)ST_0(x)}{\sqrt{x-x^2}}\,dx, \qquad c_i = \frac{2}{\pi}\int_0^1 \frac{y(x)ST_i(x)}{\sqrt{x-x^2}}\,dx, \quad i = 1,2,\ldots. \tag{17}$$

In practice, only the first $(m + 1)$-terms of shifted Chebyshev polynomials are considered. Then we have

$$y_m(x) = \sum_{i=0}^{m} c_i ST_i(x). \tag{18}$$

The analytic form of the shifted Chebyshev polynomials $ST_n(x)$ of degree n is given by

$$ST_n(x) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!}x^k, \quad n = 1,2,\ldots. \tag{19}$$

The main approximate formula of the derivative of $y_m(x)$ is given in the following theorem.

Theorem 1 *Let $y(x)$ be approximated by shifted Chebyshev polynomials as* (18) *and also suppose r is an integer number, then*

$$D^{(r)}\big(y_m(x)\big) = \sum_{i=r}^{m}\sum_{k=r}^{i} c_i \lambda_{i,k,r} x^{k-r}, \tag{20}$$

where $\lambda_{i,k,r}$ is given by

$$\lambda_{i,k,r} = (-1)^{i-k}\frac{2^{2k} i(i+k-1)!k!}{(i-k)!(2k)!(k-r)!}. \tag{21}$$

Proof The proof of this theorem can be done directly with the help of equation (19) and some properties of the shifted Chebyshev polynomials. □

Also in this subsection, special attention is given to study the convergence analysis and evaluating the upper bound of the error of the proposed formula.

Theorem 2 *The derivative of order r for the shifted Chebyshev polynomials can be expressed in terms of the shifted Chebyshev polynomials themselves in the following form*:

$$D^r\big(ST_i(x)\big) = \sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}ST_j(x), \tag{22}$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i-k}2i(i+k-1)!\Gamma(k-r+\frac{1}{2})}{h_j\Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-r-j+1)\Gamma(k+j-r+1)}, \quad h_0 = 2, h_j = 1, j = 1,2,\ldots.$$

Proof We concern the properties of the shifted Chebyshev polynomials [32] and expand x^{k-r} in the proof of the previous theorem in the following form:

$$x^{k-r} = \sum_{j=0}^{k-r} c_{kj}ST_j(x), \tag{23}$$

where c_{kj} can be obtained using (17) where $y(x) = x^{k-r}$; then

$$c_{kj} = \frac{2}{h_j\pi}\int_0^1 \frac{x^{k-r}ST_j(x)}{\sqrt{x-x^2}}\,dx, \quad h_0 = 2, h_j = 1, j = 1,2,\ldots.$$

At $j = 0$ we find,

$$c_{k0} = \frac{1}{\pi}\int_0^1 \frac{x^{k-r}ST_0(x)}{\sqrt{x-x^2}}\,dx = \frac{1}{\sqrt{\pi}}\frac{\Gamma(k-r+1/2)}{\Gamma(k-r+1)},$$

also, at any j and using equation (19) we can find that

$$c_{kj} = \frac{j}{\sqrt{\pi}}\sum_{\ell=0}^{j}(-1)^{j-\ell}\frac{(j+\ell-1)!2^{2\ell+1}\Gamma(k+\ell-r+1/2)}{(j-\ell)!(2\ell)!\Gamma(k+\ell-r+1)}, \quad j = 1,2,\ldots,$$

employing (23) gives

$$D^{(r)}\big(ST_i(x)\big) = \sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}ST_j(x), \quad i = r, r+1, \ldots,$$

where

$$\Theta_{i,j,k} = \begin{cases} i\frac{(-1)^{i-k}(i+k-1)!2^{2k}k!\Gamma(k-r+\frac{1}{2})}{(i-k)!(2k)!\sqrt{\pi}(\Gamma(k+1-r))^2}, & j = 0; \\ \frac{(-1)^{i-k}ij(i+k-1)!2^{2k+1}k!}{\sqrt{\pi}\Gamma(k+1-r)(i-k)!(2k)!} \times \sum_{l=0}^{j}\frac{(-1)^{j-l}(j+l-1)!2^{2l}\Gamma(k+l-r+\frac{1}{2})}{(j-l)!(2l)!\Gamma(k+l-r+1)}, & j = 1,2,\ldots. \end{cases}$$

After some lengthly manipulation $\Theta_{i,j,k}$ can put in the following form:

$$\Theta_{i,j,k} = \frac{(-1)^{i-k} 2i(i+k-1)!\Gamma(k-r+\frac{1}{2})}{h_j \Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-r-j+1)\Gamma(k+j-r+1)}, \quad j = 0,1,\ldots, \tag{24}$$

and this completes the proof of the theorem. □

Theorem 3 *The error* $|E_T(m)| = |D^{(r)}y(x) - D^{(r)}y_m(x)|$ *in approximating* $D^{(r)}y(x)$ *by* $D^{(r)}y_m(x)$ *is bounded by*

$$\left|E_T(m)\right| \le \left| \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=r}^{i} \sum_{j=0}^{k-r} \Theta_{i,j,k} \right) \right|. \tag{25}$$

Proof A combination of equations (16), (18), and (22) leads to

$$\left|E_T(m)\right| = \left|D^{(r)}y(x) - D^{(r)}y_m(x)\right| = \left| \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=r}^{i} \sum_{j=0}^{k-r} \Theta_{i,j,k} ST_j(x) \right) \right|,$$

but $|ST_j(x)| \le 1$, so, we can obtain

$$\left|E_T(m)\right| \le \left| \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=r}^{i} \sum_{j=0}^{k-r} \Theta_{i,j,k} \right) \right|,$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem. □

3.2 Procedure solution

In this subsection, we implement the proposed method to solve numerically the system of ordinary differential equations of the form (9)-(10). The unknown functions $f(\eta)$ and $\theta(\eta)$ may be expanded by a finite series of shifted Chebyshev polynomials as in the following approximations:

$$f_m(\eta) = \sum_{i=0}^{m} c_i ST_i(\eta), \qquad \theta_m(\eta) = \sum_{i=0}^{m} d_i ST_i(\eta). \tag{26}$$

From equations (9)-(10), (26) and Theorem 1 we have

$$\begin{aligned}
&\sum_{i=3}^{m} \sum_{k=3}^{i} c_i \gamma_{i,k,3} \eta^{k-3} - \left(\sum_{i=1}^{m} \sum_{k=1}^{i} c_i \gamma_{i,k,1} \eta^{k-1} \right)^2 + \left(\sum_{i=0}^{m} c_i ST_i(\eta) \right) \left(\sum_{i=2}^{m} \sum_{k=2}^{i} c_i \gamma_{i,k,2} \eta^{k-2} \right) \\
&\quad - \beta \left(\sum_{i=1}^{m} \sum_{k=1}^{i} c_i \gamma_{i,k,1} \eta^{k-1} \right) + K \left(\sum_{i=2}^{m} \sum_{k=2}^{i} c_i \gamma_{i,k,2} \eta^{k-2} \right)^2 \\
&\quad + K \left[-2 \left(\sum_{i=1}^{m} \sum_{k=1}^{i} c_i \gamma_{i,k,1} \eta^{k-1} \right) \left(\sum_{i=3}^{m} \sum_{k=3}^{i} c_i \gamma_{i,k,3} \eta^{k-3} \right) \right. \\
&\quad \left. + \left(\sum_{i=0}^{m} c_i ST_i(\eta) \right) \left(\sum_{i=4}^{m} \sum_{k=4}^{i} c_i \gamma_{i,k,4} \eta^{k-4} \right) \right] = 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\frac{1}{\Pr}\sum_{i=2}^{m}\sum_{k=2}^{i} d_i\gamma_{i,k,2}\eta^{k-2} + \left(\sum_{i=0}^{m} c_i ST_i(\eta)\right)\left(\sum_{i=1}^{m}\sum_{k=1}^{i} d_i\gamma_{i,k,1}\eta^{k-1}\right)\\
&\quad - 2\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta^{k-1}\right)\left(\sum_{i=0}^{m} d_i ST_i(\eta)\right)\\
&\quad + \mathrm{Ec}\left(\beta\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta^{k-1}\right)^2\right.\\
&\quad \left. + \left(\sum_{i=2}^{m}\sum_{k=2}^{i} c_i\gamma_{i,k,2}\eta^{k-2}\right)^2\right) + \frac{1}{\Pr}\left(a^* e^{-\eta} + b^*\sum_{i=0}^{m} d_i ST_i(\eta)\right) = 0. \qquad (28)
\end{aligned}$$

We now collocate equations (27)-(28) at $(m-n+1)$ points η_s, $s = 0, 1, \ldots, m-n$ as

$$\begin{aligned}
&\sum_{i=3}^{m}\sum_{k=3}^{i} c_i\gamma_{i,k,3}\eta_s^{k-3} - \left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta_s^{k-1}\right)^2 + \left(\sum_{i=0}^{m} c_i ST_i(\eta_s)\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i} c_i\gamma_{i,k,2}\eta_s^{k-2}\right)\\
&\quad - \beta\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta_s^{k-1}\right) + K\left(\sum_{i=2}^{m}\sum_{k=2}^{i} c_i\gamma_{i,k,2}\eta_s^{k-2}\right)^2\\
&\quad + K\left[-2\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta_s^{k-1}\right)\left(\sum_{i=3}^{m}\sum_{k=3}^{i} c_i\gamma_{i,k,3}\eta_s^{k-3}\right)\right.\\
&\quad \left. + \left(\sum_{i=0}^{m} c_i ST_i(\eta_s)\right)\left(\sum_{i=4}^{m}\sum_{k=4}^{i} c_i\gamma_{i,k,4}\eta_s^{k-4}\right)\right] = 0, \qquad (29)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Pr}\sum_{i=2}^{m}\sum_{k=2}^{i} d_i\gamma_{i,k,2}\eta_s^{k-2} + \left(\sum_{i=0}^{m} c_i ST_i(\eta_s)\right)\left(\sum_{i=1}^{m}\sum_{k=1}^{i} d_i\gamma_{i,k,1}\eta_s^{k-1}\right)\\
&\quad - 2\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta_s^{k-1}\right)\left(\sum_{i=0}^{m} d_i ST_i(\eta_s)\right)\\
&\quad + \mathrm{Ec}\left(\beta\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k,1}\eta_s^{k-1}\right)^2 + \left(\sum_{i=2}^{m}\sum_{k=2}^{i} c_i\gamma_{i,k,2}\eta_s^{k-2}\right)^2\right)\\
&\quad + \frac{1}{\Pr}\left(a^* e^{-\eta_s} + b^*\sum_{i=0}^{m} d_i ST_i(\eta_s)\right) = 0. \qquad (30)
\end{aligned}$$

For suitable collocation points, we use the roots of the shifted Chebyshev polynomial $ST_{m-n+1}(\eta)$. Also, by substituting formula (26) in the boundary conditions (11)-(12) we can obtain five equations as follows:

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i c_i = f_w, \qquad \sum_{i=0}^{m}\ell 1_i c_i - \lambda\left(1 - 3K\sum_{i=0}^{m}\ell 1_i c_i\right)\left(\sum_{i=0}^{m}\ell 2_i c_i\right) = 1,\\
&\sum_{i=0}^{m}(-1)^i d_i = 1, \qquad \sum_{i=0}^{m}\ell 3_i c_i = 0, \qquad \sum_{i=0}^{m}\ell 4_i c_i = 0, \qquad \sum_{i=0}^{m} d_i = 0,
\end{aligned} \qquad (31)$$

where $\ell 1_i = ST_i'(0), \ell 2_i = ST_i''(0), \ell 3_i = ST_i'(\eta_\infty), \ell 4_i = ST_i''(\eta_\infty)$.

Table 1 Comparison of $-(1-K)f''(0)$ with $f_w = \beta = \lambda = 0$ using the previous work and the Chebyshev spectral collocation method

K	Rajagopal *et al.* [33]	Present work
0.005	0.9975	0.99739823
0.010	0.9949	0.99479907
0.030	0.9846	0.98462934
0.050	0.9738	0.97216181

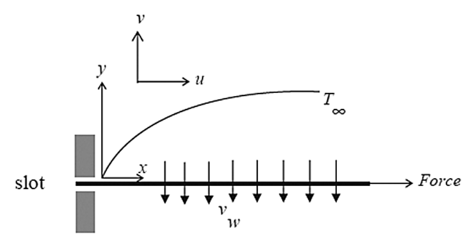

Figure 1 The physical configurations diagram.

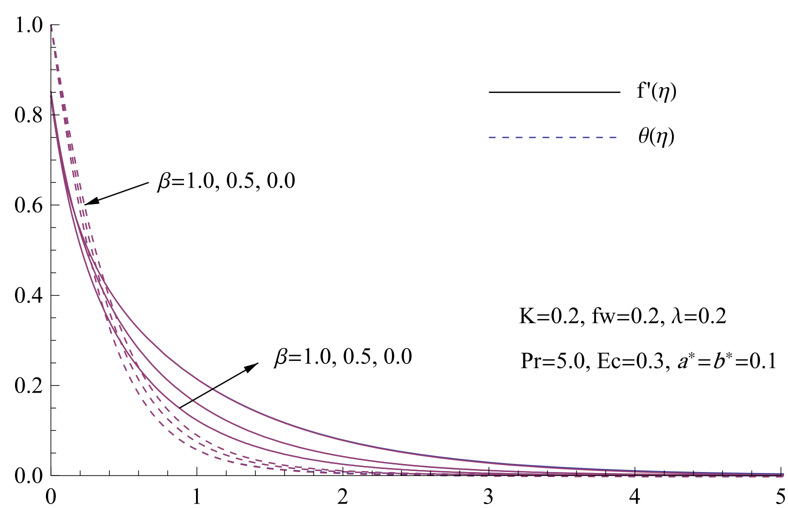

Figure 2 The behavior of the velocity and temperature distribution for various values of β.

Equations (29)-(30), together with the *six* equations of the boundary conditions (31), give a system of $(2m + 2)$ algebraic equations which can be solved, for the unknowns c_i, $d_i, i = 0, 1, \ldots, m$, using Newton iteration method.

4 Results and discussion

For the non-Newtonian viscoelastic fluid the value of the skin friction coefficient is equal to $-(1-K)f''(0)$. To validate our numerical solution, we compare that value with the earlier work of Rajagopal et al. [33]. Table 1 presents these values with those using the Chebyshev spectral collocation method. This comparison ensures that our results are in excellent agreement with this reference.

Also, in this section, computational results are shown in Figures 2-6. The velocity decreases with the increasing values of porous parameter β as shown in Figure 2. Physically, a greater of β means a high dynamic viscosity μ_e, which corresponds to porous medium and a small permeability for the porous medium, which causes the production of a resis-

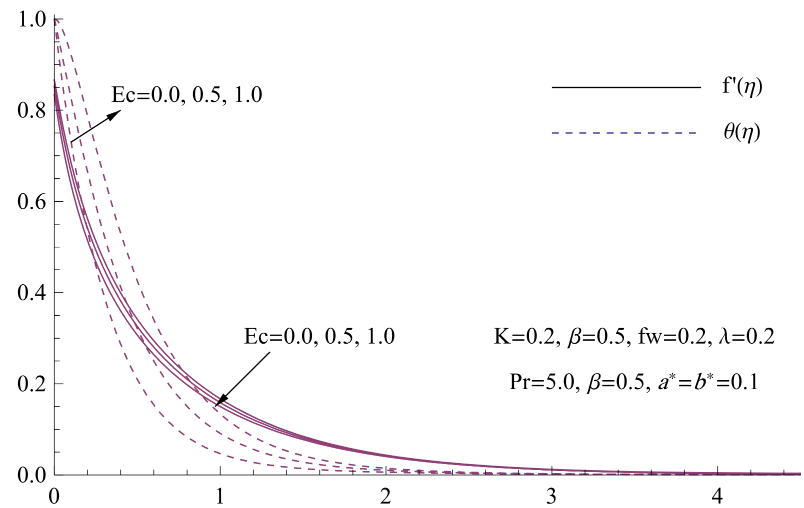

Figure 3 **The behavior of the velocity and temperature distribution for various values of Ec.**

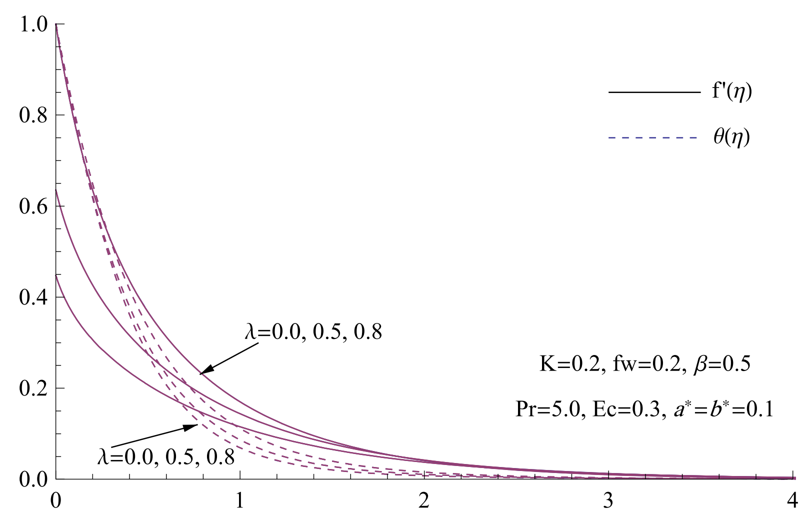

Figure 4 **The behavior of the velocity and temperature distribution for various values of** $\boldsymbol{\lambda}$**.**

tance force to the fluid flow which causes a decrease for the velocity distribution when β increases and so enhances the temperature distribution along the thermal boundary layer as we can see from Figure 2. Also, from this figure we can observe that, owing to the increase in the value of the porous parameter, and as a result, the thermal boundary layer becomes thicker but the momentum boundary layer becomes thinner.

Figure 3 depicts dimensionless velocity and temperature profiles for various values of the Eckert number. From this figure, we note that, when the Eckert number is larger the velocity curve is lower, so that the momentum effect is lower for a larger Eckert number. Also, from the same figure we can note that after increasing the value of the Eckert number, the thermal boundary layer becomes thicker but the temperature distribution enhances.

Figure 4 plots the dimensionless velocity and the dimensionless wall temperature versus similarity variable η for selected values of slip velocity parameter. It is worth noting that the

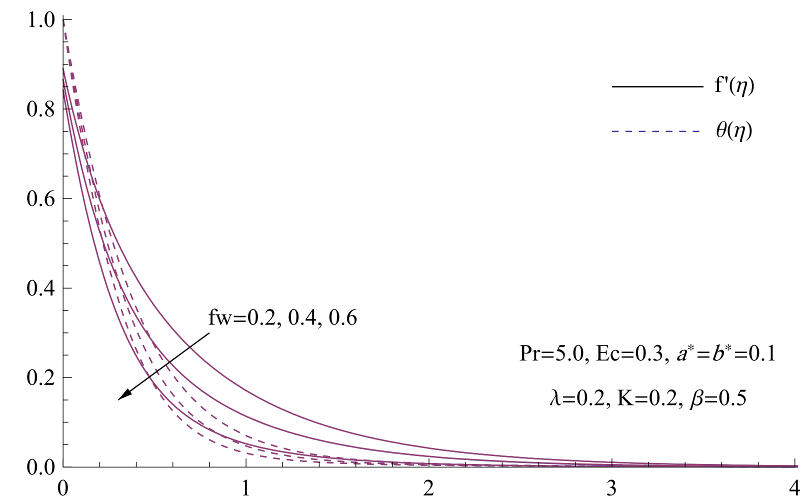

Figure 5 The behavior of the velocity and temperature distribution for various values of f_w.

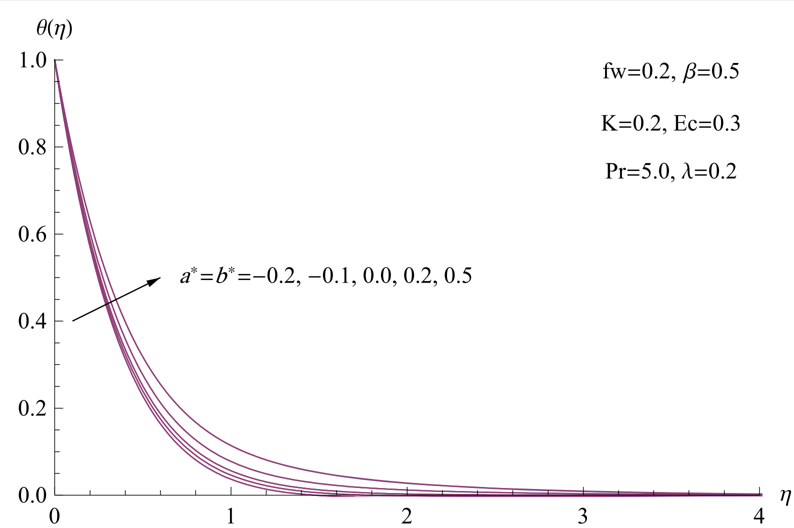

Figure 6 The behavior of the temperature distribution for various values of a^*, b^*.

velocity decreases with the increase of the velocity slip parameter, while the temperature is increased with the increase of the same parameter.

The effects of the suction parameter on the fluid flow and the temperature distribution have been analyzed and the results are presented in Figure 5. This figure shows that the suction parameter has a profound effect on the boundary layer thickness in which the suction reduces the thermal boundary layer thickness. So, the net effect for the suction parameter is to slow down both the flow velocity and the temperature distribution.

Figure 6 indicates that the thermal boundary-layer thicknesses increase when the internal heat generation parameters $a^* > 0$ and $b^* > 0$ become stronger, whereas the internal heat absorption parameters $a^* < 0$ and $b^* < 0$ have the opposite effect. Also, it is noticed that the highest temperature distribution for the fluid in the boundary layer was obtained with the greatest heat generation parameters $a^* > 0$ and $b^* > 0$. Likewise, it is shown that the effect of the heat absorption parameters $a^* < 0$ and $b^* < 0$ causes a drop in the temperature distribution as the heat following from the sheet is absorbed.

5 Conclusions

The Chebyshev spectral collocation method is used to solve the problem of flow and heat transfer of a viscoelastic fluid over a stretching sheet which is embedded in a porous medium with viscous dissipation, internal heat generation/absorption, and slip velocity. The convergence analysis and the upper bound of the error for the proposed method *are* presented. From the study, it was found that increasing the suction parameter, the porous parameter, and the slip velocity parameter lowers the momentum boundary layer thickness. Furthermore, an increasing Eckert number causes the thermal boundary layer and temperature distribution to increase.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
Both authors contributed equally in this article. They read and approved the final manuscript.

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References

1. Crane, LJ: Flow past a stretching plate. Z. Angew. Math. Phys. **21**, 645-647 (1970)
2. Vleggar, J: Laminar boundary layer behavior on continuous accelerating surfaces. Chem. Eng. Sci. **32**, 1517-1525 (1977)
3. Dutta, BK, Roy, P, Gupta, AS: Temperature field in flow over a stretching sheet with uniform heat flux. Int. Commun. Heat Mass Transf. **12**, 89-94 (1985)
4. Pop, I, Na, TY: A note on MHD flow over a stretching permeable surface. Mech. Res. Commun. **25**, 263-269 (1988)
5. Ali, ME: On thermal boundary layer on a power law stretched surface with suction or injection. Int. J. Heat Fluid Flow **16**, 280-290 (1995)
6. Vajravelu, K, Rollins, D: Heat transfer in a viscoelastic fluid over a stretching sheet. J. Math. Anal. Appl. **158**, 241-255 (1991)
7. Andersson, HI: MHD flow of a viscoelastic fluid past a stretching surface. Acta Mech. **95**, 227-230 (1992)
8. Sarma, MS, Nageswara Rao, B: Heat transfer in a viscoelastic fluid over a stretching sheet. J. Math. Anal. Appl. **222**, 268-275 (1998)
9. Subhas, A, Veena, P: Viscoelastic fluid flow and heat transfer in a porous medium over a stretching sheet. Int. J. Non-Linear Mech. **33**, 531-540 (1998)
10. Sanjayanand, E, Khan, SK: On heat and mass transfer in a viscoelastic boundary layer flow over an exponentially stretching sheet. Int. J. Therm. Sci. **45**, 819-828 (2006)
11. Nandeppanavar, MM, Abel, MS, Vajravelu, K: Flow and heat transfer characteristics of a viscoelastic fluid in a porous medium over an impermeable stretching sheet with viscous dissipation. Int. J. Heat Mass Transf. **53**, 4707-4713 (2010)
12. Subhas Abel, M, Siddheshwar, PG, Nandeppanavar, MM: Heat transfer in a viscoelastic boundary layer flow over a stretching sheet with viscous dissipation and non-uniform heat source. Int. J. Heat Mass Transf. **50**(5), 960-966 (2007)
13. Nandeppanavar, MM, Subhas Abel, M, Siddalingappa, MN: Heat transfer through a porous medium over a stretching sheet with effect of viscous dissipation. Chem. Eng. Commun. **200**, 1513-1529 (2013)
14. Hongjun, Y, Yang, Z: A class of compressible non-Newtonian fluids with external force and vacuum under no compatibility conditions. Bound. Value Probl. **2016**, 201 (2016)
15. Navier, CLMH: Mémoire sur les lois du mouvement des fluids. Mém. Acad. Sci. Inst. Fr. **6**, 389-416 (1823)
16. Thompson, PA, Troian, SM: A general boundary condition for liquid flow at solid surfaces. Nature **389**, 360-362 (1997)
17. Abbas, Z, Wang, Y, Hayat, T, Oberlack, M: Slip effects and heat transfer analysis in a viscous fluid over an oscillatory stretching surface. Int. J. Numer. Methods Fluids **59**, 443-458 (2009)
18. Turkyilmazoglu, M: Multiple solutions of heat and mass transfer of MHD slip flow for the viscoelastic fluid over a stretching sheet. Int. J. Therm. Sci. **50**, 2264-2276 (2011)
19. Ferras, LL, Afonso, AM, Alves, MA, Nobrega, JM, Carneiro, OS, Pinho, FT: Slip flows of Newtonian and viscoelastic fluids in a 4:1 contraction. J. Non-Newton. Fluid Mech. **214**, 28-37 (2014)

20. Singh, KD: Effect of slip condition on viscoelastic MHD oscillatory forced convection flow in a vertical channel with heat radiation. Int. J. Appl. Mech. Eng. **18**(4), 1-10 (2013)
21. Krishan, DS: MHD mixed convection visco-elastic slip flow through a porous medium in a vertical porous channel with thermal radiation. Kragujevac J. Sci. **35**, 27-40 (2013)
22. Khader, MM, Sweilam, NH: On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method. Appl. Math. Model. **37**, 9819-9828 (2013)
23. Khader, MM, Sweilam, NH, Mahdy, AMS: Numerical study for the fractional differential equations generated by optimization problem using Chebyshev collocation method and FDM. Appl. Math. Inf. Sci. **7**(5), 2013-2020 (2013)
24. Emran, T: Application of Chebyshev collocation method for solving two classes of non-classical parabolic PDEs. Ain Shams Eng. J. **6**(1), 373-379 (2015)
25. Rakib, FE, Hamzaga, DO, Zaki, FAE-R: Spectral analysis of wave propagation on branching strings. Bound. Value Probl. **2016**, 215 (2016)
26. Khader, MM: On the numerical solutions for the fractional diffusion equation. Commun. Nonlinear Sci. Numer. Simul. **16**, 2535-2542 (2011)
27. Ingham, DB, Pop, I, Cheng, P: Combined free and forced convection in a porous medium between two vertical walls with viscous dissipation. Transp. Porous Media **5**, 381-398 (1990)
28. Rees, D, Lage, JL: The effect of thermal stratification of natural convection in a vertical porous insulation layer. Int. J. Heat Mass Transf. **40**, 111-121 (1997)
29. Beckett, PM: Combined natural and forced convection between parallel vertical walls. SIAM J. Appl. Math. **39**, 372-384 (1980)
30. Beckett, PM, Friend, IE: Combined natural and forced convection between parallel walls: developing flow at higher Rayleigh numbers. Int. J. Heat Mass Transf. **27**, 611-621 (1984)
31. Chamkha, AJ, Khaled, AA: Similarity solutions for hydromagnetic simultaneous heat and mass transfer by natural convection from an inclined plate with internal heat generation or absorption. Heat Mass Transf. **37**, 117-123 (2001)
32. Snyder, MA: Chebyshev Methods in Numerical Approximation. Prentice-Hall, Englewood Cliffs (1966)
33. Rajagopal, KR, Na, TY, Gupta, AS: Flow of a viscoelastic fluid over a stretching sheet. Rheol. Acta **23**, 213-215 (1984)