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# Positive solutions for a system of Riemann-Liouville fractional differential equations with multi-point fractional boundary conditions

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**Abstract**

We study the existence and nonexistence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations subject to multi-point boundary conditions which contain fractional derivatives.

**MSC:** 34A08; 45G15

**Keywords:** Riemann-Liouville fractional differential equations; multi-point boundary conditions; positive solutions; existence; nonexistence

## 1 Introduction

We consider the system of nonlinear ordinary fractional differential equations

$$(\mathrm{S})\qquad \begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t),w(t))=0, & t\in(0,1),\\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t),w(t))=0, & t\in(0,1),\\ D_{0+}^{\gamma}w(t)+\nu h(t,u(t),v(t),w(t))=0, & t\in(0,1),\end{cases}$$

with the multi-point boundary conditions which contain fractional derivatives

$$(\mathrm{BC})\qquad \begin{cases} u^{(j)}(0)=0,\quad j=0,\ldots,n-2; & D_{0+}^{p_1}u(t)|_{t=1}=\sum_{i=1}^{N}a_iD_{0+}^{q_1}u(t)|_{t=\xi_i},\\ v^{(j)}(0)=0,\quad j=0,\ldots,m-2; & D_{0+}^{p_2}v(t)|_{t=1}=\sum_{i=1}^{M}b_iD_{0+}^{q_2}v(t)|_{t=\eta_i},\\ w^{(j)}(0)=0,\quad j=0,\ldots,l-2; & D_{0+}^{p_3}w(t)|_{t=1}=\sum_{i=1}^{L}c_iD_{0+}^{q_3}w(t)|_{t=\zeta_i},\end{cases}$$

where $\lambda,\mu,\nu>0$, $\alpha,\beta,\gamma\in\mathbb{R}$, $\alpha\in(n-1,n]$, $\beta\in(m-1,m]$, $\gamma\in(l-1,l]$, $n,m,l\in\mathbb{N}$, $n,m,l\geq 3$, $p_1,p_2,p_3,q_1,q_2,q_3\in\mathbb{R}$, $p_1\in[1,n-2]$, $p_2\in[1,m-2]$, $p_3\in[1,l-2]$, $q_1\in[0,p_1]$, $q_2\in[0,p_2]$, $q_3\in[0,p_3]$, $\xi_i,a_i\in\mathbb{R}$ for all $i=1,\ldots,N$ ($N\in\mathbb{N}$), $0<\xi_1<\cdots<\xi_N\leq 1$, $\eta_i,b_i\in\mathbb{R}$ for all $i=1,\ldots,M$ ($M\in\mathbb{N}$), $0<\eta_1<\cdots<\eta_M\leq 1$, $\zeta_i,c_i\in\mathbb{R}$ for all $i=1,\ldots,L$ ($L\in\mathbb{N}$), $0<\zeta_1<\cdots<\zeta_L\leq 1$, and $D_{0+}^{k}$ denotes the Riemann-Liouville derivative of order $k$.

Under some assumptions on $f$, $g$ and $h$, we give intervals for the parameters $\lambda$, $\mu$ and $\nu$ such that positive solutions of (S)-(BC) exist. By a positive solution of problem (S)-(BC),

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we mean a triplet of functions $(u,v,w) \in (C([0,1],\mathbb{R}_+))^3$, $(\mathbb{R}_+ = [0,\infty))$ satisfying (S) and (BC) with $u(t) > 0$ for all $t \in (0,1]$, or $v(t) > 0$ for all $t \in (0,1]$, or $w(t) > 0$ for all $t \in (0,1]$. The nonexistence of positive solutions for the above problem is also studied. Our results generalize the results from the paper [1], where the authors investigated a system with two fractional differential equations and multi-point boundary conditions. Besides, our results improve and extend the results from [2], where only a few cases are presented for the existence of positive solutions for a system of integral equations and, as an application, for a system with three fractional equations subject to some boundary conditions in points $t = 0$ and $t = 1$ (Application 4.3 from [2]).

Systems with two fractional differential equations with multi-point or Riemann-Stieltjes integral boundary conditions were also studied in [3–13], etc. Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [14–22]).

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations. Section 3 contains the main existence theorems for positive solutions with respect to a cone for our problem (S)-(BC). In Section 4, we investigate the nonexistence of positive solutions of (S)-(BC); and in Section 5, some examples are given to support our results. The main conclusions for our investigations from this paper are presented in Section 6.

## 2 Auxiliary results

We present firstly some auxiliary results from [23] that will be used to prove our main results.

We consider the fractional differential equation

$$D_{0+}^{\alpha} u(t) + x(t) = 0, \quad 0 < t < 1, \tag{1}$$

with the multi-point boundary conditions

$$u^{(j)}(0) = 0, \quad j = 0,\ldots,n-2; \qquad D_{0+}^{p_1} u(t)|_{t=1} = \sum_{i=1}^{N} a_i D_{0+}^{q_1} u(t)|_{t=\xi_i}, \tag{2}$$

where $\alpha \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $a_i, \xi_i \in \mathbb{R}$, $i = 1,\ldots,N$ ($N \in \mathbb{N}$), $0 < \xi_1 < \cdots < \xi_N \leq 1$, $p_1, q_1 \in \mathbb{R}$, $p_1 \in [1, n-2]$, $q_1 \in [0, p_1]$, and $x \in C[0,1]$. We denote $\Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_1)} \sum_{i=1}^{N} a_i \xi_i^{\alpha-q_1-1}$.

**Lemma 2.1** ([23]) *If* $\Delta_1 \neq 0$, *then the function* $u \in C[0,1]$ *given by*

$$u(t) = \int_0^1 G_1(t,s) x(s)\, ds, \quad t \in [0,1], \tag{3}$$

*is solution of problem* (1)-(2), *where*

$$G_1(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta_1} \sum_{i=1}^{N} a_i g_2(\xi_i, s), \quad \forall (t,s) \in [0,1] \times [0,1], \tag{4}$$

*and*

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-p_1-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-p_1-1}, & 0 \le t \le s \le 1, \end{cases}$$
$$g_2(t,s) = \frac{1}{\Gamma(\alpha-q_1)} \begin{cases} t^{\alpha-q_1-1}(1-s)^{\alpha-p_1-1} - (t-s)^{\alpha-q_1-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-q_1-1}(1-s)^{\alpha-p_1-1}, & 0 \le t \le s \le 1. \end{cases} \tag{5}$$

**Lemma 2.2** ([23]) *The functions $g_1$ and $g_2$ given by* (5) *have the properties*:

(a) $g_1(t,s) \le h_1(s)$ *for all* $t,s \in [0,1]$, *where*

$$h_1(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-p_1-1}\bigl(1-(1-s)^{p_1}\bigr), \quad s \in [0,1];$$

(b) $g_1(t,s) \ge t^{\alpha-1}h_1(s)$ *for all* $t,s \in [0,1]$;

(c) $g_1(t,s) \le \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *for all* $t,s \in [0,1]$;

(d) $g_2(t,s) \ge t^{\alpha-q_1-1}h_2(s)$ *for all* $t,s \in [0,1]$, *where*

$$h_2(s) = \frac{1}{\Gamma(\alpha-q_1)}(1-s)^{\alpha-p_1-1}\bigl(1-(1-s)^{p_1-q_1}\bigr), \quad s \in [0,1];$$

(e) $g_2(t,s) \le \frac{1}{\Gamma(\alpha-q_1)}t^{\alpha-q_1-1}$ *for all* $t,s \in [0,1]$;

(f) *The functions $g_1$ and $g_2$ are continuous on* $[0,1] \times [0,1]$; $g_1(t,s) \ge 0$, $g_2(t,s) \ge 0$ *for all* $t,s \in [0,1]$; $g_1(t,s) > 0$, $g_2(t,s) > 0$ *for all* $t,s \in (0,1)$.

**Lemma 2.3** ([23]) *Assume that $a_i \ge 0$ for all $i = 1,\dots,N$ and $\Delta_1 > 0$. Then the function $G_1$ given by* (4) *is a nonnegative continuous function on* $[0,1] \times [0,1]$ *and satisfies the inequalities*:

(a) $G_1(t,s) \le J_1(s)$ *for all* $t,s \in [0,1]$, *where* $J_1(s) = h_1(s) + \frac{1}{\Delta_1}\sum_{i=1}^N a_i g_2(\xi_i,s)$, $s \in [0,1]$;

(b) $G_1(t,s) \ge t^{\alpha-1}J_1(s)$ *for all* $t,s \in [0,1]$;

(c) $G_1(t,s) \le \sigma_1 t^{\alpha-1}$, *for all* $t,s \in [0,1]$, *where* $\sigma_1 = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta_1\Gamma(\alpha-q_1)}\sum_{i=1}^N a_i\xi_i^{\alpha-q_1-1}$.

**Lemma 2.4** ([23]) *Assume that $a_i \ge 0$ for all $i = 1,\dots,N$, $\Delta_1 > 0$, $x \in C[0,1]$ and $x(t) \ge 0$ for all $t \in [0,1]$. Then the solution $u$ of problem* (1)-(2) *given by* (3) *satisfies the inequality* $u(t) \ge t^{\alpha-1}u(t')$ *for all* $t,t' \in [0,1]$.

We can also formulate similar results as Lemmas 2.1-2.4 for the fractional boundary value problems

$$D_{0+}^{\beta}v(t) + y(t) = 0, \quad 0 < t < 1, \tag{6}$$

$$v^{(j)}(0) = 0, \quad j = 0,\dots,m-2; \qquad D_{0+}^{p_2}v(t)|_{t=1} = \sum_{i=1}^{M} b_i D_{0+}^{q_2}v(t)|_{t=\eta_i}, \tag{7}$$

and

$$D_{0+}^{\gamma}w(t) + z(t) = 0, \quad 0 < t < 1, \tag{8}$$

$$w^{(j)}(0)=0, \quad j=0,\ldots,l-2; \qquad D_{0+}^{p_3}w(t)|_{t=1}=\sum_{i=1}^{L}c_iD_{0+}^{q_3}w(t)|_{t=\zeta_i}, \tag{9}$$

where $\beta\in(m-1,m]$, $\gamma\in(l-1,l]$, $m,l\in\mathbb{N}$, $m,l\geq 3$, $b_i,\eta_i\in\mathbb{R}$, $i=1,\ldots,M$ ($M\in\mathbb{N}$), $0<\eta_1<\cdots<\eta_M\leq 1$, $c_i,\zeta_i\in\mathbb{R}$, $i=1,\ldots,L$ ($L\in\mathbb{N}$), $0<\zeta_1<\cdots<\zeta_L\leq 1$, $p_2,q_2,p_3,q_3\in\mathbb{R}$, $p_2\in[1,m-2]$, $q_2\in[0,p_2]$, $p_3\in[1,l-2]$, $q_3\in[0,p_3]$, and $y,z\in C[0,1]$.

We denote by $\Delta_2, g_3, g_4, G_2, h_3, h_4, J_2$ and $\sigma_2$, and $\Delta_3, g_5, g_6, G_3, h_5, h_6, J_3$ and $\sigma_3$ the corresponding constants and functions for problem (6)-(7) and problem (8)-(9), respectively, defined in a similar manner as $\Delta_1, g_1, g_2, G_1, h_1, h_2, J_1$ and $\sigma_1$, respectively. More precisely, we have

$$\Delta_2=\frac{\Gamma(\beta)}{\Gamma(\beta-p_2)}-\frac{\Gamma(\beta)}{\Gamma(\beta-q_2)}\sum_{i=1}^{M}b_i\eta_i^{\beta-q_2-1},$$

$$g_3(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}t^{\beta-1}(1-s)^{\beta-p_2-1}-(t-s)^{\beta-1}, & 0\leq s\leq t\leq 1,\\ t^{\beta-1}(1-s)^{\beta-p_2-1}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$g_4(t,s)=\frac{1}{\Gamma(\beta-q_2)}\begin{cases}t^{\beta-q_2-1}(1-s)^{\beta-p_2-1}-(t-s)^{\beta-q_2-1}, & 0\leq s\leq t\leq 1,\\ t^{\beta-q_2-1}(1-s)^{\beta-p_2-1}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$G_2(t,s)=g_3(t,s)+\frac{t^{\beta-1}}{\Delta_2}\sum_{i=1}^{M}b_ig_4(\eta_i,s), \quad \forall(t,s)\in[0,1]\times[0,1],$$

$$h_3(s)=\frac{1}{\Gamma(\beta)}(1-s)^{\beta-p_2-1}\big(1-(1-s)^{p_2}\big), \quad s\in[0,1],$$

$$h_4(s)=\frac{1}{\Gamma(\beta-q_2)}(1-s)^{\beta-p_2-1}\big(1-(1-s)^{p_2-q_2}\big), \quad s\in[0,1],$$

$$J_2(s)=h_3(s)+\frac{1}{\Delta_2}\sum_{i=1}^{M}b_ig_4(\eta_i,s), \quad s\in[0,1],$$

$$\sigma_2=\frac{1}{\Gamma(\beta)}+\frac{1}{\Delta_2\Gamma(\beta-q_2)}\sum_{i=1}^{M}b_i\eta_i^{\beta-q_2-1},$$

and

$$\Delta_3=\frac{\Gamma(\gamma)}{\Gamma(\gamma-p_3)}-\frac{\Gamma(\gamma)}{\Gamma(\gamma-q_3)}\sum_{i=1}^{L}c_i\zeta_i^{\gamma-q_3-1},$$

$$g_5(t,s)=\frac{1}{\Gamma(\gamma)}\begin{cases}t^{\gamma-1}(1-s)^{\gamma-p_3-1}-(t-s)^{\gamma-1}, & 0\leq s\leq t\leq 1,\\ t^{\gamma-1}(1-s)^{\gamma-p_3-1}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$g_6(t,s)=\frac{1}{\Gamma(\gamma-q_3)}\begin{cases}t^{\gamma-q_3-1}(1-s)^{\gamma-p_3-1}-(t-s)^{\gamma-q_3-1}, & 0\leq s\leq t\leq 1,\\ t^{\gamma-q_3-1}(1-s)^{\gamma-p_3-1}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$G_3(t,s)=g_5(t,s)+\frac{t^{\gamma-1}}{\Delta_3}\sum_{i=1}^{L}c_ig_6(\zeta_i,s), \quad \forall(t,s)\in[0,1]\times[0,1],$$

$$h_5(s)=\frac{1}{\Gamma(\gamma)}(1-s)^{\gamma-p_3-1}\big(1-(1-s)^{p_3}\big), \quad s\in[0,1],$$

$$h_6(s) = \frac{1}{\Gamma(\gamma - q_3)}(1-s)^{\gamma-p_3-1}\big(1-(1-s)^{p_3-q_3}\big), \quad s \in [0,1],$$

$$J_3(s) = h_5(s) + \frac{1}{\Delta_3}\sum_{i=1}^{L} c_i g_6(\zeta_i, s), \quad s \in [0,1],$$

$$\sigma_3 = \frac{1}{\Gamma(\gamma)} + \frac{1}{\Delta_3\Gamma(\gamma - q_3)}\sum_{i=1}^{L} c_i\zeta_i^{\gamma-q_3-1}.$$

The inequalities from Lemmas 2.3 and 2.4 for the functions $G_2$, $G_3$, $v$ and $w$ are the following $G_2(t,s) \le J_2(s)$, $G_2(t,s) \ge t^{\beta-1}J_2(s)$, $G_2(t,s) \le \sigma_2 t^{\beta-1}$, $G_3(t,s) \le J_3(s)$, $G_3(t,s) \ge t^{\gamma-1}J_3(s)$, $G_3(t,s) \le \sigma_3 t^{\gamma-1}$ for all $t,s \in [0,1]$, and $v(t) \ge t^{\beta-1}v(t')$, $w(t) \ge t^{\gamma-1}w(t')$ for all $t,t' \in [0,1]$.

In the proof of our main existence results, we shall use the following theorem (the Guo-Krasnosel'skii fixed point theorem, see [24]).

**Theorem 2.1** *Let $X$ be a Banach space, and let $C \subset X$ be a cone in $X$. Assume $\Omega_1$ and $\Omega_2$ are bounded open subsets of $X$ with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \le \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.
*Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

## 3 Existence of positive solutions

In this section, we give sufficient conditions on $\lambda$, $\mu$, $\nu$, $f$, $g$ and $h$ such that positive solutions with respect to a cone for our problem (S)-(BC) exist.

We present the assumptions that we shall use in the sequel.

(H1) $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $\gamma \in (l-1, l]$, $n,m,l \in \mathbb{N}$, $n,m,l \ge 3$, $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$, $p_1 \in [1, n-2]$, $p_2 \in [1, m-2]$, $p_3 \in [1, l-2]$, $q_1 \in [0, p_1]$, $q_2 \in [0, p_2]$, $q_3 \in [0, p_3]$, $\xi_i \in \mathbb{R}$, $a_i \ge 0$ for all $i = 1, \ldots, N$ ($N \in \mathbb{N}$), $0 < \xi_1 < \cdots < \xi_N \le 1$, $\eta_i \in \mathbb{R}$, $b_i \ge 0$ for all $i = 1, \ldots, M$ ($M \in \mathbb{N}$), $0 < \eta_1 < \cdots < \eta_M \le 1$, and $\zeta_i \in \mathbb{R}$, $c_i \ge 0$ for all $i = 1, \ldots, L$ ($L \in \mathbb{N}$), $0 < \zeta_1 < \cdots < \zeta_L \le 1$; $\lambda, \mu, \nu > 0$, $\Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_1)}\sum_{i=1}^{N} a_i\xi_i^{\alpha-q_1-1} > 0$, $\Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta-p_2)} - \frac{\Gamma(\beta)}{\Gamma(\beta-q_2)}\sum_{i=1}^{M} b_i\eta_i^{\beta-q_2-1} > 0$, $\Delta_3 = \frac{\Gamma(\gamma)}{\Gamma(\gamma-p_3)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-q_3)}\sum_{i=1}^{L} c_i\zeta_i^{\gamma-q_3-1} > 0$.
(H2) The functions $f, g, h : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.

For $\sigma \in (0,1)$, we introduce the following extreme limits:

$$f_0^s = \limsup_{u+v+w\to 0+} \max_{t\in[0,1]} \frac{f(t,u,v,w)}{u+v+w}, \qquad g_0^s = \limsup_{u+v+w\to 0+} \max_{t\in[0,1]} \frac{g(t,u,v,w)}{u+v+w},$$

$$h_0^s = \limsup_{u+v+w\to 0+} \max_{t\in[0,1]} \frac{h(t,u,v,w)}{u+v+w}, \qquad f_0^i = \liminf_{u+v+w\to 0+} \min_{t\in[\sigma,1]} \frac{f(t,u,v,w)}{u+v+w},$$

$$g_0^i = \liminf_{u+v+w\to 0+} \min_{t\in[\sigma,1]} \frac{g(t,u,v,w)}{u+v+w}, \qquad h_0^i = \liminf_{u+v+w\to 0+} \min_{t\in[\sigma,1]} \frac{h(t,u,v,w)}{u+v+w},$$

$$f_\infty^s = \limsup_{u+v+w\to \infty} \max_{t\in[0,1]} \frac{f(t,u,v,w)}{u+v+w}, \qquad g_\infty^s = \limsup_{u+v+w\to \infty} \max_{t\in[0,1]} \frac{g(t,u,v,w)}{u+v+w},$$

$$h_\infty^s = \limsup_{u+v+w\to \infty} \max_{t\in[0,1]} \frac{h(t,u,v,w)}{u+v+w}, \qquad f_\infty^i = \liminf_{u+v+w\to \infty} \min_{t\in[\sigma,1]} \frac{f(t,u,v,w)}{u+v+w},$$

$$g_\infty^i = \liminf_{u+v+w\to \infty} \min_{t\in[\sigma,1]} \frac{g(t,u,v,w)}{u+v+w}, \qquad h_\infty^i = \liminf_{u+v+w\to \infty} \min_{t\in[\sigma,1]} \frac{h(t,u,v,w)}{u+v+w}.$$

In the definition of the extreme limits above, the variables $u, v$ and $w$ are nonnegative.

By using the Green functions $G_i$, $i = 1, 2, 3$, from Section 2, we consider the following nonlinear system of integral equations:

$$\begin{cases} u(t) = \lambda \int_0^1 G_1(t,s)f(s,u(s),v(s),w(s))\,ds, & t \in [0,1], \\ v(t) = \mu \int_0^1 G_2(t,s)g(s,u(s),v(s),w(s))\,ds, & t \in [0,1], \\ w(t) = \nu \int_0^1 G_3(t,s)h(s,u(s),v(s),w(s))\,ds, & t \in [0,1]. \end{cases}$$

If $(u, v, w)$ is a solution of the above system, then by Lemma 2.1 and the corresponding lemmas for problems (6)-(7) and (8)-(9), we deduce that $(u, v, w)$ is a solution of problem (S)-(BC).

We consider the Banach space $X = C[0, 1]$ with the supremum norm $\|\cdot\|$ and the Banach space $Y = X \times X \times X$ with the norm $\|(u, v, w)\|_Y = \|u\| + \|v\| + \|w\|$. We define the cones

$$P_1 = \{u \in X, u(t) \ge t^{\alpha-1}\|u\|, \forall t \in [0,1]\} \subset X,$$
$$P_2 = \{v \in X, v(t) \ge t^{\beta-1}\|v\|, \forall t \in [0,1]\} \subset X,$$
$$P_3 = \{w \in X, w(t) \ge t^{\gamma-1}\|w\|, \forall t \in [0,1]\} \subset X,$$

and $P = P_1 \times P_2 \times P_3 \subset Y$.

For $\lambda, \mu, \nu > 0$, we define now the operator $Q : P \to Y$ by $Q(u, v, w) = (Q_1(u, v, w), Q_2(u, v, w), Q_3(u, v, w))$ with

$$Q_1(u,v,w)(t) = \lambda \int_0^1 G_1(t,s)f\bigl(s,u(s),v(s),w(s)\bigr)\,ds, \quad t \in [0,1], (u,v,w) \in P,$$
$$Q_2(u,v,w)(t) = \mu \int_0^1 G_2(t,s)g\bigl(s,u(s),v(s),w(s)\bigr)\,ds, \quad t \in [0,1], (u,v,w) \in P,$$
$$Q_3(u,v,w)(t) = \nu \int_0^1 G_3(t,s)h\bigl(s,u(s),v(s),w(s)\bigr)\,ds, \quad t \in [0,1], (u,v,w) \in P.$$

**Lemma 3.1** *If* (H1)-(H2) *hold, then* $Q : P \to P$ *is a completely continuous operator.*

*Proof* Let $(u, v, w) \in P$ be an arbitrary element. Because $Q_1(u, v, w)$, $Q_2(u, v, w)$ and $Q_3(u, v, w)$ satisfy problem (1)-(2) for $x(t) = \lambda f(t, u(t), v(t), w(t))$, $t \in [0, 1]$, problem (6)-(7) for $y(t) = \mu g(t, u(t), v(t), w(t))$, $t \in [0, 1]$, and problem (8)-(9) for $z(t) = \nu h(t, u(t), v(t), w(t))$, $t \in [0, 1]$, respectively, then by Lemma 2.4 and the corresponding ones for problems (6)-(7) and (8)-(9), we obtain

$$Q_1(u,v,w)\bigl(t'\bigr) \ge t^{\alpha-1}Q_1(u,v,w)\bigl(t'\bigr), \qquad Q_2(u,v,w)\bigl(t'\bigr) \ge t^{\beta-1}Q_2(u,v,w)\bigl(t'\bigr),$$
$$Q_3(u,v,w)\bigl(t'\bigr) \ge t^{\gamma-1}Q_3(u,v,w)\bigl(t'\bigr), \quad \forall t, t' \in [0,1], (u,v,w) \in P,$$

and so

$$\begin{aligned} Q_1(u,v,w)(t) &\ge t^{\alpha-1} \max_{t' \in [0,1]} Q_1(u,v,w)\bigl(t'\bigr) \\ &= t^{\alpha-1}\bigl\|Q_1(u,v,w)\bigr\|, \quad \forall t \in [0,1], (u,v,w) \in P, \end{aligned}$$

$$\begin{aligned} Q_2(u,v,w)(t) &\geq t^{\beta-1} \max_{t'\in[0,1]} Q_2(u,v,w)\bigl(t'\bigr) \\ &= t^{\beta-1}\bigl\| Q_2(u,v,w)\bigr\|, \quad \forall t\in[0,1], (u,v,w)\in P, \\ Q_3(u,v,w)(t) &\geq t^{\gamma-1} \max_{t'\in[0,1]} Q_3(u,v,w)\bigl(t'\bigr) \\ &= t^{\gamma-1}\bigl\| Q_3(u,v,w)\bigr\|, \quad \forall t\in[0,1], (u,v,w)\in P. \end{aligned}$$

Therefore, $Q(u,v,w) = (Q_1(u,v,w), Q_2(u,v,w), Q_3(u,v,w)) \in P$, and then $Q(P) \subset P$. By using standard arguments, we can easily show that $Q_1$, $Q_2$ and $Q_3$ are completely continuous (continuous and compact, that is, map bounded sets into relatively compact sets), and then $Q$ is a completely continuous operator. □

If $(u,v,w) \in P$ is a fixed point of operator $Q$, then $(u,v,w)$ is a solution of problem (S)-(BC). So, we will investigate the existence of fixed points of operator $Q$.

For $\sigma \in (0,1)$, we denote $A = \int_\sigma^1 J_1(s)\,ds$, $B = \int_0^1 J_1(s)\,ds$, $C = \int_\sigma^1 J_2(s)\,ds$, $D = \int_0^1 J_2(s)\,ds$, $E = \int_\sigma^1 J_3(s)\,ds$, $F = \int_0^1 J_3(s)\,ds$, where $J_1$, $J_2$ and $J_3$ are defined in Section 2.

First, for $f_0^s, g_0^s, h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$ and numbers $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 > 0$ with $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3 = 1$, $\widetilde{\alpha}_2', \widetilde{\alpha}_3' > 0$ with $\widetilde{\alpha}_2' + \widetilde{\alpha}_3' = 1$, $\widetilde{\alpha}_1'', \widetilde{\alpha}_3'' > 0$ with $\widetilde{\alpha}_1'' + \widetilde{\alpha}_3'' = 1$, $\widetilde{\alpha}_1''', \widetilde{\alpha}_2''' > 0$ with $\widetilde{\alpha}_1''' + \widetilde{\alpha}_2''' = 1$, we define the numbers

$$\begin{aligned} &L_1 = \frac{\alpha_1}{\theta\sigma^{\alpha-1} f_\infty^i A}, \qquad L_3 = \frac{\alpha_2}{\theta\sigma^{\beta-1} g_\infty^i C}, \qquad L_5 = \frac{\alpha_3}{\theta\sigma^{\gamma-1} h_\infty^i E}, \qquad L_2 = \frac{\widetilde{\alpha}_1}{f_0^s B}, \\ &L_4 = \frac{\widetilde{\alpha}_2}{g_0^s D}, \qquad L_6 = \frac{\widetilde{\alpha}_3}{h_0^s F}, \qquad L_4' = \frac{\widetilde{\alpha}_2'}{g_0^s D}, \qquad L_6' = \frac{\widetilde{\alpha}_3'}{h_0^s F}, \qquad L_2'' = \frac{\widetilde{\alpha}_1''}{f_0^s B}, \qquad L_6'' = \frac{\widetilde{\alpha}_3''}{h_0^s F}, \\ &L_2''' = \frac{\widetilde{\alpha}_1'''}{f_0^s B}, \qquad L_4''' = \frac{\widetilde{\alpha}_2'''}{g_0^s D}, \qquad \widetilde{L}_2 = \frac{1}{f_0^s B}, \qquad \widetilde{L}_4 = \frac{1}{g_0^s D}, \qquad \widetilde{L}_6 = \frac{1}{h_0^s F}, \end{aligned}$$

where $\theta = \min\{\sigma^{\alpha-1}, \sigma^{\beta-1}, \sigma^{\gamma-1}\}$.

**Theorem 3.1** *Assume that* (H1) *and* (H2) *hold*, $\sigma \in (0,1)$, $\alpha_1, \alpha_2, \alpha_3 \geq 0$ *with* $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 > 0$ *with* $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3 = 1$, $\widetilde{\alpha}_2', \widetilde{\alpha}_3' > 0$ *with* $\widetilde{\alpha}_2' + \widetilde{\alpha}_3' = 1$, $\widetilde{\alpha}_1'', \widetilde{\alpha}_3'' > 0$ *with* $\widetilde{\alpha}_1'' + \widetilde{\alpha}_3'' = 1$, $\widetilde{\alpha}_1''', \widetilde{\alpha}_2''' > 0$ *with* $\widetilde{\alpha}_1''' + \widetilde{\alpha}_2''' = 1$.

(1) *If* $f_0^s, g_0^s, h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_1 < L_2$, $L_3 < L_4$ *and* $L_5 < L_6$, *then for each* $\lambda \in (L_1, L_2)$, $\mu \in (L_3, L_4)$, $\nu \in (L_5, L_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(2) *If* $f_0^s = 0$, $g_0^s, h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_3 < L_4'$ *and* $L_5 < L_6'$, *then for each* $\lambda \in (L_1, \infty)$, $\mu \in (L_3, L_4')$, $\nu \in (L_5, L_6')$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(3) *If* $g_0^s = 0$, $f_0^s, h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_1 < L_2''$ *and* $L_5 < L_6''$, *then for each* $\lambda \in (L_1, L_2'')$, $\mu \in (L_3, \infty)$, $\nu \in (L_5, L_6'')$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(4) *If* $h_0^s = 0$, $f_0^s, g_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_1 < L_2'''$ *and* $L_3 < L_4'''$, *then for each* $\lambda \in (L_1, L_2''')$, $\mu \in (L_3, L_4''')$, $\nu \in (L_5, \infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(5) *If* $f_0^s = g_0^s = 0$, $h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_5 < \widetilde{L}_6$, *then for each* $\lambda \in (L_1, \infty)$, $\mu \in (L_3, \infty)$, $\nu \in (L_5, \widetilde{L}_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(6) *If* $f_0^s = h_0^s = 0$, $g_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_3 < \widetilde{L}_4$, *then for each* $\lambda \in (L_1,\infty)$, $\mu \in (L_3,\widetilde{L}_4)$, $\nu \in (L_5,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(7) *If* $g_0^s = h_0^s = 0$, $f_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, $L_1 < \widetilde{L}_2$, *then for each* $\lambda \in (L_1,\widetilde{L}_2)$, $\mu \in (L_3,\infty)$, $\nu \in (L_5,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(8) *If* $f_0^s = g_0^s = h_0^s = 0$, $f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$, *then for each* $\lambda \in (L_1,\infty)$, $\mu \in (L_3,\infty)$, $\nu \in (L_5,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(9) *If* $f_0^s, g_0^s, h_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,L_2)$, $\mu \in (0,L_4)$, $\nu \in (0,L_6)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(10) *If* $f_0^s = 0$, $g_0^s, h_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,L_4')$, $\nu \in (0,L_6')$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(11) *If* $g_0^s = 0$, $f_0^s, h_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,L_2'')$, $\mu \in (0,\infty)$, $\nu \in (0,L_6'')$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(12) *If* $h_0^s = 0$, $f_0^s, g_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,L_2''')$, $\mu \in (0,L_4''')$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(13) *If* $f_0^s = g_0^s = 0$, $h_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$, $\nu \in (0,\widetilde{L}_6)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(14) *If* $f_0^s = h_0^s = 0$, $g_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\widetilde{L}_4)$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(15) *If* $g_0^s = h_0^s = 0$, $f_0^s \in (0,\infty)$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,\widetilde{L}_2)$, $\mu \in (0,\infty)$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(16) *If* $f_0^s = g_0^s = h_0^s = 0$ *and at least one of* $f_\infty^i$, $g_\infty^i$, $h_\infty^i$ *is* $\infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

*Proof* We consider the above cone $P \subset Y$ and the operators $Q_1$, $Q_2$, $Q_3$ and $Q$. We will prove some illustrative cases of this theorem.

*Case* (1). We consider $f_0^s, g_0^s, h_0^s, f_\infty^i, g_\infty^i, h_\infty^i \in (0,\infty)$. Let $\lambda \in (L_1,L_2)$, $\mu \in (L_3,L_4)$ and $\nu \in (L_5,L_6)$. We choose $\varepsilon > 0$ a positive number such that $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$, $\varepsilon < h_\infty^i$ and

$$\frac{\widetilde{\alpha}_1}{(f_0^s + \varepsilon)B} \geq \lambda, \qquad \frac{\widetilde{\alpha}_2}{(g_0^s + \varepsilon)D} \geq \mu, \qquad \frac{\widetilde{\alpha}_3}{(h_0^s + \varepsilon)F} \geq \nu,$$

$$\frac{\alpha_1}{\theta\sigma^{\alpha-1}(f_\infty^i - \varepsilon)A} \leq \lambda, \qquad \frac{\alpha_2}{\theta\sigma^{\beta-1}(g_\infty^i - \varepsilon)C} \leq \mu, \qquad \frac{\alpha_3}{\theta\sigma^{\gamma-1}(h_\infty^i - \varepsilon)E} \leq \nu.$$

By using (H2) and the definition of $f_0^s$, $g_0^s$ and $h_0^s$, we deduce that there exists $R_1 > 0$ such that $f(t,u,v,w) \leq (f_0^s + \varepsilon)(u + v + w)$, $g(t,u,v,w) \leq (g_0^s + \varepsilon)(u + v + w)$, $h(t,u,v,w) \leq$

$(h_0^s + \varepsilon)(u + v + w)$ for all $t \in [0,1]$ and $u, v, w \geq 0$ with $u + v + w \leq R_1$. We define the set $\Omega_1 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_1\}$.

Now let $(u,v,w) \in P \cap \partial\Omega_1$, that is, $\|(u,v,w)\|_Y = R_1$ or, equivalently, $\|u\| + \|v\| + \|w\| = R_1$. Then $u(t) + v(t) + w(t) \leq R_1$ for all $t \in [0,1]$, and by Lemma 2.3, we obtain

$$\begin{aligned}
Q_1(u,v,w)(t) &\leq \lambda \int_0^1 J_1(s) f\bigl(s, u(s), v(s), w(s)\bigr)\, ds \\
&\leq \lambda \int_0^1 J_1(s)\bigl(f_0^s + \varepsilon\bigr)\bigl(u(s) + v(s) + w(s)\bigr)\, ds \\
&\leq \lambda\bigl(f_0^s + \varepsilon\bigr) \int_0^1 J_1(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\, ds \\
&= \lambda\bigl(f_0^s + \varepsilon\bigr) B \bigl\|(u,v,w)\bigr\|_Y \leq \widetilde{\alpha}_1 \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1], \\
Q_2(u,v,w)(t) &\leq \mu \int_0^1 J_2(s) g\bigl(s, u(s), v(s), w(s)\bigr)\, ds \\
&\leq \mu \int_0^1 J_2(s)\bigl(g_0^s + \varepsilon\bigr)\bigl(u(s) + v(s) + w(s)\bigr)\, ds \\
&\leq \mu\bigl(g_0^s + \varepsilon\bigr) \int_0^1 J_2(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\, ds \\
&= \mu\bigl(g_0^s + \varepsilon\bigr) D \bigl\|(u,v,w)\bigr\|_Y \leq \widetilde{\alpha}_2 \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1], \\
Q_3(u,v,w)(t) &\leq \nu \int_0^1 J_3(s) h\bigl(s, u(s), v(s), w(s)\bigr)\, ds \\
&\leq \nu \int_0^1 J_3(s)\bigl(h_0^s + \varepsilon\bigr)\bigl(u(s) + v(s) + w(s)\bigr)\, ds \\
&\leq \nu\bigl(h_0^s + \varepsilon\bigr) \int_0^1 J_3(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\, ds \\
&= \nu\bigl(h_0^s + \varepsilon\bigr) F \bigl\|(u,v,w)\bigr\|_Y \leq \widetilde{\alpha}_3 \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1].
\end{aligned}$$

Therefore, $\|Q_1(u,v,w)\| \leq \widetilde{\alpha}_1 \|(u,v,w)\|_Y$, $\|Q_2(u,v,w)\| \leq \widetilde{\alpha}_2 \|(u,v,w)\|_Y$, $\|Q_3(u,v,w)\| \leq \widetilde{\alpha}_3 \|(u,v,w)\|_Y$.

Then, for $(u,v,w) \in P \cap \partial\Omega_1$, we deduce

$$\begin{aligned}
\bigl\|Q(u,v,w)\bigr\|_Y &= \bigl\|Q_1(u,v,w)\bigr\| + \bigl\|Q_2(u,v,w)\bigr\| + \bigl\|Q_3(u,v,w)\bigr\| \\
&\leq (\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3) \bigl\|(u,v,w)\bigr\|_Y = \bigl\|(u,v,w)\bigr\|_Y. 
\end{aligned} \tag{10}$$

By the definition of $f_\infty^i$, $g_\infty^i$ and $h_\infty^i$, there exists $\overline{R}_2 > 0$ such that $f(t,u,v,w) \geq (f_\infty^i - \varepsilon)(u + v + w)$, $g(t,u,v,w) \geq (g_\infty^i - \varepsilon)(u + v + w)$, $h(t,u,v,w) \geq (h_\infty^i - \varepsilon)(u + v + w)$ for all $u, v, w \geq 0$ with $u + v + w \geq \overline{R}_2$ and $t \in [\sigma, 1]$. We consider $R_2 = \max\{2R_1, \overline{R}_2/\theta\}$, and we define $\Omega_2 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_2\}$. Then, for $(u,v,w) \in P$ with $\|(u,v,w)\|_Y = R_2$, we obtain

$$\begin{aligned}
u(t) + v(t) + w(t) &\geq \sigma^{\alpha-1}\|u\| + \sigma^{\beta-1}\|v\| + \sigma^{\gamma-1}\|w\| \geq \theta\bigl(\|u\| + \|v\| + \|w\|\bigr) \\
&= \theta \bigl\|(u,v,w)\bigr\|_Y = \theta R_2 \geq \overline{R}_2, \quad \forall t \in [\sigma, 1].
\end{aligned}$$

Then, by Lemma 2.3, we conclude

$$\begin{aligned}
Q_1(u,v,w)(t) &\geq \lambda \int_0^1 t^{\alpha-1} J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \lambda \sigma^{\alpha-1} \int_\sigma^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \lambda \sigma^{\alpha-1} \int_\sigma^1 J_1(s)\bigl(f_\infty^i - \varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\geq \lambda \sigma^{\alpha-1} \theta\bigl(f_\infty^i - \varepsilon\bigr) \int_\sigma^1 J_1(s) \bigl\|(u,v,w)\bigr\|_Y\,ds \\
&= \lambda \sigma^{\alpha-1} \theta\bigl(f_\infty^i - \varepsilon\bigr) A \bigl\|(u,v,w)\bigr\|_Y \\
&\geq \alpha_1 \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [\sigma,1], \\
Q_2(u,v,w)(t) &\geq \mu \int_0^1 t^{\beta-1} J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \mu \sigma^{\beta-1} \int_\sigma^1 J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \mu \sigma^{\beta-1} \int_\sigma^1 J_2(s)\bigl(g_\infty^i - \varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\geq \mu \sigma^{\beta-1} \theta\bigl(g_\infty^i - \varepsilon\bigr) \int_\sigma^1 J_2(s) \bigl\|(u,v,w)\bigr\|_Y\,ds \\
&= \mu \sigma^{\beta-1} \theta\bigl(g_\infty^i - \varepsilon\bigr) C \bigl\|(u,v,w)\bigr\|_Y \\
&\geq \alpha_2 \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [\sigma,1], \\
Q_3(u,v,w)(t) &\geq \nu \int_0^1 t^{\gamma-1} J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \nu \sigma^{\gamma-1} \int_\sigma^1 J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \nu \sigma^{\gamma-1} \int_\sigma^1 J_3(s)\bigl(h_\infty^i - \varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\geq \nu \sigma^{\gamma-1} \theta\bigl(h_\infty^i - \varepsilon\bigr) \int_\sigma^1 J_3(s) \bigl\|(u,v,w)\bigr\|_Y\,ds \\
&= \nu \sigma^{\gamma-1} \theta\bigl(h_\infty^i - \varepsilon\bigr) F \bigl\|(u,v,w)\bigr\|_Y \\
&\geq \alpha_3 \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [\sigma,1].
\end{aligned}$$

So $\|Q_1(u,v,w)\| \geq Q_1(u,v,w)(\sigma) \geq \alpha_1\|(u,v,w)\|_Y$, $\|Q_2(u,v,w)\| \geq Q_2(u,v,w)(\sigma) \geq \alpha_2\|(u,v,w)\|_Y$, $\|Q_3(u,v,w)\| \geq Q_3(u,v,w)(\sigma) \geq \alpha_3\|(u,v,w)\|_Y$.

Hence, for $(u,v,w) \in P \cap \partial\Omega_2$, we obtain

$$\begin{aligned}
\bigl\|Q(u,v,w)\bigr\|_Y &= \bigl\|Q_1(u,v,w)\bigr\| + \bigl\|Q_2(u,v,w)\bigr\| + \bigl\|Q_3(u,v,w)\bigr\| \\
&\geq (\alpha_1+\alpha_2+\alpha_3)\bigl\|(u,v,w)\bigr\|_Y = \bigl\|(u,v,w)\bigr\|_Y. \qquad (11)
\end{aligned}$$

By using Lemma 3.1, Theorem 2.1 i) and relations (10), (11), we deduce that $Q$ has a fixed point $(u,v,w) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, $u(t) \geq t^{\alpha-1}\|u\|$, $v(t) \geq t^{\beta-1}\|v\|$, $w(t) \geq t^{\gamma-1}\|w\|$ for all

$t \in [0,1]$, and $R_1 \leq \|u\| + \|v\| + \|w\| \leq R_2$. If $\|u\| > 0$, then $u(t) > 0$ for all $t \in (0,1]$, if $\|v\| > 0$, then $v(t) > 0$ for all $t \in (0,1]$, and if $\|w\| > 0$, then $w(t) > 0$ for all $t \in (0,1]$. So, $(u,v,w)$ is a positive solution for our problem (S)-(BC).

*Case* (10). We consider $f_0^s = 0, f_\infty^i = \infty, g_0^s, h_0^s, g_\infty^i, h_\infty^i \in (0,\infty)$. Let $\lambda \in (0,\infty)$, $\mu \in (0,L_4')$ and $\nu \in (0,L_6')$. We choose $\varepsilon > 0$ a positive number such that $\varepsilon \leq \lambda\theta\sigma^{\alpha-1}A$ and

$$\varepsilon \leq \frac{1-\mu g_0^s D - \nu h_0^s F}{2\lambda B}, \qquad \varepsilon \leq \frac{\widetilde{\alpha}_2' - \mu g_0^s D}{2\mu D}, \qquad \varepsilon \leq \frac{\widetilde{\alpha}_3' - \nu h_0^s F}{2\nu F}.$$

The numerators of the above fractions are positive because $\mu < \frac{\widetilde{\alpha}_2'}{g_0^s D}$, that is, $\widetilde{\alpha}_2' > \mu g_0^s D$, $\nu < \frac{\widetilde{\alpha}_3'}{h_0^s F}$, that is, $\widetilde{\alpha}_3' > \nu h_0^s F$, and $1 - \mu g_0^s D - \nu h_0^s F = \widetilde{\alpha}_2' + \widetilde{\alpha}_3' - \mu g_0^s D - \nu h_0^s F = (\widetilde{\alpha}_2' - \mu g_0^s D) + (\widetilde{\alpha}_3' - \nu h_0^s F) > 0$.

By using (H2) and the definition of $f_0^s$, $g_0^s$, $h_0^s$, we deduce that there exists $R_1 > 0$ such that $f(t,u,v,w) \leq \varepsilon(u+v+w)$, $g(t,u,v,w) \leq (g_0^s + \varepsilon)(u+v+w)$, $h(t,u,v,w) \leq (h_0^s + \varepsilon)(u+v+w)$ for all $t \in [0,1]$, $u,v,w \geq 0$ with $u+v+w \leq R_1$. We define the set $\Omega_1 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_1\}$.

Now let $(u,v,w) \in P \cap \partial\Omega_1$, that is, $\|(u,v,w)\|_Y = R_1$. Then $u(t) + v(t) + w(t) \leq R_1$ for all $t \in [0,1]$, and by Lemma 2.3 we obtain

$$\begin{aligned}
Q_1(u,v,w)(t) &\leq \lambda \int_0^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\leq \lambda \int_0^1 J_1(s)\varepsilon\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\leq \lambda\varepsilon \int_0^1 J_1(s)\bigl(\|u\|+\|v\|+\|w\|\bigr)\,ds \\
&= \lambda\varepsilon B\bigl\|(u,v,w)\bigr\|_Y \leq \frac{1}{2}\bigl(1-\mu g_0^s D - \nu h_0^s F\bigr)\bigl\|(u,v,w)\bigr\|_Y, \\
Q_2(u,v,w)(t) &\leq \mu \int_0^1 J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\leq \mu \int_0^1 J_2(s)\bigl(g_0^s+\varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\leq \mu\bigl(g_0^s+\varepsilon\bigr) \int_0^1 J_2(s)\bigl(\|u\|+\|v\|+\|w\|\bigr)\,ds \\
&= \mu\bigl(g_0^s+\varepsilon\bigr)D\bigl\|(u,v,w)\bigr\|_Y \\
&\leq \mu\left(g_0^s + \frac{\widetilde{\alpha}_2' - \mu g_0^s D}{2\mu D}\right)D\bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{2}\bigl(\mu g_0^s D + \widetilde{\alpha}_2'\bigr)\bigl\|(u,v,w)\bigr\|_Y, \\
Q_3(u,v,w)(t) &\leq \nu \int_0^1 J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\leq \nu \int_0^1 J_3(s)\bigl(h_0^s+\varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\leq \nu\bigl(h_0^s+\varepsilon\bigr) \int_0^1 J_3(s)\bigl(\|u\|+\|v\|+\|w\|\bigr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \nu\left(h_0^s + \varepsilon\right)F\left\|(u,v,w)\right\|_Y \le \nu\left(h_0^s + \frac{\widetilde{\alpha}_3' - \nu h_0^s F}{2\nu F}\right)F\left\|(u,v,w)\right\|_Y \\
&= \frac{1}{2}\left(\nu h_0^s F + \widetilde{\alpha}_3'\right)\left\|(u,v,w)\right\|_Y, \quad \forall t \in [0,1].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left\|Q_1(u,v,w)\right\| \le \frac{1}{2}\left(1 - \mu g_0^s D - \nu h_0^s F\right)\left\|(u,v,w)\right\|_Y, \\
&\left\|Q_2(u,v,w)\right\| \le \frac{1}{2}\left(\mu g_0^s D + \widetilde{\alpha}_2'\right)\left\|(u,v,w)\right\|_Y, \\
&\left\|Q_3(u,v,w)\right\| \le \frac{1}{2}\left(\nu h_0^s F + \widetilde{\alpha}_3'\right)\left\|(u,v,w)\right\|_Y.
\end{aligned}
$$

Then, for $(u,v,w) \in P \cap \partial\Omega_1$, we conclude

$$
\begin{aligned}
\left\|Q(u,v,w)\right\|_Y &= \left\|Q_1(u,v,w)\right\| + \left\|Q_2(u,v,w)\right\| + \left\|Q_3(u,v,w)\right\| \\
&\le \frac{1}{2}\left(1 - \mu g_0^s D - \nu h_0^s F + \mu g_0^s D + \widetilde{\alpha}_2' + \nu h_0^s F + \widetilde{\alpha}_3'\right)\left\|(u,v,w)\right\|_Y \\
&= \left\|(u,v,w)\right\|_Y. \qquad (12)
\end{aligned}
$$

By the definition of $f_\infty^i$, there exists $\overline{R}_2 > 0$ such that $f(t,u,v,w) \ge \frac{1}{\varepsilon}(u+v+w)$ for all $u,v,w \ge 0$ with $u+v+w \ge \overline{R}_2$ and $t \in [\sigma,1]$. We consider $R_2 = \max\{2R_1, \overline{R}_2/\theta\}$, and we define $\Omega_2 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_2\}$. Then, for $(u,v,w) \in P$ with $\|(u,v,w)\|_Y = R_2$, we obtain $u(t)+v(t)+w(t) \ge \theta\|(u,v,w)\|_Y = \theta R_2 \ge \overline{R}_2$ for all $t \in [\sigma,1]$. Then by Lemma 2.3 we deduce

$$
\begin{aligned}
Q_1(u,v,w)(t) &\ge \lambda \int_0^1 t^{\alpha-1} J_1(s) f\left(s,u(s),v(s),w(s)\right) ds \\
&\ge \lambda\sigma^{\alpha-1} \int_\sigma^1 J_1(s) f\left(s,u(s),v(s),w(s)\right) ds \\
&\ge \lambda\sigma^{\alpha-1} \int_\sigma^1 J_1(s) \frac{1}{\varepsilon}\left(u(s)+v(s)+w(s)\right) ds \\
&\ge \lambda\sigma^{\alpha-1}\theta \frac{1}{\varepsilon} \int_\sigma^1 J_1(s) \left\|(u,v,w)\right\|_Y ds \\
&= \lambda\sigma^{\alpha-1}\theta \frac{1}{\varepsilon} A \left\|(u,v,w)\right\|_Y \ge \left\|(u,v,w)\right\|_Y, \quad \forall t \in [\sigma,1].
\end{aligned}
$$

Then $\|Q_1(u,v,w)\| \ge Q_1(u,v,w)(\sigma) \ge \|(u,v,w)\|_Y$, and

$$
\left\|Q(u,v,w)\right\|_Y \ge \left\|Q_1(u,v,w)\right\| \ge \left\|(u,v,w)\right\|_Y. \qquad (13)
$$

By using Lemma 3.1, Theorem 2.1(i) and inequalities (12), (13), we conclude that $Q$ has a fixed point $(u,v,w) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a positive solution of problem (S)-(BC).

*Case* (15). We consider $g_0^s = h_0^s = 0$, $g_\infty^i = \infty$, $f_0^s, f_\infty^i, h_\infty^i \in (0,\infty)$. Let $\lambda \in (0,\widetilde{L}_2)$, $\mu \in (0,\infty)$, $\nu \in (0,\infty)$. We choose $\varepsilon > 0$ a positive number such that $\varepsilon \le \mu\theta\sigma^{\beta-1}C$ and

$$
\varepsilon \le \frac{1 - \lambda f_0^s B}{2\lambda B}, \qquad \varepsilon \le \frac{1 - \lambda f_0^s B}{4\mu D}, \qquad \varepsilon \le \frac{1 - \lambda f_0^s B}{4\nu F}.
$$

The numerator of the above fractions is positive because $\lambda < \frac{1}{f_0^s B}$, that is, $1 - \lambda f_0^s B > 0$.

By using (H2) and the definition of $f_0^s, g_0^s, h_0^s$, we deduce that there exists $R_1 > 0$ such that $f(t,u,v,w) \leq (f_0^s + \varepsilon)(u+v+w)$, $g(t,u,v,w) \leq \varepsilon(u+v+w)$, $h(t,u,v,w) \leq \varepsilon(u+v+w)$ for all $t \in [0,1]$, $u,v,w \geq 0$ with $u+v+w \leq R_1$. We define the set $\Omega_1 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_1\}$.

Now let $(u,v,w) \in P \cap \partial\Omega_1$, that is, $\|(u,v,w)\|_Y = R_1$. Then $u(t)+v(t)+w(t) \leq R_1$ for all $t \in [0,1]$, and by Lemma 2.3, we obtain

$$\begin{aligned}
Q_1(u,v,w)(t) &\leq \lambda \int_0^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\leq \lambda \int_0^1 J_1(s)\bigl(f_0^s + \varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\leq \lambda\bigl(f_0^s + \varepsilon\bigr) \int_0^1 J_1(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\,ds \\
&= \lambda\bigl(f_0^s + \varepsilon\bigr)B\bigl\|(u,v,w)\bigr\|_Y \leq \lambda\left(f_0^s + \frac{1-\lambda f_0^s B}{2\lambda B}\right)B\bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{2}\bigl(\lambda f_0^s B + 1\bigr)\bigl\|(u,v,w)\bigr\|_Y, \\
Q_2(u,v,w)(t) &\leq \mu \int_0^1 J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\leq \mu \int_0^1 J_2(s)\varepsilon\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\leq \mu\varepsilon \int_0^1 J_2(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\,ds \\
&= \mu\varepsilon D\bigl\|(u,v,w)\bigr\|_Y \leq \mu\frac{1-\lambda f_0^s B}{4\mu D} D\bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{4}\bigl(1 - \lambda f_0^s B\bigr)\bigl\|(u,v,w)\bigr\|_Y, \\
Q_3(u,v,w)(t) &\leq \nu \int_0^1 J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\leq \nu \int_0^1 J_3(s)\varepsilon\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\leq \nu\varepsilon \int_0^1 J_3(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\,ds \\
&= \nu\varepsilon F\bigl\|(u,v,w)\bigr\|_Y \leq \nu\frac{1-\lambda f_0^s B}{4\nu F} F\bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{4}\bigl(1 - \lambda f_0^s B\bigr)\bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1].
\end{aligned}$$

Therefore

$$\begin{aligned}
\bigl\|Q_1(u,v,w)\bigr\| &\leq \frac{1}{2}\bigl(\lambda f_0^s B + 1\bigr)\bigl\|(u,v,w)\bigr\|_Y, \\
\bigl\|Q_2(u,v,w)\bigr\| &\leq \frac{1}{4}\bigl(1 - \lambda f_0^s B\bigr)\bigl\|(u,v,w)\bigr\|_Y, \\
\bigl\|Q_3(u,v,w)\bigr\| &\leq \frac{1}{4}\bigl(1 - \lambda f_0^s B\bigr)\bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Then, for $(u,v,w) \in P \cap \partial\Omega_1$, we deduce

$$\begin{aligned} \|Q(u,v,w)\|_Y &= \|Q_1(u,v,w)\| + \|Q_2(u,v,w)\| + \|Q_3(u,v,w)\| \\ &\leq \frac{1}{4}\big(2 + 2\lambda f_0^s B + 1 - \lambda f_0^s B + 1 - \lambda f_0^s B\big)\|(u,v,w)\|_Y = \|(u,v,w)\|_Y. \end{aligned} \tag{14}$$

By the definition of $g_\infty^i$, there exists $\overline{R}_2 > 0$ such that $g(t,u,v,w) \geq \frac{1}{\varepsilon}(u+v+w)$ for all $u,v,w \geq 0$ with $u+v+w \geq \overline{R}_2$ and $t \in [\sigma,1]$. We consider $R_2 = \max\{2R_1, \overline{R}_2/\theta\}$, and we define $\Omega_2 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_2\}$. Then, for $(u,v,w) \in P$ with $\|(u,v,w)\|_Y = R_2$, we obtain $u(t)+v(t)+w(t) \geq \theta\|(u,v,w)\|_Y = \theta R_2 \geq \overline{R}_2$ for all $t \in [\sigma,1]$.

Then, by Lemma 2.3, we conclude

$$\begin{aligned} Q_2(u,v,w)(t) &\geq \mu \int_0^1 t^{\beta-1} J_2(s) g\big(s,u(s),v(s),w(s)\big)\,ds \\ &\geq \mu\sigma^{\beta-1}\int_\sigma^1 J_2(s) g\big(s,u(s),v(s),w(s)\big)\,ds \\ &\geq \mu\sigma^{\beta-1}\int_\sigma^1 J_2(s)\frac{1}{\varepsilon}\big(u(s)+v(s)+w(s)\big)\,ds \\ &\geq \mu\sigma^{\beta-1}\theta\frac{1}{\varepsilon}\int_\sigma^1 J_2(s)\|(u,v,w)\|_Y\,ds \\ &= \mu\sigma^{\beta-1}\theta\frac{1}{\varepsilon}C\|(u,v,w)\|_Y \geq \|(u,v,w)\|_Y, \quad \forall t \in [\sigma,1]. \end{aligned}$$

Then $\|Q_2(u,v,w)\| \geq Q_2(u,v,w)(\sigma) \geq \|(u,v,w)\|_Y$, and

$$\|Q(u,v,w)\|_Y \geq \|Q_2(u,v,w)\| \geq \|(u,v,w)\|_Y. \tag{15}$$

By using Lemma 3.1, Theorem 2.1(i) and inequalities (14), (15), we deduce that $Q$ has a fixed point $(u,v,w) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a positive solution of problem (S)-(BC).

*Case* (16) We consider $f_0^s = g_0^s = h_0^s = 0$, $h_\infty^i = \infty$, $f_\infty^i, g_\infty^i \in (0,\infty)$. Let $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$ and $\nu \in (0,\infty)$. We choose $\varepsilon > 0$ such that

$$\varepsilon \leq \nu\theta\sigma^{\gamma-1}E, \qquad \varepsilon \leq \frac{1}{3\lambda B}, \qquad \varepsilon \leq \frac{1}{3\mu D}, \qquad \varepsilon \leq \frac{1}{3\nu F}.$$

By using (H2) and the definition of $f_0^s, g_0^s, h_0^s$, we deduce that there exists $R_1 > 0$ such that $f(t,u,v,w) \leq \varepsilon(u+v+w)$, $g(t,u,v,w) \leq \varepsilon(u+v+w)$, $h(t,u,v,w) \leq \varepsilon(u+v+w)$ for all $t \in [0,1]$, $u,v,w \geq 0$ with $u+v+w \leq R_1$. We define the set $\Omega_1 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_1\}$.

Now let $(u,v,w) \in P \cap \partial\Omega_1$, that is, $\|(u,v,w)\|_Y = R_1$. Then $u(t)+v(t)+w(t) \leq R_1$ for all $t \in [0,1]$, and by Lemma 2.3 we obtain

$$\begin{aligned} Q_1(u,v,w)(t) &\leq \lambda\int_0^1 J_1(s) f\big(s,u(s),v(s),w(s)\big)\,ds \\ &\leq \lambda\int_0^1 J_1(s)\varepsilon\big(u(s)+v(s)+w(s)\big)\,ds \\ &\leq \lambda\varepsilon\int_0^1 J_1(s)\big(\|u\|+\|v\|+\|w\|\big)\,ds \end{aligned}$$

$$\begin{aligned}
&= \lambda\varepsilon B\big\|(u,v,w)\big\|_Y \le \frac{1}{3}\big\|(u,v,w)\big\|_Y, \\
Q_2(u,v,w)(t) &\le \mu\int_0^1 J_2(s)g\big(s,u(s),v(s),w(s)\big)\,ds \\
&\le \mu\int_0^1 J_2(s)\varepsilon\big(u(s)+v(s)+w(s)\big)\,ds \\
&\le \mu\varepsilon\int_0^1 J_2(s)\big(\|u\|+\|v\|+\|w\|\big)\,ds \\
&= \mu\varepsilon D\big\|(u,v,w)\big\|_Y \le \frac{1}{3}\big\|(u,v,w)\big\|_Y, \\
Q_3(u,v,w)(t) &\le \nu\int_0^1 J_3(s)h\big(s,u(s),v(s),w(s)\big)\,ds \\
&\le \nu\int_0^1 J_3(s)\varepsilon\big(u(s)+v(s)+w(s)\big)\,ds \\
&\le \nu\varepsilon\int_0^1 J_3(s)\big(\|u\|+\|v\|+\|w\|\big)\,ds \\
&= \nu\varepsilon F\big\|(u,v,w)\big\|_Y \le \frac{1}{3}\big\|(u,v,w)\big\|_Y, \quad \forall t\in[0,1].
\end{aligned}$$

Therefore $\|Q_1(u,v,w)\| \le \frac{1}{3}\|(u,v,w)\|_Y$, $\|Q_2(u,v,w)\| \le \frac{1}{3}\|(u,v,w)\|_Y$, $\|Q_3(u,v,w)\| \le \frac{1}{3}\|(u,v,w)\|_Y$.

Then, for $(u,v,w)\in P\cap\partial\Omega_1$, we conclude

$$\big\|Q(u,v,w)\big\|_Y \le \big\|(u,v,w)\big\|_Y. \tag{16}$$

By the definition of $h_\infty^i$, there exists $\overline{R}_2>0$ such that $h(t,u,v,w)\ge\frac{1}{\varepsilon}(u+v+w)$ for all $u,v,w\ge 0$ with $u+v+w\ge\overline{R}_2$ and $t\in[\sigma,1]$. We consider $R_2=\max\{2R_1,\overline{R}_2/\theta\}$, and we define $\Omega_2=\{(u,v,w)\in Y, \|(u,v,w)\|_Y<R_2\}$. Then, for $(u,v,w)\in P$ with $\|(u,v,w)\|_Y=R_2$, we obtain $u(t)+v(t)+w(t)\ge\theta\|(u,v,w)\|_Y=\theta R_2\ge\overline{R}_2$ for all $t\in[\sigma,1]$.

Then, by Lemma 2.3, we deduce

$$\begin{aligned}
Q_3(u,v,w)(t) &\ge \nu\int_0^1 t^{\gamma-1}J_3(s)h\big(s,u(s),v(s),w(s)\big)\,ds \\
&\ge \nu\sigma^{\gamma-1}\int_\sigma^1 J_3(s)h\big(s,u(s),v(s),w(s)\big)\,ds \\
&\ge \nu\sigma^{\gamma-1}\int_\sigma^1 J_3(s)\frac{1}{\varepsilon}\big(u(s)+v(s)+w(s)\big)\,ds \\
&\ge \nu\sigma^{\gamma-1}\theta\frac{1}{\varepsilon}\int_\sigma^1 J_3(s)\big\|(u,v,w)\big\|_Y\,ds \\
&= \nu\sigma^{\gamma-1}\theta\frac{1}{\varepsilon}E\big\|(u,v,w)\big\|_Y \ge \big\|(u,v,w)\big\|_Y, \quad \forall t\in[\sigma,1].
\end{aligned}$$

Then $\|Q_3(u,v,w)\|\ge Q_3(u,v,w)(\sigma)\ge\|(u,v,w)\|_Y$, and

$$\big\|Q(u,v,w)\big\|_Y \ge \big\|Q_3(u,v,w)\big\| \ge \big\|(u,v,w)\big\|_Y. \tag{17}$$

By using Lemma 3.1, Theorem 2.1(i) and inequalities (16), (17), we conclude that $Q$ has a fixed point $(u,v,w) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a positive solution of problem (S)-(BC). $\square$

**Remark 3.1** Each of the cases (9)-(16) of Theorem 3.1 contains seven cases as follows: $\{f_\infty^i = \infty, g_\infty^i, h_\infty^i \in (0,\infty)\}$, or $\{g_\infty^i = \infty, f_\infty^i, h_\infty^i \in (0,\infty)\}$, or $\{h_\infty^i = \infty, f_\infty^i, g_\infty^i \in (0,\infty)\}$, or $\{f_\infty^i = g_\infty^i = \infty, h_\infty^i \in (0,\infty)\}$, or $\{f_\infty^i = h_\infty^i = \infty, g_\infty^i \in (0,\infty)\}$, or $\{g_\infty^i = h_\infty^i = \infty, f_\infty^i \in (0,\infty)\}$, or $\{f_\infty^i = g_\infty^i = h_\infty^i = \infty\}$. So the total number of cases from Theorem 3.1 is 64, which we grouped in 16 cases.

Each of the cases (1)-(8) contains four subcases because $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$, or $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, or $\alpha_2 = 1$ and $\alpha_1 = \alpha_3 = 0$, or $\alpha_3 = 1$ and $\alpha_1 = \alpha_2 = 0$.

**Remark 3.2** In the paper [2], the authors present only 15 cases (Theorems 2.1-2.15 from [2]) from 64 cases, namely the first nine cases of our Theorem 3.1. They did not study the cases when some extreme limits are 0 and other are $\infty$. Besides, compared to Theorems 2.2-2.7 and 2.9-2.15 from [2], our intervals for parameters $\lambda, \mu, \nu$ presented in Theorem 3.1 (our cases (2)-(7) and (9)) are better than the corresponding ones from [2]. In addition, the cone used in [2] implies the existence of nonnegative solutions which satisfy the condition $\inf_{t\in[\xi,\eta]}(u(t) + v(t) + w(t)) > 0$, which is different from our definition of positive solutions.

**Remark 3.3** One can formulate existence results for the general case of the system of $n$ fractional differential equations

$$(\widetilde{\mathrm{S}}) \quad D_{0+}^{\alpha_j} u_j(t) + \lambda_j f_j\big(t, u_1(t), \ldots, u_n(t)\big) = 0, \quad j = 1, \ldots, n,$$

with the boundary conditions

$$(\widetilde{\mathrm{BC}}) \quad \begin{cases} u_j^{(k)}(0) = 0, \quad k = 0, \ldots, m_j - 2, j = 1, \ldots, n, \\ D_{0+}^{p_j} u_j(t)|_{t=1} = \sum_{k=1}^{N_j} a_{jk} D_{0+}^{q_j} u_j(t)|_{t=\xi_{jk}}, \quad j = 1, \ldots, n, \end{cases}$$

where $\alpha_j \in (m_j - 1, m_j]$, $m_j \in \mathbb{N}$, $m_j \geq 3$; $\xi_{jk}, a_{jk} \in \mathbb{R}$ for all $k = 1, \ldots, N_j$, $(N_j \in \mathbb{N})$; $0 < \xi_{j1} < \xi_{j2} < \cdots \leq \xi_{jN_j}$, $p_j \in [1, n_j - 2]$, $q_j \in [0, p_j]$, $j = 1, \ldots, N$.

According to the values of $f_{j0}^s = \limsup_{u_1+\cdots+u_n \to 0+} \sup_{t\in[0,1]} \frac{f_j(t,u_1,\ldots,u_n)}{u_1+\cdots+u_n} \in [0,\infty)$, and $f_{j\infty}^i = \liminf_{u_1+\cdots+u_n \to \infty} \inf_{t\in[\sigma,1]} \frac{f_j(t,u_1,\ldots,u_n)}{u_1+\cdots+u_n} \in (0,\infty]$, $j = 1, \ldots, n$, we have $2^{2n}$ cases, which can be grouped in $2^{n+1}$ cases.

In what follows, for $f_0^i, g_0^i, h_0^i, f_\infty^s, g_\infty^s, h_\infty^s \in (0,\infty)$ and numbers $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 > 0$ with $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3 = 1$, $\widetilde{\alpha}_2', \widetilde{\alpha}_3' > 0$ with $\widetilde{\alpha}_2' + \widetilde{\alpha}_3' = 1$, $\widetilde{\alpha}_1'', \widetilde{\alpha}_3'' > 0$ with $\widetilde{\alpha}_1'' + \widetilde{\alpha}_3'' = 1$, $\widetilde{\alpha}_1''', \widetilde{\alpha}_2''' > 0$ with $\widetilde{\alpha}_1''' + \widetilde{\alpha}_2''' = 1$, we define the numbers

$$M_1 = \frac{\alpha_1}{\theta\sigma^{\alpha-1} f_0^i A}, \qquad M_3 = \frac{\alpha_2}{\theta\sigma^{\beta-1} g_0^i C}, \qquad M_5 = \frac{\alpha_3}{\theta\sigma^{\gamma-1} h_0^i E},$$

$$M_2 = \frac{\widetilde{\alpha}_1}{f_\infty^s B}, \qquad M_4 = \frac{\widetilde{\alpha}_2}{g_\infty^s D}, \qquad M_6 = \frac{\widetilde{\alpha}_3}{h_\infty^s F}, \qquad M_4' = \frac{\widetilde{\alpha}_2'}{g_\infty^s D},$$

$$M_6' = \frac{\widetilde{\alpha}_3'}{h_\infty^s F}, \qquad M_2'' = \frac{\widetilde{\alpha}_1''}{f_\infty^s B}, \qquad M_6'' = \frac{\widetilde{\alpha}_3''}{h_\infty^s F}, \qquad M_2''' = \frac{\widetilde{\alpha}_1'''}{f_\infty^s B},$$

$$M_4''' = \frac{\widetilde{\alpha}_2'''}{g_\infty^s D}, \qquad \widetilde{M}_2 = \frac{1}{f_\infty^s B}, \qquad \widetilde{M}_4 = \frac{1}{g_\infty^s D}, \qquad \widetilde{M}_6 = \frac{1}{h_\infty^s F},$$

where $\theta = \min\{\sigma^{\alpha-1}, \sigma^{\beta-1}, \sigma^{\gamma-1}\}$.

**Theorem 3.2** *Assume that* (H1) *and* (H2) *hold*, $\sigma \in (0,1)$, $\alpha_1, \alpha_2, \alpha_3 \geq 0$ *with* $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 > 0$ *with* $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3 = 1$, $\widetilde{\alpha}_2', \widetilde{\alpha}_3' > 0$ *with* $\widetilde{\alpha}_2' + \widetilde{\alpha}_3' = 1$, $\widetilde{\alpha}_1'', \widetilde{\alpha}_3'' > 0$ *with* $\widetilde{\alpha}_1'' + \widetilde{\alpha}_3'' = 1$, $\widetilde{\alpha}_1''', \widetilde{\alpha}_2''' > 0$ *with* $\widetilde{\alpha}_1''' + \widetilde{\alpha}_2''' = 1$.

(1) *If* $f_0^i, g_0^i, h_0^i, f_\infty^s, g_\infty^s, h_\infty^s \in (0,\infty)$, $M_1 < M_2$, $M_3 < M_4$ *and* $M_5 < M_6$, *then for each* $\lambda \in (M_1, M_2)$, $\mu \in (M_3, M_4)$, $\nu \in (M_5, M_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(2) *If* $f_\infty^s = 0$, $g_\infty^s, h_\infty^s, f_0^i, g_0^i, h_0^i \in (0,\infty)$, $M_3 < M_4'$ *and* $M_5 < M_6'$, *then for each* $\lambda \in (M_1, \infty)$, $\mu \in (M_3, M_4')$, $\nu \in (M_5, M_6')$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(3) *If* $g_\infty^s = 0$, $f_\infty^s, h_\infty^s, f_0^i, g_0^i, h_0^i \in (0,\infty)$, $M_1 < M_2''$ *and* $M_5 < M_6''$, *then for each* $\lambda \in (M_1, M_2'')$, $\mu \in (M_3, \infty)$, $\nu \in (M_5, M_6'')$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(4) *If* $h_\infty^s = 0$, $f_\infty^s, g_\infty^s, f_0^i, g_0^i, h_0^i \in (0,\infty)$, $M_1 < M_2'''$ *and* $M_3 < M_4'''$, *then for each* $\lambda \in (M_1, M_2''')$, $\mu \in (M_3, M_4''')$, $\nu \in (M_5, \infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(5) *If* $f_\infty^s = g_\infty^s = 0$, $h_\infty^s, f_0^i, g_0^i, h_0^i \in (0,\infty)$, $M_5 < \widetilde{M}_6$, *then for each* $\lambda \in (M_1, \infty)$, $\mu \in (M_3, \infty)$, $\nu \in (M_5, \widetilde{M}_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(6) *If* $f_\infty^s = h_\infty^s = 0$, $g_\infty^s, f_0^i, g_0^i, h_0^i \in (0,\infty)$, $M_3 < \widetilde{M}_4$, *then for each* $\lambda \in (M_1, \infty)$, $\mu \in (M_3, \widetilde{M}_4)$, $\nu \in (M_5, \infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(7) *If* $g_\infty^s = h_\infty^s = 0$, $f_\infty^s, f_0^i, g_0^i, h_0^i \in (0,\infty)$, $M_1 < \widetilde{M}_2$, *then for each* $\lambda \in (M_1, \widetilde{M}_2)$, $\mu \in (M_3, \infty)$, $\nu \in (M_5, \infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(8) *If* $f_\infty^s = g_\infty^s = h_\infty^s = 0$, $f_0^i, g_0^i, h_0^i \in (0,\infty)$, *then for each* $\lambda \in (M_1, \infty)$, $\mu \in (M_3, \infty)$, $\nu \in (M_5, \infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(9) *If* $f_\infty^s, g_\infty^s, h_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0, M_2)$, $\mu \in (0, M_4)$, $\nu \in (0, M_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(10) *If* $f_\infty^s = 0$, $g_\infty^s, h_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0, \infty)$, $\mu \in (0, M_4')$, $\nu \in (0, M_6')$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(11) *If* $g_\infty^s = 0$, $f_\infty^s, h_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0, M_2'')$, $\mu \in (0, \infty)$, $\nu \in (0, M_6'')$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(12) *If* $h_\infty^s = 0$, $f_\infty^s, g_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0, M_2''')$, $\mu \in (0, M_4''')$, $\nu \in (0, \infty)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(13) *If* $f_\infty^s = g_\infty^s = 0$, $h_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0, \infty)$, $\mu \in (0, \infty)$, $\nu \in (0, \widetilde{M}_6)$ *there exists a positive solution* $(u(t), v(t), w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(14) *If* $f_\infty^s = h_\infty^s = 0$, $g_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\widetilde{M}_4)$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(15) *If* $g_\infty^s = h_\infty^s = 0$, $f_\infty^s \in (0,\infty)$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0,\widetilde{M}_2)$, $\mu \in (0,\infty)$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

(16) *If* $f_\infty^s = g_\infty^s = h_\infty^s = 0$ *and at least one of* $f_0^i$, $g_0^i$, $h_0^i$ *is* $\infty$, *then for each* $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$, $\nu \in (0,\infty)$ *there exists a positive solution* $(u(t),v(t),w(t))$, $t \in [0,1]$ *for problem* (S)-(BC).

*Proof* We consider again the above cone $P \subset Y$ and the operators $Q_1$, $Q_2$, $Q_3$ and $Q$. We will also prove for this theorem some illustrative cases.

*Case* (1) We consider $f_0^i, g_0^i, h_0^i, f_\infty^s, g_\infty^s, h_\infty^s \in (0,\infty)$. Let $\lambda \in (M_1, M_2)$, $\mu \in (M_3, M_4)$, $\nu \in (M_5, M_6)$. We choose $\varepsilon > 0$ a positive number such that $\varepsilon < f_0^i$, $\varepsilon < g_0^i$, $\varepsilon < h_0^i$ and

$$\frac{\alpha_1}{\theta\sigma^{\alpha-1}(f_0^i-\varepsilon)A} \le \lambda, \qquad \frac{\alpha_2}{\theta\sigma^{\beta-1}(g_0^i-\varepsilon)C} \le \mu, \qquad \frac{\alpha_3}{\theta\sigma^{\gamma-1}(h_0^i-\varepsilon)E} \le \nu,$$
$$\frac{\widetilde{\alpha}_1}{(f_\infty^s+\varepsilon)B} \ge \lambda, \qquad \frac{\widetilde{\alpha}_2}{(g_\infty^s+\varepsilon)D} \ge \mu, \qquad \frac{\widetilde{\alpha}_3}{(h_\infty^s+\varepsilon)F} \ge \nu.$$

By using (H2) and the definition of $f_0^i$, $g_0^i$, $h_0^i$, we deduce that there exists $R_3 > 0$ such that $f(t,u,v,w) \ge (f_0^i-\varepsilon)(u+v+w)$, $g(t,u,v,w) \ge (g_0^i-\varepsilon)(u+v+w)$, $h(t,u,v,w) \ge (h_0^i-\varepsilon)(u+v+w)$ for all $u,v,w \ge 0$ with $u+v+w \le R_3$ and $t \in [\sigma,1]$. We denote $\Omega_3 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_3\}$.

Let $(u,v,w) \in P \cap \partial\Omega_3$, that is, $\|(u,v,w)\|_Y = R_3$ or, equivalently, $\|u\| + \|v\| + \|w\| = R_3$. Because $u(t)+v(t)+w(t) \le R_3$ for all $t \in [0,1]$, then by Lemma 2.3 we obtain for all $t \in [\sigma,1]$

$$\begin{aligned}
Q_1(u,v,w)(t) &\ge \lambda \int_0^1 t^{\alpha-1} J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\ge \lambda\sigma^{\alpha-1} \int_\sigma^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\ge \lambda\sigma^{\alpha-1} \int_\sigma^1 J_1(s)\bigl(f_0^i-\varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\ge \lambda\sigma^{\alpha-1}\theta\bigl(f_0^i-\varepsilon\bigr) \int_\sigma^1 J_1(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \\
&= \lambda\sigma^{\alpha-1}\theta\bigl(f_0^i-\varepsilon\bigr)A\bigl\|(u,v,w)\bigr\|_Y \ge \alpha_1\bigl\|(u,v,w)\bigr\|_Y, \\
Q_2(u,v,w)(t) &\ge \mu \int_0^1 t^{\beta-1} J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\ge \mu\sigma^{\beta-1} \int_\sigma^1 J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\ge \mu\sigma^{\beta-1} \int_\sigma^1 J_2(s)\bigl(g_0^i-\varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\ge \mu\sigma^{\beta-1}\theta\bigl(g_0^i-\varepsilon\bigr) \int_\sigma^1 J_2(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \\
&= \mu\sigma^{\beta-1}\theta\bigl(g_0^i-\varepsilon\bigr)C\bigl\|(u,v,w)\bigr\|_Y \ge \alpha_2\bigl\|(u,v,w)\bigr\|_Y,
\end{aligned}$$

$$\begin{aligned}
Q_3(u,v,w)(t) &\geq \nu \int_0^1 t^{\gamma-1} J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \nu\sigma^{\gamma-1} \int_\sigma^1 J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\geq \nu\sigma^{\gamma-1} \int_\sigma^1 J_3(s)\bigl(h_0^i-\varepsilon\bigr)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\geq \nu\sigma^{\gamma-1}\theta\bigl(h_0^i-\varepsilon\bigr) \int_\sigma^1 J_3(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \\
&= \nu\sigma^{\gamma-1}\theta\bigl(h_0^i-\varepsilon\bigr)E\bigl\|(u,v,w)\bigr\|_Y \geq \alpha_3\bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

So

$$\begin{aligned}
&\bigl\|Q_1(u,v,w)\bigr\| \geq Q_1(u,v,w)(\sigma) \geq \alpha_1\bigl\|(u,v,w)\bigr\|_Y, \\
&\bigl\|Q_2(u,v,w)\bigr\| \geq Q_2(u,v,w)(\sigma) \geq \alpha_2\bigl\|(u,v,w)\bigr\|_Y, \\
&\bigl\|Q_3(u,v,w)\bigr\| \geq Q_3(u,v,w)(\sigma) \geq \alpha_3\bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Then, for an arbitrary element $(u,v,w) \in P \cap \partial\Omega_3$, we deduce

$$\bigl\|Q(u,v,w)\bigr\|_Y \geq (\alpha_1+\alpha_2+\alpha_3)\bigl\|(u,v,w)\bigr\|_Y = \bigl\|(u,v,w)\bigr\|_Y. \tag{18}$$

Now we define the functions $f^*, g^*, h^* : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$, $f^*(t,x) = \max_{0\leq u+v+w\leq x} f(t,u,v,w)$, $g^*(t,x) = \max_{0\leq u+v+w\leq x} g(t,u,v,w)$, $h^*(t,x) = \max_{0\leq u+v+w\leq x} h(t,u,v,w)$, $t \in [0,1]$, $x \in \mathbb{R}_+$. Then $f(t,u,v,w) \leq f^*(t,x)$, $g(t,u,v,w) \leq g^*(t,x)$, $h(t,u,v,w) \leq h^*(t,x)$ for all $t \in [0,1]$, $u,v,w \geq 0$ and $u+v+w \leq x$. The functions $f^*(t,\cdot)$, $g^*(t,\cdot)$, $h^*(t,\cdot)$ are nondecreasing for every $t \in [0,1]$, and they satisfy the conditions

$$\begin{aligned}
&\limsup_{x\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x} \leq f_\infty^s, \qquad \limsup_{x\to\infty} \max_{t\in[0,1]} \frac{g^*(t,x)}{x} \leq g_\infty^s, \\
&\limsup_{x\to\infty} \max_{t\in[0,1]} \frac{h^*(t,x)}{x} \leq h_\infty^s.
\end{aligned}$$

Therefore, for $\varepsilon > 0$, there exists $\bar{R}_4 > 0$ such that, for all $x \geq \bar{R}_4$ and $t \in [0,1]$, we have $f^*(t,x) \leq (f_\infty^s+\varepsilon)x$, $g^*(t,x) \leq (g_\infty^s+\varepsilon)x$, $h^*(t,x) \leq (h_\infty^s+\varepsilon)x$.

We consider $R_4 = \max\{2R_3, \bar{R}_4\}$, and we denote $\Omega_4 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_4\}$. Let $(u,v,w) \in P \cap \partial\Omega_4$. By the definition of $f^*$, $g^*$, $h^*$, we conclude

$$\begin{aligned}
&f\bigl(t,u(t),v(t),w(t)\bigr) \leq f^*\bigl(t,\bigl\|(u,v,w)\bigr\|_Y\bigr), \qquad g\bigl(t,u(t),v(t),w(t)\bigr) \leq g^*\bigl(t,\bigl\|(u,v,w)\bigr\|_Y\bigr), \\
&h\bigl(t,u(t),v(t),w(t)\bigr) \leq h^*\bigl(t,\bigl\|(u,v,w)\bigr\|_Y\bigr), \quad \forall t \in [0,1].
\end{aligned}$$

Then, for all $t \in [0,1]$, we obtain

$$\begin{aligned}
Q_1(u,v,w)(t) &\leq \lambda \int_0^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \leq \lambda \int_0^1 J_1(s) f^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds \\
&\leq \lambda\bigl(f_\infty^s+\varepsilon\bigr) \int_0^1 J_1(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \leq \widetilde{\alpha}_1\bigl\|(u,v,w)\bigr\|_Y,
\end{aligned}$$

$$\begin{aligned} Q_2(u,v,w)(t) &\le \mu \int_0^1 J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \le \mu \int_0^1 J_2(s) g^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds \\ &\le \mu\bigl(g_\infty^s+\varepsilon\bigr)\int_0^1 J_2(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \le \widetilde{\alpha}_2\bigl\|(u,v,w)\bigr\|_Y, \\ Q_3(u,v,w)(t) &\le \nu \int_0^1 J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \le \nu \int_0^1 J_3(s) h^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds \\ &\le \nu\bigl(h_\infty^s+\varepsilon\bigr)\int_0^1 J_3(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \le \widetilde{\alpha}_3\bigl\|(u,v,w)\bigr\|_Y. \end{aligned}$$

Therefore, we deduce $\|Q_1(u,v,w)\| \le \widetilde{\alpha}_1\|(u,v,w)\|_Y$, $\|Q_2(u,v,w)\| \le \widetilde{\alpha}_2\|(u,v,w)\|_Y$, $\|Q_3(u,v,w)\| \le \widetilde{\alpha}_3\|(u,v,w)\|_Y$.

Hence, for $(u,v,w) \in P \cap \partial\Omega_4$, we conclude that

$$\bigl\|Q(u,v,w)\bigr\|_Y \le (\widetilde{\alpha}_1+\widetilde{\alpha}_2+\widetilde{\alpha}_3)\bigl\|(u,v,w)\bigr\|_Y = \bigl\|(u,v,w)\bigr\|_Y. \tag{19}$$

By using Lemma 3.1, Theorem 2.1(ii) and relations (18), (19), we deduce that $Q$ has a fixed point $(u,v,w) \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which is a positive solution for our problem (S)-(BC).

*Case* (11). We consider $g_\infty^s = 0$, $h_0^i = \infty$, $f_\infty^s, h_\infty^s, f_0^i, g_0^i \in (0,\infty)$. Let $\lambda \in (0, M_2'')$, $\mu \in (0,\infty)$, $\nu \in (0, M_6'')$. We choose $\varepsilon > 0$ such that $\varepsilon \le \nu\theta\sigma^{\gamma-1}E$ and

$$\varepsilon \le \frac{\widetilde{\alpha}_1'' - \lambda f_\infty^s B}{2\lambda B}, \qquad \varepsilon \le \frac{1-\lambda f_\infty^s B - \nu h_\infty^s F}{2\mu D}, \qquad \varepsilon \le \frac{\widetilde{\alpha}_3'' - \nu h_\infty^s F}{2\nu F}.$$

The numerators of the above fractions are positive because $\lambda < \frac{\widetilde{\alpha}_1''}{f_\infty^s B}$, that is, $\widetilde{\alpha}_1'' > \lambda f_\infty^s B$, $\nu < \frac{\widetilde{\alpha}_3''}{h_\infty^s F}$, that is, $\widetilde{\alpha}_3'' > \nu h_\infty^s F$, and $1-\lambda f_\infty^s B - \nu h_\infty^s F = \widetilde{\alpha}_1'' + \widetilde{\alpha}_3'' - \lambda f_\infty^s B - \nu h_\infty^s F = (\widetilde{\alpha}_1'' - \lambda f_\infty^s B) + (\widetilde{\alpha}_3'' - \nu h_\infty^s F) > 0$.

By using (H2) and the definition of $h_0^i$, we deduce that there exists $R_3 > 0$ such that $h(t,u,v,w) \ge \frac{1}{\varepsilon}(u+v+w)$ for all $u,v,w \ge 0$ with $u+v+w \le R_3$ and $t \in [\sigma,1]$. We denote $\Omega_3 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_3\}$.

Let $(u,v,w) \in P \cap \partial\Omega_3$, that is, $\|(u,v,w)\|_Y = R_3$. Because $u(t)+v(t)+w(t) \le R_3$ for all $t \in [0,1]$, then by using Lemma 2.3, we obtain

$$\begin{aligned} Q_3(u,v,w)(t) &\ge \nu \int_0^1 t^{\gamma-1} J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\ &\ge \nu\sigma^{\gamma-1}\int_\sigma^1 J_3(s) h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\ &\ge \nu\sigma^{\gamma-1}\int_\sigma^1 J_3(s)\frac{1}{\varepsilon}\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\ &\ge \nu\sigma^{\gamma-1}\theta\frac{1}{\varepsilon}\int_\sigma^1 J_3(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \\ &= \nu\sigma^{\gamma-1}\theta\frac{1}{\varepsilon}E\bigl\|(u,v,w)\bigr\|_Y \ge \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [\sigma,1]. \end{aligned}$$

Then $\|Q_3(u,v,w)\| \ge Q_3(u,v,w)(\sigma) \ge \|(u,v,w)\|_Y$, and

$$\bigl\|Q(u,v,w)\bigr\|_Y \ge \bigl\|Q_3(u,v,w)\bigr\| \ge \bigl\|(u,v,w)\bigr\|_Y. \tag{20}$$

Now, using the functions $f^*$, $g^*$, $h^*$ defined in the proof of case (1), we have

$$\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x}\le f^s_\infty,\qquad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x}=0,$$
$$\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{h^*(t,x)}{x}\le h^s_\infty.$$

Therefore, for $\varepsilon>0$, there exists $\bar{R}_4>0$ such that, for all $x\ge\bar{R}_4$ and $t\in[0,1]$, we deduce $f^*(t,x)\le(f^s_\infty+\varepsilon)x$, $g^*(t,x)\le\varepsilon x$, $h^*(t,x)\le(h^s_\infty+\varepsilon)x$. We consider $R_4=\max\{2R_3,\bar{R}_4\}$, and we denote $\Omega_4=\{(u,v,w)\in Y,\|(u,v,w)\|_Y<R_4\}$. Let $(u,v,w)\in P\cap\partial\Omega_4$. Then, for all $t\in[0,1]$, we obtain

$$\begin{aligned}
Q_1(u,v,w)(t)&\le\lambda\int_0^1 J_1(s)f\bigl(s,u(s),v(s),w(s)\bigr)\,ds\\
&\le\lambda\int_0^1 J_1(s)f^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds\\
&\le\lambda\bigl(f^s_\infty+\varepsilon\bigr)\int_0^1 J_1(s)\bigl\|(u,v,w)\bigr\|_Y\,ds\\
&\le\lambda\left(f^s_\infty+\frac{\widetilde{\alpha}_1''-\lambda f^s_\infty B}{2\lambda B}\right)B\bigl\|(u,v,w)\bigr\|_Y\\
&=\frac{1}{2}\bigl(\lambda f^s_\infty B+\widetilde{\alpha}_1''\bigr)\bigl\|(u,v,w)\bigr\|_Y,\\
Q_2(u,v,w)(t)&\le\mu\int_0^1 J_2(s)g\bigl(s,u(s),v(s),w(s)\bigr)\,ds\\
&\le\mu\int_0^1 J_2(s)g^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds\\
&\le\mu\varepsilon\int_0^1 J_2(s)\bigl\|(u,v,w)\bigr\|_Y\,ds\\
&\le\mu\frac{1-\lambda f^s_\infty B-\nu h^s_\infty F}{2\mu D}D\bigl\|(u,v,w)\bigr\|_Y\\
&=\frac{1}{2}\bigl(1-\lambda f^s_\infty B-\nu h^s_\infty F\bigr)\bigl\|(u,v,w)\bigr\|_Y,\\
Q_3(u,v,w)(t)&\le\nu\int_0^1 J_3(s)h\bigl(s,u(s),v(s),w(s)\bigr)\,ds\\
&\le\nu\int_0^1 J_3(s)h^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds\\
&\le\nu\bigl(h^s_\infty+\varepsilon\bigr)\int_0^1 J_3(s)\bigl\|(u,v,w)\bigr\|_Y\,ds\\
&\le\nu\left(h^s_\infty+\frac{\widetilde{\alpha}_3''-\nu h^s_\infty F}{2\nu F}\right)F\bigl\|(u,v,w)\bigr\|_Y\\
&=\frac{1}{2}\bigl(\nu h^s_\infty F+\widetilde{\alpha}_3''\bigr)\bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Therefore

$$\bigl\|Q_1(u,v,w)\bigr\|\le\frac{1}{2}\bigl(\lambda f^s_\infty B+\widetilde{\alpha}_1''\bigr)\bigl\|(u,v,w)\bigr\|_Y,$$

$$\begin{aligned}\left\|Q_2(u,v,w)\right\| &\le \frac{1}{2}\left(1-\lambda f_\infty^s B-\nu h_\infty^s F\right)\left\|(u,v,w)\right\|_Y,\\ \left\|Q_3(u,v,w)\right\|_Y &\le \frac{1}{2}\left(\nu h_\infty^s F+\widetilde{\alpha}_3''\right)\left\|(u,v,w)\right\|_Y.\end{aligned}$$

Then, for $(u,v,w)\in P\cap\partial\Omega_4$, we conclude that

$$\begin{aligned}\left\|Q(u,v,w)\right\|_Y &\le \frac{1}{2}\left(\lambda f_\infty^s B+\widetilde{\alpha}_1''+1-\lambda f_\infty^s B-\nu h_\infty^s F+\nu h_\infty^s F+\widetilde{\alpha}_3''\right)\left\|(u,v,w)\right\|_Y\\ &= \left\|(u,v,w)\right\|_Y.\end{aligned}\tag{21}$$

By using Lemma 3.1, Theorem 2.1(ii) and relations (20), (21), we deduce that $Q$ has a fixed point $(u,v,w)\in P\cap(\overline{\Omega}_4\setminus\Omega_3)$, which is a positive solution for our problem (S)-(BC).

*Case* (14). We consider $f_\infty^s=h_\infty^s=0$, $g_0^i=\infty$, $g_\infty^s, f_0^i, h_0^i\in(0,\infty)$. Let $\lambda\in(0,\infty)$, $\mu\in(0,\widetilde{M}_4)$, $\nu\in(0,\infty)$. We choose $\varepsilon>0$ such that $\varepsilon\le\mu\theta\sigma^{\beta-1}C$ and

$$\varepsilon\le\frac{1-\mu g_\infty^s D}{4\lambda B},\qquad \varepsilon\le\frac{1-\mu g_\infty^s D}{2\mu D},\qquad \varepsilon\le\frac{1-\mu g_\infty^s D}{4\nu F}.$$

The numerator of the above fractions is positive because $\mu<\frac{1}{g_0^s D}$, that is, $1-\mu g_\infty^s D>0$.

By using (H2) and the definition of $g_0^i$, we deduce that there exists $R_3>0$ such that $g(t,u,v,w)\ge\frac{1}{\varepsilon}(u+v+w)$ for all $u,v,w\ge0$ with $u+v+w\le R_3$ and $t\in[\sigma,1]$. We denote $\Omega_3=\{(u,v,w)\in Y, \|(u,v,w)\|_Y<R_3\}$.

Let $(u,v,w)\in P\cap\partial\Omega_3$, that is, $\|(u,v,w)\|_Y=R_3$. Because $u(t)+v(t)+w(t)\le R_3$ for all $t\in[0,1]$, then by using Lemma 2.3, we obtain

$$\begin{aligned}Q_2(u,v,w)(t) &\ge \mu\int_0^1 t^{\beta-1}J_2(s)g\left(s,u(s),v(s),w(s)\right)ds\\ &\ge \mu\sigma^{\beta-1}\int_\sigma^1 J_2(s)g\left(s,u(s),v(s),w(s)\right)ds\\ &\ge \mu\sigma^{\beta-1}\int_\sigma^1 J_2(s)\frac{1}{\varepsilon}\left(u(s)+v(s)+w(s)\right)ds\\ &\ge \mu\sigma^{\beta-1}\theta\frac{1}{\varepsilon}\int_\sigma^1 J_2(s)\left\|(u,v,w)\right\|_Y ds\\ &= \mu\sigma^{\beta-1}\theta\frac{1}{\varepsilon}C\left\|(u,v,w)\right\|_Y\ge\left\|(u,v,w)\right\|_Y,\quad \forall t\in[\sigma,1].\end{aligned}$$

Then $\|Q_2(u,v,w)\|\ge Q_2(u,v,w)(\sigma)\ge\|(u,v,w)\|_Y$, and

$$\left\|Q(u,v,w)\right\|_Y\ge\left\|Q_2(u,v,w)\right\|\ge\left\|(u,v,w)\right\|_Y.\tag{22}$$

Now, using the functions $f^*$, $g^*$, $h^*$ defined in the proof of case (1), we have

$$\lim_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x}=0,\qquad \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x}\le g_\infty^s,\qquad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{h^*(t,x)}{x}=0.$$

Therefore, for $\varepsilon>0$, there exists $\bar{R}_4>0$ such that, for all $x\ge\bar{R}_4$ and $t\in[0,1]$, we deduce $f^*(t,x)\le\varepsilon x$, $g^*(t,x)\le(g_\infty^s+\varepsilon)x$, $h^*(t,x)\le\varepsilon x$.

We consider $R_4 = \max\{2R_3, \bar{R}_4\}$, and we denote $\Omega_4 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_4\}$. Let $(u,v,w) \in P \cap \partial\Omega_4$. Then, for all $t \in [0,1]$, we obtain

$$\begin{aligned}
Q_1(u,v,w)(t) &\le \lambda \int_0^1 J_1(s) f\bigl(s, u(s), v(s), w(s)\bigr)\, ds \\
&\le \lambda \int_0^1 J_1(s) f^*\bigl(s, \bigl\|(u,v,w)\bigr\|_Y\bigr)\, ds \\
&\le \lambda\varepsilon \int_0^1 J_1(s) \bigl\|(u,v,w)\bigr\|_Y\, ds \\
&= \lambda\varepsilon B \bigl\|(u,v,w)\bigr\|_Y \le \lambda \frac{1-\mu g_\infty^s D}{4\lambda B} B \bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{4}\bigl(1-\mu g_\infty^s D\bigr) \bigl\|(u,v,w)\bigr\|_Y, \\
Q_2(u,v,w)(t) &\le \mu \int_0^1 J_2(s) g\bigl(s, u(s), v(s), w(s)\bigr)\, ds \\
&\le \mu \int_0^1 J_2(s) g^*\bigl(s, \bigl\|(u,v,w)\bigr\|_Y\bigr)\, ds \\
&\le \mu\bigl(g_\infty^s + \varepsilon\bigr) \int_0^1 J_2(s) \bigl\|(u,v,w)\bigr\|_Y\, ds \\
&= \mu\bigl(g_\infty^s + \varepsilon\bigr) D \bigl\|(u,v,w)\bigr\|_Y \\
&\le \mu\left(g_\infty^s + \frac{1-\mu g_\infty^s D}{2\mu D}\right) D \bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{2}\bigl(\mu g_\infty^s D + 1\bigr) \bigl\|(u,v,w)\bigr\|_Y, \\
Q_3(u,v,w)(t) &\le \nu \int_0^1 J_3(s) h\bigl(s, u(s), v(s), w(s)\bigr)\, ds \\
&\le \nu \int_0^1 J_3(s) h^*\bigl(s, \bigl\|(u,v,w)\bigr\|_Y\bigr)\, ds \\
&\le \nu\varepsilon \int_0^1 J_3(s) \bigl\|(u,v,w)\bigr\|_Y\, ds \\
&= \nu\varepsilon F \bigl\|(u,v,w)\bigr\|_Y \le \nu \frac{1-\nu g_\infty^s D}{4\nu F} F \bigl\|(u,v,w)\bigr\|_Y \\
&= \frac{1}{4}\bigl(1-\mu g_\infty^s D\bigr) \bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Therefore

$$\begin{aligned}
\bigl\|Q_1(u,v,w)\bigr\| &\le \frac{1}{4}\bigl(1-\mu g_\infty^s D\bigr) \bigl\|(u,v,w)\bigr\|_Y, \\
\bigl\|Q_2(u,v,w)\bigr\| &\le \frac{1}{2}\bigl(1+\mu g_\infty^s D\bigr) \bigl\|(u,v,w)\bigr\|_Y, \\
\bigl\|Q_3(u,v,w)\bigr\| &\le \frac{1}{4}\bigl(1-\mu g_\infty^s D\bigr) \bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Then, for $(u,v,w) \in P \cap \partial\Omega_4$, we conclude that

$$\bigl\|Q(u,v,w)\bigr\|_Y \le \frac{1}{4}\bigl(1-\mu g_\infty^s D + 2 + 2\mu g_\infty^s D + 1 - \mu g_\infty^s D\bigr) \bigl\|(u,v,w)\bigr\|_Y = \bigl\|(u,v,w)\bigr\|_Y. \quad (23)$$

By using Lemma 3.1, Theorem 2.1(ii) and relations (22) and (23), we deduce that $Q$ has a fixed point $(u,v,w) \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which is a positive solution for our problem (S)-(BC).

*Case* (16). We consider $f_\infty^s = g_\infty^s = h_\infty^s = 0$, $f_0^i = g_0^i = \infty$ and $h_0^i \in (0,\infty)$. Let $\lambda \in (0,\infty)$, $\mu \in (0,\infty)$, $\nu \in (0,\infty)$. We choose $\varepsilon > 0$ such that

$$\varepsilon \le \lambda\theta\sigma^{\alpha-1}A, \qquad \varepsilon \le \frac{1}{3\lambda B}, \qquad \varepsilon \le \frac{1}{3\mu D}, \qquad \varepsilon \le \frac{1}{3\nu F}.$$

By using (H2) and the definition of $f_0^i$, we deduce that there exists $R_3 > 0$ such that $f(t,u,v,w) \ge \frac{1}{\varepsilon}(u+v+w)$ for all $u,v,w \ge 0$ with $u+v+w \le R_3$ and $t \in [\sigma,1]$. We denote $\Omega_3 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_3\}$.

Let $(u,v,w) \in P \cap \partial\Omega_3$, that is, $\|(u,v,w)\|_Y = R_3$. Because $u(t)+v(t)+w(t) \le R_3$ for all $t \in [0,1]$, then by using Lemma 2.3, we obtain

$$\begin{aligned} Q_1(u,v,w)(t) &\ge \lambda \int_0^1 t^{\alpha-1} J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\ &\ge \lambda\sigma^{\alpha-1} \int_\sigma^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\ &\ge \lambda\sigma^{\alpha-1} \int_\sigma^1 J_1(s)\frac{1}{\varepsilon}\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\ &\ge \lambda\sigma^{\alpha-1}\theta\frac{1}{\varepsilon} \int_\sigma^1 J_1(s)\bigl\|(u,v,w)\bigr\|_Y\,ds \\ &= \lambda\sigma^{\alpha-1}\theta\frac{1}{\varepsilon}A\bigl\|(u,v,w)\bigr\|_Y \ge \bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [\sigma,1]. \end{aligned}$$

Then $\|Q_1(u,v,w)\| \ge Q_1(u,v,w)(\sigma) \ge \|(u,v,w)\|_Y$, and

$$\bigl\|Q(u,v,w)\bigr\|_Y \ge \bigl\|Q_1(u,v,w)\bigr\| \ge \bigl\|(u,v,w)\bigr\|_Y. \tag{24}$$

Now, using the functions $f^*$, $g^*$, $h^*$ defined in the proof of case (1), we have

$$\lim_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x} = 0, \qquad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x} = 0, \qquad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{h^*(t,x)}{x} = 0.$$

Therefore, for $\varepsilon > 0$, there exists $\bar{R}_4 > 0$ such that $f^*(t,x) \le \varepsilon x$, $g^*(t,x) \le \varepsilon x$, $h^*(t,x) \le \varepsilon x$ for all $x \ge \bar{R}_4$ and $t \in [0,1]$.

We consider $R_4 = \max\{2R_3, \bar{R}_4\}$, and we denote $\Omega_4 = \{(u,v,w) \in Y, \|(u,v,w)\|_Y < R_4\}$. Let $(u,v,w) \in P \cap \partial\Omega_4$. Then, for all $t \in [0,1]$, we obtain

$$\begin{aligned} Q_1(u,v,w)(t) &\le \lambda \int_0^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \le \lambda \int_0^1 J_1(s) f^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds \\ &\le \lambda\varepsilon \int_0^1 J_1(s)\bigl\|(u,v,w)\bigr\|_Y\,ds = \lambda\varepsilon B\bigl\|(u,v,w)\bigr\|_Y \le \frac{1}{3}\bigl\|(u,v,w)\bigr\|_Y, \\ Q_2(u,v,w)(t) &\le \mu \int_0^1 J_2(s) g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \le \mu \int_0^1 J_2(s) g^*\bigl(s,\bigl\|(u,v,w)\bigr\|_Y\bigr)\,ds \\ &\le \mu\varepsilon \int_0^1 J_2(s)\bigl\|(u,v,w)\bigr\|_Y\,ds = \mu\varepsilon D\bigl\|(u,v,w)\bigr\|_Y \le \frac{1}{3}\bigl\|(u,v,w)\bigr\|_Y, \end{aligned}$$

$$\begin{aligned} Q_3(u,v,w)(t) &\le \nu \int_0^1 J_3(s)h\big(s,u(s),v(s),w(s)\big)\,ds \le \nu \int_0^1 J_3(s)h^*\big(s,\big\|(u,v,w)\big\|_Y\big)\,ds \\ &\le \nu\varepsilon \int_0^1 J_3(s)\big\|(u,v,w)\big\|_Y\,ds = \nu\varepsilon F\big\|(u,v,w)\big\|_Y \le \frac{1}{3}\big\|(u,v,w)\big\|_Y. \end{aligned}$$

Therefore $\|Q_1(u,v,w)\| \le \frac{1}{3}\|(u,v,w)\|_Y$, $\|Q_2(u,v,w)\| \le \frac{1}{3}\|(u,v,w)\|_Y$, $\|Q_3(u,v,w)\| \le \frac{1}{3}\|(u,v,w)\|_Y$. Then, for $(u,v,w) \in P \cap \partial\Omega_4$, we conclude that

$$\big\|Q(u,v,w)\big\|_Y \le \big\|(u,v,w)\big\|_Y. \tag{25}$$

By using Lemma 3.1, Theorem 2.1(ii) and relations (24) and (25), we deduce that $Q$ has a fixed point $(u,v,w) \in P \cap (\bar{\Omega}_4 \setminus \Omega_3)$, which is a positive solution for our problem (S)-(BC). □

**Remark 3.4** Each of the cases (9)-(16) of Theorem 3.2 contains seven cases as follows: $\{f_0^i = \infty, g_0^i, h_0^i \in (0,\infty)\}$, or $\{g_0^i = \infty, f_0^i, h_0^i \in (0,\infty)\}$, or $\{h_0^i = \infty, f_0^i, g_0^i \in (0,\infty)\}$, or $\{f_0^i = g_0^i = \infty, h_0^i \in (0,\infty)\}$, or $\{f_0^i = h_0^i = \infty, g_0^i \in (0,\infty)\}$, or $\{g_0^i = h_0^i = \infty, f_0^i \in (0,\infty)\}$, or $\{f_0^i = g_0^i = h_0^i = \infty\}$. So the total number of cases from Theorem 3.2 is 64, which we grouped in 16 cases.

Each of the cases (1)-(8) contains four subcases because $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$, or $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, or $\alpha_2 = 1$ and $\alpha_1 = \alpha_3 = 0$, or $\alpha_3 = 1$ and $\alpha_1 = \alpha_2 = 0$.

**Remark 3.5** In the paper [2], the authors present only 15 cases (Theorems 2.16-2.30 from [2]) from 64 cases, namely the first nine cases of our Theorem 3.2. They did not study the cases when some extreme limits are 0 and other are $\infty$. Besides, compared to Theorems 2.17-2.22 and 2.24-2.30 from [2], our intervals for parameters $\lambda, \mu, \nu$ presented in Theorem 3.2 (our cases (2)-(7) and (9)) are better than the corresponding ones from [2].

**Remark 3.6** One can formulate existence results for the general case of the system of $n$ fractional differential equations $(\widetilde{\mathrm{S}})$ with the boundary conditions $(\widetilde{\mathrm{BC}})$ from Remark 3.3. According to the values of $f_{j\infty}^s = \limsup_{u_1+\cdots+u_n\to\infty} \sup_{t\in[0,1]} \frac{f_j(t,u_1,\ldots,u_n)}{u_1+\cdots+u_n} \in [0,\infty)$, and $f_{j0}^i = \liminf_{u_1+\cdots+u_n\to 0} \inf_{t\in[\sigma,1]} \frac{f_j(t,u_1,\ldots,u_n)}{u_1+\cdots+u_n} \in (0,\infty]$, $j = 1,\ldots,n$, we have $2^{2n}$ cases, which can be grouped in $2^{n+1}$ cases.

## 4 Nonexistence of positive solutions

We present in this section intervals for $\lambda$, $\mu$ and $\nu$, for which there exist no positive solutions of problem (S)-(BC), viewed as fixed points of operator $Q$.

**Theorem 4.1** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $A_1, A_2, A_3$ *such that*

$$\begin{aligned} &f(t,u,v,w) \le A_1(u+v+w), \qquad g(t,u,v,w) \le A_2(u+v+w), \\ &h(t,u,v,w) \le A_3(u+v+w), \quad \forall t \in [0,1],\ u,v,w \ge 0, \end{aligned} \tag{26}$$

*then there exist positive constants* $\lambda_0$, $\mu_0$, $\nu_0$ *such that, for every* $\lambda \in (0,\lambda_0)$, $\mu \in (0,\mu_0)$, $\nu \in (0,\nu_0)$ *the boundary value problem* (S)-(BC) *has no positive solution.*

*Proof* We define $\lambda_0 = \frac{1}{3A_1B}$, $\mu_0 = \frac{1}{3A_2D}$, $\nu_0 = \frac{1}{3A_3F}$, where $B = \int_0^1 J_1(s)\,ds$, $D = \int_0^1 J_2(s)\,ds$, $F = \int_0^1 J_3(s)\,ds$. We will show that for any $\lambda \in (0,\lambda_0)$, $\mu \in (0,\mu_0)$, $\nu \in (0,\nu_0)$, problem (S)-(BC) has no positive solution.

Let $\lambda \in (0,\lambda_0)$, $\mu \in (0,\mu_0)$, $\nu \in (0,\nu_0)$. We suppose that (S)-(BC) has a positive solution $(u(t),v(t),w(t))$, $t \in [0,1]$. Then we have

$$\begin{aligned}
u(t) &= Q_1(u,v,w)(t) = \lambda \int_0^1 G_1(t,s)f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\le \lambda \int_0^1 J_1(s)f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\le \lambda A_1 \int_0^1 J_1(s)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\le \lambda A_1\bigl(\|u\|+\|v\|+\|w\|\bigr)\int_0^1 J_1(s)\,ds \\
&= \lambda A_1 B\bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1], \\
v(t) &= Q_2(u,v,w)(t) = \mu \int_0^1 G_2(t,s)g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\le \mu \int_0^1 J_2(s)g\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\le \mu A_2 \int_0^1 J_2(s)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\le \mu A_2\bigl(\|u\|+\|v\|+\|w\|\bigr)\int_0^1 J_2(s)\,ds \\
&= \mu A_2 D\bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1], \\
w(t) &= Q_3(u,v,w)(t) = \nu \int_0^1 G_3(t,s)h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\le \nu \int_0^1 J_3(s)h\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\
&\le \nu A_3 \int_0^1 J_3(s)\bigl(u(s)+v(s)+w(s)\bigr)\,ds \\
&\le \nu A_3\bigl(\|u\|+\|v\|+\|w\|\bigr)\int_0^1 J_3(s)\,ds \\
&= \nu A_3 F\bigl\|(u,v,w)\bigr\|_Y, \quad \forall t \in [0,1].
\end{aligned}$$

Therefore we conclude

$$\begin{aligned}
&\|u\| \le \lambda A_1 B\bigl\|(u,v,w)\bigr\|_Y < \lambda_0 A_1 B\bigl\|(u,v,w)\bigr\|_Y = \frac{1}{3}\bigl\|(u,v,w)\bigr\|_Y, \\
&\|v\| \le \mu A_2 D\bigl\|(u,v,w)\bigr\|_Y < \mu_0 A_2 D\bigl\|(u,v,w)\bigr\|_Y = \frac{1}{3}\bigl\|(u,v,w)\bigr\|_Y, \\
&\|w\| \le \nu A_3 F\bigl\|(u,v,w)\bigr\|_Y < \nu_0 A_3 F\bigl\|(u,v,w)\bigr\|_Y = \frac{1}{3}\bigl\|(u,v,w)\bigr\|_Y.
\end{aligned}$$

Hence we deduce $\|(u,v,w)\|_Y = \|u\|+\|v\|+\|w\| < \|(u,v,w)\|_Y$, which is a contradiction. So the boundary value problem (S)-(BC) has no positive solution. □

**Remark 4.1** In the proof of Theorem 4.1 we can also define $\lambda_0 = \frac{\alpha_1}{A_1 B}$, $\mu_0 = \frac{\alpha_2}{A_2 D}$, $\nu_0 = \frac{\alpha_3}{A_3 F}$ with $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

**Remark 4.2** If $f_0^s, g_0^s, h_0^s, f_\infty^s, g_\infty^s, h_\infty^s < \infty$, then there exist positive constants $A_1, A_2, A_3$ such that (26) holds (see also [3] for a system with two equations), and then we obtain the conclusion of Theorem 4.1.

**Theorem 4.2** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_1 > 0$ *such that*

$$f(t,u,v,w) \geq m_1(u + v + w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \tag{27}$$

*then there exists a positive constant* $\widetilde{\lambda}_0$ *such that, for every* $\lambda > \widetilde{\lambda}_0$, $\mu > 0$ *and* $\nu > 0$, *the boundary value problem* (S)-(BC) *has no positive solution.*

*Proof* We define $\widetilde{\lambda}_0 = \frac{1}{\theta\sigma^{\alpha-1} m_1 A}$, where $A = \int_\sigma^1 J_1(s)\,ds$. We will show that for every $\lambda > \widetilde{\lambda}_0$, $\mu > 0$ and $\nu > 0$, problem (S)-(BC) has no positive solution.

Let $\lambda > \widetilde{\lambda}_0$, $\mu > 0$ and $\nu > 0$. We suppose that (S)-(BC) has a positive solution $(u(t), v(t), w(t))$, $t \in [0,1]$. Then we obtain

$$\begin{aligned} u(t) &= Q_1(u,v,w)(t) = \lambda \int_0^1 G_1(t,s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\ &\geq \lambda t^{\alpha-1} \int_\sigma^1 J_1(s) f\bigl(s,u(s),v(s),w(s)\bigr)\,ds \\ &\geq \lambda \sigma^{\alpha-1} \int_\sigma^1 J_1(s) m_1\bigl(u(s) + v(s) + w(s)\bigr)\,ds \\ &\geq \lambda\theta\sigma^{\alpha-1} m_1 \int_\sigma^1 J_1(s)\bigl(\|u\| + \|v\| + \|w\|\bigr)\,ds \\ &= \lambda\theta\sigma^{\alpha-1} m_1 A \bigl\|(u,v,w)\bigr\|_Y. \end{aligned}$$

Therefore we deduce

$$\|u\| \geq u(\sigma) \geq \lambda\theta\sigma^{\alpha-1} m_1 A \bigl\|(u,v,w)\bigr\|_Y > \widetilde{\lambda}_0 \theta\sigma^{\alpha-1} m_1 A \bigl\|(u,v,w)\bigr\|_Y = \bigl\|(u,v,w)\bigr\|_Y,$$

and so, $\|(u,v,w)\|_Y = \|u\| + \|v\| + \|w\| > \|(u,v,w)\|_Y$, which is a contradiction. Therefore the boundary value problem (S)-(BC) has no positive solution. □

In a similar manner, we obtain the following theorems.

**Theorem 4.3** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_2 > 0$ *such that*

$$g(t,u,v,w) \geq m_2(u + v + w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \tag{28}$$

*then there exists a positive constant* $\widetilde{\mu}_0$ *such that, for every* $\lambda > 0$, $\mu > \widetilde{\mu}_0$ *and* $\nu > 0$, *the boundary value problem* (S)-(BC) *has no positive solution.*

In Theorem 4.3 we define $\widetilde{\mu}_0 = \frac{1}{\theta\sigma^{\beta-1}m_2C}$, where $C = \int_\sigma^1 J_2(s)\,ds$.

**Theorem 4.4** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_3 > 0$ *such that*

$$h(t,u,v,w) \geq m_3(u+v+w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \tag{29}$$

*then there exists a positive constant* $\widetilde{\nu}_0$ *such that, for every* $\lambda > 0$, $\mu > 0$ *and* $\nu > \widetilde{\nu}_0$, *the boundary value problem* (S)-(BC) *has no positive solution.*

In Theorem 4.4 we define $\widetilde{\nu}_0 = \frac{1}{\theta\sigma^{\gamma-1}m_3E}$, where $E = \int_\sigma^1 J_3(s)\,ds$.

**Remark 4.3**

(a) If for $\sigma \in (0,1)$, $f_0^i, f_\infty^i > 0$ and $f(t,u,v,w) > 0$ for all $t \in [\sigma,1]$ and $u,v,w \geq 0$ with $u+v+w > 0$, then relation (27) holds, and we obtain the conclusion of Theorem 4.2.

(b) If for $\sigma \in (0,1)$, $g_0^i, g_\infty^i > 0$ and $g(t,u,v,w) > 0$ for all $t \in [\sigma,1]$ and $u,v,w \geq 0$ with $u+v+w > 0$, then relation (28) holds, and we obtain the conclusion of Theorem 4.3.

(c) If for $\sigma \in (0,1)$, $h_0^i, h_\infty^i > 0$ and $h(t,u,v,w) > 0$ for all $t \in [\sigma,1]$ and $u,v,w \geq 0$ with $u+v+w > 0$, then relation (29) holds, and we obtain the conclusion of Theorem 4.4.

**Theorem 4.5** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_1, m_2 > 0$ *such that*

$$\begin{aligned} &f(t,u,v,w) \geq m_1(u+v+w), \\ &g(t,u,v,w) \geq m_2(u+v+w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \end{aligned} \tag{30}$$

*then there exist positive constants* $\widetilde{\widetilde{\lambda}}_0$ *and* $\widetilde{\widetilde{\mu}}_0$ *such that, for every* $\lambda > \widetilde{\widetilde{\lambda}}_0$, $\mu > \widetilde{\widetilde{\mu}}_0$ *and* $\nu > 0$, *the boundary value problem* (*S*)-(*BC*) *has no positive solution.*

*Proof* We define $\widetilde{\widetilde{\lambda}}_0 = \frac{1}{2\theta\sigma^{\alpha-1}m_1A}$ $(= \frac{\widetilde{\lambda}_0}{2})$ and $\widetilde{\widetilde{\mu}}_0 = \frac{1}{2\theta\sigma^{\beta-1}m_2C}$ $(= \frac{\widetilde{\mu}_0}{2})$. Then, for every $\lambda > \widetilde{\widetilde{\lambda}}_0$, $\mu > \widetilde{\widetilde{\mu}}_0$ and $\nu > 0$, problem (S)-(BC) has no positive solution. Indeed, let $\lambda > \widetilde{\widetilde{\lambda}}_0$, $\mu > \widetilde{\widetilde{\mu}}_0$ and $\nu > 0$. We suppose that (S)-(BC) has a positive solution $(u(t),v(t),w(t))$, $t \in [0,1]$. Then, in a similar manner as in the proof of Theorem 4.2, we deduce

$$\|u\| \geq \lambda\theta\sigma^{\alpha-1}m_1A\big\|(u,v,w)\big\|_Y, \qquad \|v\| \geq \mu\theta\sigma^{\beta-1}m_2C\big\|(u,v,w)\big\|_Y,$$

and so

$$\begin{aligned} \big\|(u,v,w)\big\|_Y &= \|u\| + \|v\| + \|w\| \geq \|u\| + \|v\| \\ &\geq \big(\lambda\theta\sigma^{\alpha-1}m_1A + \mu\theta\sigma^{\beta-1}m_2C\big)\big\|(u,v,w)\big\|_Y \\ &> \big(\widetilde{\widetilde{\lambda}}_0\theta\sigma^{\alpha-1}m_1A + \widetilde{\widetilde{\mu}}_0\theta\sigma^{\beta-1}m_2C\big)\big\|(u,v,w)\big\|_Y \\ &= \left(\frac{1}{2} + \frac{1}{2}\right)\big\|(u,v,w)\big\|_Y = \big\|(u,v,w)\big\|_Y, \end{aligned}$$

which is a contradiction. Therefore the boundary value problem (S)-(BC) has no positive solution. □

**Remark 4.4** In the proof of Theorem 4.5 we can also define $\widetilde{\widetilde{\lambda}}_0 = \frac{\widetilde{\alpha}_1}{\theta\sigma^{\alpha-1}m_1A}$, $\widetilde{\widetilde{\mu}}_0 = \frac{\widetilde{\alpha}_2}{\theta\sigma^{\beta-1}m_2C}$ with $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ with $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

In a similar manner we obtain the following theorems.

**Theorem 4.6** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_1, m_3 > 0$ *such that*

$$\begin{aligned} &f(t,u,v,w) \geq m_1(u+v+w), \\ &h(t,u,v,w) \geq m_3(u+v+w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \end{aligned} \tag{31}$$

*then there exist positive constants* $\widetilde{\widetilde{\lambda}}'_0$ *and* $\widetilde{\widetilde{\nu}}'_0$ *such that, for every* $\lambda > \widetilde{\widetilde{\lambda}}'_0$, $\mu > 0$ *and* $\nu > \widetilde{\widetilde{\nu}}'_0$, *the boundary value problem* (S)-(BC) *has no positive solution.*

In Theorem 4.6 we define $\widetilde{\widetilde{\lambda}}'_0 = \frac{1}{2\theta\sigma^{\alpha-1}m_1A}$ $(= \frac{\widetilde{\lambda}_0}{2})$ and $\widetilde{\widetilde{\nu}}'_0 = \frac{1}{2\theta\sigma^{\gamma-1}m_3E}$ $(= \frac{\widetilde{\nu}_0}{2})$, or in general $\widetilde{\widetilde{\lambda}}'_0 = \frac{\widetilde{\alpha}_1}{\theta\sigma^{\alpha-1}m_1A}$ and $\widetilde{\widetilde{\nu}}'_0 = \frac{\widetilde{\alpha}_2}{\theta\sigma^{\gamma-1}m_3E}$ with $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

**Theorem 4.7** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_2, m_3 > 0$ *such that*

$$\begin{aligned} &g(t,u,v,w) \geq m_2(u+v+w), \\ &h(t,u,v,w) \geq m_3(u+v+w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \end{aligned} \tag{32}$$

*then there exist positive constants* $\widetilde{\widetilde{\mu}}''_0$ *and* $\widetilde{\widetilde{\nu}}''_0$ *such that, for every* $\lambda > 0$, $\mu > \widetilde{\widetilde{\mu}}''_0$ *and* $\nu > \widetilde{\widetilde{\nu}}''_0$, *the boundary value problem* (S)-(BC) *has no positive solution.*

In Theorem 4.7 we define $\widetilde{\widetilde{\mu}}''_0 = \frac{1}{2\theta\sigma^{\beta-1}m_2C}$ $(= \frac{\widetilde{\mu}_0}{2})$ and $\widetilde{\widetilde{\nu}}''_0 = \frac{1}{2\theta\sigma^{\gamma-1}m_3E}$ $(= \frac{\widetilde{\nu}_0}{2})$, or in general $\widetilde{\widetilde{\mu}}''_0 = \frac{\widetilde{\alpha}_1}{\theta\sigma^{\beta-1}m_2C}$ and $\widetilde{\widetilde{\nu}}''_0 = \frac{\widetilde{\alpha}_2}{\theta\sigma^{\gamma-1}m_3E}$ with $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

**Remark 4.5**

(a) If for $\sigma \in (0,1)$, $f_0^i, f_\infty^i, g_0^i, g_\infty^i > 0$ and $f(t,u,v,w) > 0$, $g(t,u,v,w) > 0$ for all $t \in [\sigma,1]$ and $u,v,w \geq 0$ with $u+v+w > 0$, then relation (30) holds, and we obtain the conclusion of Theorem 4.5.

(b) If for $\sigma \in (0,1)$, $f_0^i, f_\infty^i, h_0^i, h_\infty^i > 0$ and $f(t,u,v,w) > 0$, $h(t,u,v,w) > 0$ for all $t \in [\sigma,1]$ and $u,v,w \geq 0$ with $u+v+w > 0$, then relation (31) holds, and we obtain the conclusion of Theorem 4.6.

(c) If for $\sigma \in (0,1)$, $g_0^i, g_\infty^i, h_0^i, h_\infty^i > 0$ and $g(t,u,v,w) > 0$, $h(t,u,v,w) > 0$ for all $t \in [\sigma,1]$ and $u,v,w \geq 0$ with $u+v+w > 0$, then relation (32) holds, and we obtain the conclusion of Theorem 4.7.

**Theorem 4.8** *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* $\sigma \in (0,1)$ *and* $m_1, m_2, m_3 > 0$ *such that*

$$\begin{aligned} &f(t,u,v,w) \geq m_1(u+v+w), \qquad g(t,u,v,w) \geq m_2(u+v+w), \\ &h(t,u,v,w) \geq m_3(u+v+w), \quad \forall t \in [\sigma,1],\ u,v,w \geq 0, \end{aligned} \tag{33}$$

*then there exist positive constants* $\hat{\lambda}_0$, $\hat{\mu}_0$ *and* $\hat{\nu}_0$ *such that, for every* $\lambda > \hat{\lambda}_0$, $\mu > \hat{\mu}_0$ *and* $\nu > \hat{\nu}_0$, *the boundary value problem* (S)-(BC) *has no positive solution.*

*Proof* We define $\hat{\lambda}_0 = \frac{1}{3\theta\sigma^{\alpha-1}m_1A}$, $\hat{\mu}_0 = \frac{1}{3\theta\sigma^{\beta-1}m_2C}$, $\hat{\nu}_0 = \frac{1}{3\theta\sigma^{\gamma-1}m_3E}$. Then, for every $\lambda > \hat{\lambda}_0$, $\mu > \hat{\mu}_0$, $\nu > \hat{\nu}_0$, problem (S)-(BC) has no positive solution. Indeed, let $\lambda > \hat{\lambda}_0$, $\mu > \hat{\mu}_0$ and $\nu > \hat{\nu}_0$. We suppose that (S)-(BC) has a positive solution $(u(t), v(t), w(t))$, $t \in [0,1]$. Then, in a similar manner as in the proof of Theorem 4.5, we deduce

$$\begin{aligned}
&\|u\| \ge \lambda\theta\sigma^{\alpha-1}m_1A\big\|(u,v,w)\big\|_Y, \qquad \|v\| \ge \mu\theta\sigma^{\beta-1}m_2C\big\|(u,v,w)\big\|_Y, \\
&\|w\| \ge \nu\theta\sigma^{\gamma-1}m_3E\big\|(u,v,w)\big\|_Y,
\end{aligned}$$

and so

$$\begin{aligned}
\big\|(u,v,w)\big\|_Y &= \|u\| + \|v\| + \|w\| \\
&\ge \big(\lambda\theta\sigma^{\alpha-1}m_1A + \mu\theta\sigma^{\beta-1}m_2C + \nu\theta\sigma^{\gamma-1}m_3E\big)\big\|(u,v,w)\big\|_Y \\
&> \big(\hat{\lambda}_0\theta\sigma^{\alpha-1}m_1A + \hat{\mu}_0\theta\sigma^{\beta-1}m_2C + \hat{\nu}_0\theta\sigma^{\gamma-1}m_3E\big)\big\|(u,v,w)\big\|_Y \\
&= \big\|(u,v,w)\big\|_Y,
\end{aligned}$$

which is a contradiction. Therefore, the boundary value problem (S)-(BC) has no positive solution. □

**Remark 4.6** In the proof of Theorem 4.8, we can also define $\hat{\lambda}_0 = \frac{\alpha_1'}{\theta\sigma^{\alpha-1}m_1A}$, $\hat{\mu}_0 = \frac{\alpha_2'}{\theta\sigma^{\beta-1}m_2C}$, $\hat{\nu}_0 = \frac{\alpha_3'}{\theta\sigma^{\gamma-1}m_3F}$, where $\alpha_1', \alpha_2', \alpha_3' > 0$ with $\alpha_1' + \alpha_2' + \alpha_3' = 1$.

**Remark 4.7** If for $\sigma \in (0,1)$, $f_0^i, f_\infty^i, g_0^i, g_\infty^i, h_0^i, h_\infty^i > 0$ and $f(t,u,v,w) > 0$, $g(t,u,v,w) > 0$, $h(t,u,v,w) > 0$ for all $t \in [\sigma,1]$, $u,v,w \ge 0$, $u+v+w > 0$, then relation (33) holds, and we have the conclusion of Theorem 4.8.

**Remark 4.8** The conclusions of Theorems 3.1-3.2 and 4.1-4.8 remain valid for general systems of Hammerstein integral equations of the form

$$\begin{cases}
u(t) = \lambda\int_0^1 G_1(t,s)f(s,u(s),v(s),w(s))\,ds, & t \in [0,1], \\
v(t) = \mu\int_0^1 G_2(t,s)g(s,u(s),v(s),w(s))\,ds, & t \in [0,1], \\
w(t) = \nu\int_0^1 G_3(t,s)h(s,u(s),v(s),w(s))\,ds, & t \in [0,1],
\end{cases} \tag{34}$$

with positive parameters $\lambda, \mu, \nu$, and instead of assumptions (H1)-(H2), the following assumptions are satisfied:

- $(\widetilde{\mathrm{H1}})$ The functions $G_1, G_2, G_3 : [0,1] \times [0,1] \to \mathbb{R}$ are continuous, and there exist the continuous functions $J_1, J_2, J_3 : [0,1] \to \mathbb{R}$ and $\sigma \in (0,1)$, $\alpha, \beta, \gamma > 2$ such that
  - (a) $0 \le G_i(t,s) \le J_i(s)$, $\forall t,s \in [0,1]$, $i = 1,2,3$;
  - (b) $G_1(t,s) \ge t^{\alpha-1}J_1(s)$, $G_2(t,s) \ge t^{\beta-1}J_2(s)$, $G_3(t,s) \ge t^{\gamma-1}J_3(s)$, $\forall t,s \in [0,1]$;
  - (c) $\int_\sigma^1 J_i(s)\,ds > 0$, $i = 1,2,3$.
- $(\widetilde{\mathrm{H2}})$ The functions $f, g, h : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.

## 5 Examples

Let $n = 3$, $m = 5$, $l = 4$, $\alpha = \frac{5}{2}$, $\beta = \frac{17}{4}$, $\gamma = \frac{10}{3}$, $p_1 = 1$, $q_1 = \frac{1}{2}$, $p_2 = \frac{7}{3}$, $q_2 = \frac{3}{2}$, $p_3 = \frac{7}{4}$, $q_3 = \frac{2}{3}$, $N = 2$, $M = 1$, $L = 3$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{2}{3}$, $a_1 = 2$, $a_2 = \frac{1}{2}$, $\eta_1 = \frac{1}{2}$, $b_1 = 4$, $\zeta_1 = \frac{1}{4}$, $\zeta_2 = \frac{1}{2}$, $\zeta_3 = \frac{3}{4}$, $c_1 = 3$, $c_2 = 2$, $c_3 = 1$.

We consider the system of fractional differential equations

$$(\mathrm{S}_0)\quad \begin{cases} D_{0+}^{5/2}u(t) + \lambda f(t,u(t),v(t),w(t)) = 0, & t \in (0,1), \\ D_{0+}^{17/4}v(t) + \mu g(t,u(t),v(t),w(t)) = 0, & t \in (0,1), \\ D_{0+}^{10/3}w(t) + \nu h(t,u(t),v(t),w(t)) = 0, & t \in (0,1), \end{cases}$$

with the multi-point boundary conditions

$$(\mathrm{BC}_0)\quad \begin{cases} u(0) = u'(0) = 0, \qquad u'(1) = 2D_{0+}^{1/2}u(t)|_{t=\frac{1}{3}} + \frac{1}{2}D_{0+}^{1/2}u(t)|_{t=\frac{2}{3}}, \\ v(0) = v'(0) = v''(0) = v'''(0) = 0, \qquad D_{0+}^{7/3}v(t)|_{t=1} = 4D_{0+}^{3/2}v(t)|_{t=\frac{1}{2}}, \\ w(0) = w'(0) = w''(0) = 0, \\ D_{0+}^{7/4}w(t)|_{t=1} = 3D_{0+}^{2/3}w(t)|_{t=\frac{1}{4}} + 2D_{0+}^{2/3}w(t)|_{t=\frac{1}{2}} + D_{0+}^{2/3}w(t)|_{t=\frac{3}{4}}. \end{cases}$$

We have $\Delta_1 = \frac{6-3\sqrt{\pi}}{4} \approx 0.17065961 > 0$, $\Delta_2 = \frac{\Gamma(17/4)}{\Gamma(23/12)} - \frac{2^{1/4}\Gamma(17/4)}{\Gamma(11/4)} \approx 2.43672831 > 0$, $\Delta_3 = \frac{\Gamma(10/3)}{\Gamma(19/12)} - (3 + 2^{8/3} + 3^{5/3})\frac{\Gamma(10/3)}{4^{5/3}\Gamma(8/3)} \approx 0.25945301 > 0$. So assumption (H1) is satisfied.

Besides we deduce

$$g_1(t,s) = \frac{1}{\Gamma(5/2)} \begin{cases} t^{3/2}(1-s)^{1/2} - (t-s)^{3/2}, & 0 \le s \le t \le 1, \\ t^{3/2}(1-s)^{1/2}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_2(t,s) = \begin{cases} t(1-s)^{1/2} - (t-s), & 0 \le s \le t \le 1, \\ t(1-s)^{1/2}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_3(t,s) = \frac{1}{\Gamma(17/4)} \begin{cases} t^{13/4}(1-s)^{11/12} - (t-s)^{13/4}, & 0 \le s \le t \le 1, \\ t^{13/4}(1-s)^{11/12}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_4(t,s) = \frac{1}{\Gamma(11/4)} \begin{cases} t^{7/4}(1-s)^{11/12} - (t-s)^{7/4}, & 0 \le s \le t \le 1, \\ t^{7/4}(1-s)^{11/12}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_5(t,s) = \frac{1}{\Gamma(10/3)} \begin{cases} t^{7/3}(1-s)^{7/12} - (t-s)^{7/3}, & 0 \le s \le t \le 1, \\ t^{7/3}(1-s)^{7/12}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_6(t,s) = \frac{1}{\Gamma(8/3)} \begin{cases} t^{5/3}(1-s)^{7/12} - (t-s)^{5/3}, & 0 \le s \le t \le 1, \\ t^{5/3}(1-s)^{7/12}, & 0 \le t \le s \le 1. \end{cases}$$

Then we obtain

$$G_1(t,s) = g_1(t,s) + \frac{t^{3/2}}{\Delta_1}\left(2g_2\left(\frac{1}{3},s\right) + \frac{1}{2}g_2\left(\frac{2}{3},s\right)\right),$$

$$G_2(t,s) = g_3(t,s) + \frac{4t^{13/4}}{\Delta_2}g_4\left(\frac{1}{2},s\right),$$

$$
\begin{aligned}
G_3(t,s) &= g_5(t,s) + \frac{t^{7/3}}{\Delta_3}\left(3g_6\left(\frac{1}{4},s\right) + 2g_6\left(\frac{1}{2},s\right) + g_6\left(\frac{3}{4},s\right)\right),\\
h_1(s) &= \frac{4}{3\sqrt{\pi}}s(1-s)^{1/2}, \qquad h_3(s) = \frac{1}{\Gamma(17/4)}(1-s)^{11/12}\bigl(1-(1-s)^{7/3}\bigr),\\
h_5(s) &= \frac{1}{\Gamma(10/3)}(1-s)^{7/12}\bigl(1-(1-s)^{7/4}\bigr),\\
J_1(s) &= \frac{4}{3\sqrt{\pi}}s(1-s)^{1/2} + \frac{1}{\Delta_1}\left(2g_2\left(\frac{1}{3},s\right) + \frac{1}{2}g_2\left(\frac{2}{3},s\right)\right)\\
&= \begin{cases}
\frac{4}{3\sqrt{\pi}}s(1-s)^{1/2} + \frac{1}{2\Delta_1}[2(1-s)^{1/2} + 5s - 2], & 0 \le s < \frac{1}{3},\\
\frac{4}{3\sqrt{\pi}}s(1-s)^{1/2} + \frac{1}{6\Delta_1}[6(1-s)^{1/2} + 3s - 2], & \frac{1}{3} \le s < \frac{2}{3},\\
\frac{4}{3\sqrt{\pi}}s(1-s)^{1/2} + \frac{1}{\Delta_1}(1-s)^{1/2}, & \frac{2}{3} \le s \le 1,
\end{cases}\\
J_2(s) &= \frac{1}{\Gamma(17/4)}(1-s)^{11/12}\bigl(1-(1-s)^{7/3}\bigr) + \frac{4}{\Delta_2}g_4\left(\frac{1}{2},s\right)\\
&= \begin{cases}
\frac{1}{\Gamma(17/4)}(1-s)^{11/12}(1-(1-s)^{7/3}) + \frac{2^{1/4}}{\Delta_2\Gamma(11/4)}[(1-s)^{11/12} - (1-2s)^{7/4}], & 0 \le s < \frac{1}{2},\\
\frac{1}{\Gamma(17/4)}(1-s)^{11/12}(1-(1-s)^{7/3}) + \frac{2^{1/4}}{\Delta_2\Gamma(11/4)}(1-s)^{11/12}, & \frac{1}{2} \le s \le 1,
\end{cases}\\
J_3(s) &= \frac{1}{\Gamma(10/3)}(1-s)^{7/12}\bigl(1-(1-s)^{7/4}\bigr) + \frac{1}{\Delta_3}\left(3g_6\left(\frac{1}{4},s\right) + 2g_6\left(\frac{1}{2},s\right) + g_6\left(\frac{3}{4},s\right)\right)\\
&= \begin{cases}
\frac{1}{\Gamma(10/3)}(1-s)^{7/12}(1-(1-s)^{7/4}) + \frac{1}{2^{10/3}\Delta_3\Gamma(8/3)} & \\
\quad \times [(3 + 2^{8/3} + 3^{5/3})(1-s)^{7/12} - 3(1-4s)^{5/3} & \\
\quad - 2^{8/3}(1-2s)^{5/3} - (3-4s)^{5/3}], & 0 \le s < \frac{1}{4},\\
\frac{1}{\Gamma(10/3)}(1-s)^{7/12}(1-(1-s)^{7/4}) + \frac{1}{2^{10/3}\Delta_3\Gamma(8/3)} & \\
\quad \times [(3 + 2^{8/3} + 3^{5/3})(1-s)^{7/12} - 2^{8/3}(1-2s)^{5/3} - (3-4s)^{5/3}], & \frac{1}{4} \le s < \frac{1}{2},\\
\frac{1}{\Gamma(10/3)}(1-s)^{7/12}(1-(1-s)^{7/4}) + \frac{1}{2^{10/3}\Delta_3\Gamma(8/3)} & \\
\quad \times [(3 + 2^{8/3} + 3^{5/3})(1-s)^{7/12} - (3-4s)^{5/3}], & \frac{1}{2} \le s < \frac{3}{4},\\
\frac{1}{\Gamma(10/3)}(1-s)^{7/12}(1-(1-s)^{7/4}) & \\
\quad + \frac{1}{2^{10/3}\Delta_3\Gamma(8/3)}(3 + 2^{8/3} + 3^{5/3})(1-s)^{7/12}, & \frac{3}{4} \le s \le 1.
\end{cases}
\end{aligned}
$$

Now we choose $\sigma = \frac{1}{4} \in (0,1)$ and then $\theta = 2^{-13/2} \approx 0.01104854$. We also obtain $A = \int_{1/4}^1 J_1(s)\,ds \approx 2.42142749$, $B = \int_0^1 J_1(s)\,ds \approx 2.80487506$, $C = \int_{1/4}^1 J_2(s)\,ds \approx 0.11093116$, $D = \int_0^1 J_2(s)\,ds \approx 0.13771787$, $E = \int_{1/4}^1 J_3(s)\,ds \approx 1.49070723$, $F = \int_0^1 J_3(s)\,ds \approx 1.80167568$.

**Example 1** We consider the functions

$$
\begin{aligned}
f(t,u,v,w) &= \frac{(2t+1)[\widetilde{p}_1(u+v+w)+1](u+v+w)(\widetilde{q}_1 + \sin v)}{u+v+w+1},\\
g(t,u,v,w) &= \frac{\sqrt{t+1}[\widetilde{p}_2(u+v+w)+1](u+v+w)(\widetilde{q}_2 + \cos w)}{u+v+w+1},\\
h(t,u,v,w) &= \frac{t^2[\widetilde{p}_3(u+v+w)+1](u+v+w)(\widetilde{q}_3 + \sin u)}{u+v+w+1},
\end{aligned}
$$

for $t \in [0,1]$, $u,v,w \ge 0$, where $\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3 > 0$, $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3 > 1$.

We have $f_0^s = 3\widetilde{q}_1$, $g_0^s = \sqrt{2}(\widetilde{q}_2 + 1)$, $h_0^s = \widetilde{q}_3$, $f_\infty^i = \frac{3}{2}\widetilde{p}_1(\widetilde{q}_1 - 1)$, $g_\infty^i = \frac{\sqrt{5}}{2}\widetilde{p}_2(\widetilde{q}_2 - 1)$, $h_\infty^i = \frac{1}{16}\widetilde{p}_3(\widetilde{q}_3 - 1)$. For $\alpha_1 = \alpha_2 = \alpha_3 = \widetilde{\alpha}_1 = \widetilde{\alpha}_2 = \widetilde{\alpha}_3 = \frac{1}{3}$, we obtain $L_1 = \frac{2^{21/2}}{9\widetilde{p}_1(\widetilde{q}_1-1)A}$, $L_2 = \frac{1}{9\widetilde{q}_1 B}$, $L_3 = \frac{2^{14}}{3\sqrt{5}\widetilde{p}_2(\widetilde{q}_2-1)C}$, $L_4 = \frac{1}{3\sqrt{2}(\widetilde{q}_2+1)D}$, $L_5 = \frac{2^{91/6}}{3\widetilde{p}_3(\widetilde{q}_3-1)E}$, and $L_6 = \frac{1}{3\widetilde{q}_3 F}$.

The conditions $L_1 < L_2$, $L_3 < L_4$ and $L_5 < L_6$ become

$$\frac{\widetilde{p}_1(\widetilde{q}_1 - 1)}{\widetilde{q}_1} > \frac{2^{21/2}B}{A}, \qquad \frac{\widetilde{p}_2(\widetilde{q}_2 - 1)}{\widetilde{q}_2 + 1} > \frac{2^{29/2}D}{5^{1/2}C}, \qquad \frac{\widetilde{p}_3(\widetilde{q}_3 - 1)}{\widetilde{q}_3} > \frac{2^{91/6}F}{E}.$$

For example, if $\frac{\widetilde{p}_1(\widetilde{q}_1-1)}{\widetilde{q}_1} \geq 1678$, $\frac{\widetilde{p}_2(\widetilde{q}_2-1)}{\widetilde{q}_2+1} \geq 12865$ and $\frac{\widetilde{p}_3(\widetilde{q}_3-1)}{\widetilde{q}_3} \geq 44454$, then the above conditions are satisfied.

As an example, we consider $\widetilde{q}_1 = 2$, $\widetilde{q}_2 = 3$, $\widetilde{q}_3 = 4$, $\widetilde{p}_1 = 3356$, $\widetilde{p}_2 = 25730$, $\widetilde{p}_3 = 59272$, and then the inequalities $L_1 < L_2$, $L_3 < L_4$ and $L_5 < L_6$ are satisfied. In this case, $L_1 \approx 0.01980063$, $L_2 \approx 0.01980678$, $L_3 \approx 0.42784885$, $L_4 \approx 0.4278716$, $L_5 \approx 0.04625271$, $L_6 \approx 0.04625324$. By Theorem 3.1(1) we deduce that for every $\lambda \in (L_1, L_2)$, $\mu \in (L_3, L_4)$ and $\nu \in (L_5, L_6)$ there exists a positive solution $(u(t), v(t), w(t))$, $t \in [0,1]$ of problem $(S_0)$-$(BC_0)$.

Because $f_0^s = 3\widetilde{q}_1$, $f_\infty^s = 3\widetilde{p}_1(\widetilde{q}_1 + 1)$, $g_0^s = \sqrt{2}(\widetilde{q}_2 + 1)$, $g_\infty^s = \sqrt{2}\widetilde{p}_2(\widetilde{q}_2 + 1)$, $h_0^s = \widetilde{q}_3$, $h_\infty^s = \widetilde{p}_3(\widetilde{q}_3 + 1)$, then by Theorem 4.1 and Remark 4.2, we conclude that for any $\lambda \in (0, \lambda_0)$, $\mu \in (0, \mu_0)$ and $\nu \in (0, \nu_0)$, problem $(S_0)$-$(BC_0)$ has no positive solution, where $\lambda_0 = \frac{1}{3A_1 B}$, $\mu_0 = \frac{1}{3A_2 D}$, $\nu_0 = \frac{1}{3A_3 F}$. If we consider as above $\widetilde{p}_1 = 3356$, $\widetilde{q}_1 = 2$, $\widetilde{p}_2 = 36386$, $\widetilde{q}_2 = 3$, $\widetilde{p}_3 = 59272$, $\widetilde{q}_3 = 4$, then $A_1 = 30204$, $A_2 = 102920\sqrt{2} \approx 145551$, $A_3 = 296360$. Therefore we obtain $\lambda_0 \approx 3.9346 \times 10^{-6}$, $\mu_0 \approx 1.6629 \times 10^{-5}$, $\nu_0 \approx 6.24284 \times 10^{-7}$.

Because $f_0^i, f_\infty^i, g_0^i, g_\infty^i, h_0^i, h_\infty^i > 0$ and $f(t,u,v,w) > 0$, $g(t,u,v,w) > 0$, $h(t,u,v,w) > 0$ for all $t \in [1/4, 1]$ and $u, v, w \geq 0$ with $u+v+w > 0$, we can also apply Theorem 4.8 and Remark 4.7. Here $\hat{\lambda}_0 = \frac{1}{3\theta\sigma^{\alpha-1}m_1 A}$, $\hat{\mu}_0 = \frac{1}{3\theta\sigma^{\beta-1}m_2 C}$ and $\hat{\nu}_0 = \frac{1}{3\theta\sigma^{\gamma-1}m_3 E}$. For the functions $f, g, h$ presented above, we have $m_1 = 3$, $m_2 = 2\sqrt{5}$, $m_3 = \frac{1}{4}$, $\hat{\lambda}_0 \approx 33.22545838$, $\hat{\mu}_0 \approx 5504.275396$, $\hat{\nu}_0 \approx 2056.117822$. So, if $\lambda > 33.23$, $\mu > 5504.28$ and $\nu > 2056.12$, problem $(S_0)$-$(BC_0)$ has no positive solution.

**Example 2** We consider the functions

$$\begin{aligned} &f(t,u,v,w) = t^a\left(u^2 + v^2 + w^2\right), \qquad g(t,u,v,w) = (2-t)^b\left(e^{u+v+w} - 1\right), \\ &h(t,u,v,w) = (u+v+w)^c, \quad t \in [0,1], u, v, w \geq 0, \end{aligned}$$

where $a, b > 0$, $c > 1$. We have $f_0^s = 0$, $f_\infty^i = \infty$, $g_0^s = 2^b$, $g_\infty^i = \infty$, $h_0^s = 0$, $h_\infty^i = \infty$.

By Theorem 3.1(14), for any $\lambda \in (0, \infty)$, $\mu \in (0, \widetilde{L}_4)$ and $\nu \in (0, \infty)$, with $\widetilde{L}_4 = \frac{1}{2^b D}$, problem $(S_0)$-$(BC_0)$ has a positive solution. Here $D = \int_0^1 J_2(s)\,ds \approx 0.13771787$. For example, if $b = 2$, we obtain $\widetilde{L}_4 = \frac{1}{4D} \approx 1.8153054$.

We can also use Theorem 4.3, because $g(t,u,v,w) \geq u + v + w$ for all $t \in [1/4, 1]$ and $u, v, w \geq 0$, that is, $m_2 = 1$. Because $\widetilde{\mu}_0 = \frac{1}{\theta\sigma^{\beta-1}m_2 C} \approx 73847.6037$, we deduce that for every $\lambda > 0$, $\mu > 73847.61$ and $\nu > 0$, the boundary value problem $(S_0)$-$(BC_0)$ has no positive solution.

## 6 Conclusion

By using the Guo-Krasnosel'skii fixed point theorem, in this paper, we present conditions for the nonlinearities $f$, $g$ and $h$, and intervals for the positive parameters $\lambda$, $\mu$ and $\nu$ such that problem (S)-(BC) has positive solutions. In addition, we investigate the nonexistence of positive solutions for this problem. The novelties of our paper are the system (S) (a system with three fractional differential equations, unlike the well-studied case of a system with two equations) and the boundary conditions (BC) which, in contrast with other recent papers, contain fractional derivatives in $t = 1$ and in various intermediate points. The obtained theorems improve and extend the results from paper [2], where only a few cases are presented for the existence of positive solutions. Our results remain valid, with similar proofs, for general systems of Hammerstein integral equations of the form (34) under assumptions ($\widetilde{\mathrm{H}1}$) and ($\widetilde{\mathrm{H}2}$).

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