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Boundedness in a quasilinear fully parabolic two-species chemotaxis system of higher dimension

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Abstract

This paper considers the following coupled chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot(\phi(u)\nabla u) - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1-u-a_1 v), \\ v_t = \nabla\cdot(\psi(v)\nabla v) - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1-a_2 u - v), \\ w_t = \Delta w - \gamma w + \alpha u + \beta v, \end{cases}$$

with homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) with smooth boundaries, where χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, β and γ are positive. Based on the maximal Sobolev regularity, the existence of a unique global bounded classical solution of the problem is established under the assumption that both μ_1 and μ_2 are sufficiently large.

MSC: 92C17; 35K55; 35K35; 35B40

Keywords: boundedness; chemotaxis; two-species; quasilinear fully parabolic

1 Introduction

In this paper, we consider the higher dimension quasilinear fully parabolic two-species chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(\phi(u)\nabla u) - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1-u-a_1 v), & (x,t)\in\Omega\times(0,T), \\ v_t = \nabla\cdot(\psi(v)\nabla v) - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1-a_2 u-v), & (x,t)\in\Omega\times(0,T), \\ w_t = \Delta w - \gamma w + \alpha u + \beta v, & (x,t)\in\Omega\times(0,T), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundaries $\partial\Omega$, and the constants χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, β and γ are positive. The functions $\phi, \psi \in C^2([0,\infty))$

satisfy

$$\begin{cases} \phi(s) > 0, \quad s \geq 0, \qquad k_1 s^p \leq \phi(s), \quad s \geq s_0, \\ \psi(s) > 0, \quad s \geq 0, \qquad k_2 s^q \leq \psi(s), \quad s \geq s_0, \end{cases} \tag{1.2}$$

with $k_1, k_2 > 0$, $s_0 > 1$, $p, q \in \mathbb{R}$.

The system (1.1) arises in mathematical biology as a model for two biological species which move in the direction of higher concentration of a signal produced by themselves. Here, $u = u(x,t)$ and $v = v(x,t)$ represent the densities of the two populations, respectively, and $w = w(x,t)$ denotes the concentration of the chemical.

There are many results about the one-species chemotaxis systems with logistic source when $v \equiv 0$ in (1.1), that is,

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1-u), & (x,t) \in \Omega \times (0,T), \\ w_t = \Delta w - \gamma w + \alpha u, & (x,t) \in \Omega \times (0,T). \end{cases} \tag{1.3}$$

All solutions are global in time and remain bounded whenever $n \leq 2$ and $\mu_1 > 0$ is arbitrary [1], or $n \geq 3$ and $\mu_1 > \mu_0$ with some constant $\mu_0(\chi_1) > 0$ [2, 3]. Especially, the convexity of Ω which is required in [2] is unnecessary in [3].

As for two-species models without logistic-type growth restrictions, that is, when $\mu_1 = \mu_2 = 0$, the resulting system inherits some important properties from the original Keller-Segel model for single-species chemotaxis; see [4, 5] and the references therein. In particular, the striking phenomenon of finite-time blow-up, known to occur in both parabolic-elliptic and fully parabolic versions of the latter ([6, 7]), has also been detected in parabolic-parabolic-elliptic two-species systems ([8–12]).

Apart from the aforementioned system, a source of logistic type is included in (1.1). For the semilinear parabolic-parabolic-elliptic version of (1.1), in the case of weak competition when both $a_1 < 1$ and $a_2 < 1$, the large time behavior has been addressed in [13], and also in [14]. Here we point out that the smallness condition on the chemotactic strengths in [14] seems more natural than that in [13]. When $a_1 > 1$ and $0 \leq a_2 < 1$, it has been shown in [15] that the solution (u, v, w) converges to $(0, 1, \frac{\beta}{\gamma})$ as $t \to \infty$ under some assumptions on χ_1, χ_2, a_1, a_2. For the currently considered fully parabolic system (1.1), when $\phi(u) \equiv u$, $\psi(v) \equiv v$, the authors in [16] have proved that the system (1.1) possesses a global solution for $n \leq 2$ and any positive constant μ_1, μ_2. For the case $n \geq 3$, the large time behavior has been obtained but there lacks a proof of the existence of a global solution. Especially, the authors in [17] proved that for the bounded convex domain Ω and $\gamma \geq \frac{1}{2}$, the problem (1.1) possesses a global solution with large μ_1 and μ_2.

Our goal in this paper is to investigate the global existence and boundedness of solutions to (1.1). The main result of the present paper is the following theorem.

Theorem 1.1 *Suppose that* $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ *is a bounded domain with smooth boundaries, and* ϕ *and* ψ *satisfy* (1.2). *Then there is* $\mu_0 > 0$ *such that if* $\max\{\mu_1, \mu_2\} > \mu_0$, *for each nonnegative* $u_0(x), v_0(x) \in C^0(\bar{\Omega})$ *and* $w_0(x) \in W^{1,r}(\Omega)$ *with* $r > N$, *system* (1.1) *admits a* *unique classical solution* (u,v,w) *such that*

$$u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)),$$
$$v \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)),$$
$$w \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{\mathrm{Loc}}([0,\infty); W^{1,r}(\Omega)).$$

Moreover, (u,v,w) *is bounded in* $\Omega \times (0,\infty)$.

2 Preliminaries

The local existence of solutions to (1.1) can be addressed by methods involving standard parabolic regularity theory in a suitable fixed point approach.

Lemma 2.1 *Suppose* $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ *is a bounded domain with smooth boundaries, and* ϕ *and* ψ *satisfy* (1.2); *let* $r > N$. *Then for each nonnegative* $u_0(x), v_0(x) \in C^0(\bar{\Omega})$ *and* $w_0(x) \in W^{1,r}(\Omega)$, *there exists* $T_{\max} \in (0,\infty]$ *and a uniquely determined triple* (u,v,w) *of functions*

$$u \in C^0(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max})),$$
$$v \in C^0(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max})),$$
$$w \in C^0(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max})) \cap L^\infty_{\mathrm{Loc}}([0,T_{\max}); W^{1,r}(\Omega)),$$

which solves (1.1) *classically in* $\Omega \times (0,T_{\max})$. *Moreover,* $T_{\max} < \infty$ *if and only if*

$$\limsup_{t \nearrow T_{\max}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \right) = \infty.$$

Let us cite the following auxiliary statement from [3].

Lemma 2.2 *Let* $r \in (1,\infty)$. *Consider the following evolution equation*:

$$\begin{cases} w_t = \Delta w - \gamma w + \alpha u + \beta v, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial w}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

For each $w_0 \in W^{2,r}(\Omega)$ $(r > N)$ *with* $\frac{\partial w_0}{\partial n} = 0$ *on* $\partial\Omega$ *and any* $u, v \in L^r((0,T); L^r(\Omega))$, *there exists a unique solution*

$$w \in W^{1,r}((0,T); L^r(\Omega)) \cap L^r((0,T); W^{2,r}(\Omega)). \tag{2.2}$$

Moreover, there exists $C_r > 0$, *such that if* $s_0 \in [0,T)$, $w(\cdot,s_0) \in W^{2,r}(\Omega)$ $(r > N)$ *with* $\frac{\partial w(\cdot,s_0)}{\partial n} = 0$, *then*

$$\int_{s_0}^T \int_\Omega e^{\gamma rs} |\Delta w|^r \leq C_r \int_{s_0}^T \int_\Omega e^{\gamma rs} u^r + C_r \int_{s_0}^T \int_\Omega e^{\gamma rs} v^r + C_r \left(\|w(\cdot,s_0)\|^r_{L^r(\Omega)} + \|\Delta w(\cdot,s_0)\|^r_{L^r(\Omega)} \right). \tag{2.3}$$

Proof Let $h(x,s) = e^{\gamma s}w(x,s)$. We derive that h satisfies

$$\begin{cases} h_s(x,s) = \Delta h(x,s) + \alpha e^{\gamma s}u(x,s) + \beta e^{\gamma s}v(x,s), & (x,s) \in \Omega \times (0,T), \\ \frac{\partial h}{\partial n} = 0, & (x,s) \in \partial\Omega \times (0,T), \\ h(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

Applying the maximal Sobolev regularity ([18], Theorem 3.1) to h, and using the Hölder inequality, we have

$$\begin{aligned} \int_0^T \int_\Omega \left|\Delta h(x,s)\right|^r &\le C_r \int_0^T \int_\Omega e^{\gamma rs}u^r + C_r \int_0^T \int_\Omega e^{\gamma rs}v^r \\ &\quad + C_r\left(\|w_0\|^r_{L^r(\Omega)} + \|\Delta w_0\|^r_{L^r(\Omega)}\right). \end{aligned} \tag{2.5}$$

Consequently, for any $s_0 > 0$, replacing $v(t)$ by $v(t + s_0)$, we prove (2.3). □

The following lemma, which can be proved by applying Moser-type iteration techniques, which can be found in [19], will be used to prove global existence and boundedness:

Lemma 2.3 *Let $N \ge 1$, and suppose that there exists $k_0 \ge 1$ such that $k_0 > N/2$ and*

$$\sup_{t\in(0,T_{\max})} \left(\left\|u(\cdot,t)\right\|_{L^{k_0}(\Omega)} + \left\|v(\cdot,t)\right\|_{L^{k_0}(\Omega)}\right) < \infty. \tag{2.6}$$

Then $T_{\max} = \infty$ and

$$\sup_{t>0}\left(\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|v(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|w(\cdot,t)\right\|_{L^\infty(\Omega)}\right) < \infty. \tag{2.7}$$

3 Proof of Theorem 1.1

In this section, we prove our main result.

Lemma 3.1 *Suppose $\Omega \subset \mathbb{R}^N$ ($N \ge 3$) is a bounded domain with smooth boundaries, $\chi_1, \chi_2 \in \mathbb{R}^+$. For any $k > 1$, $\eta > 0$ and $s_0 > 0$, there exists $\mu_{k,\eta} > 0$ and $C = C(k, |\Omega|, \mu_1, \mu_2, \chi_1, \chi_2, \eta, u_0, v_0, w_0) > 0$ such that if $\min\{\mu_1, \mu_2\} > \mu_{k,\eta}$, then*

$$\left\|u(\cdot,t)\right\|_{L^k(\Omega)} + \left\|v(\cdot,t)\right\|_{L^k(\Omega)} \le C \tag{3.1}$$

for all $t \in (s_0, \infty)$.

Proof We fix $s_0 \in (0, T_{\max})$ such that $s_0 \le 1$. For any constant $k > 1$, we take u^{k-1} as a test function for the first equation in (1.1) and integrate by parts. Then we have

$$\begin{aligned} \frac{1}{k}\frac{d}{dt}\int_\Omega u^k &= -(k-1)\int_\Omega u^{k-2}\phi(u)|\nabla u|^2 + \chi_1(k-1)\int_\Omega u^{k-1}\nabla u \cdot \nabla w \\ &\quad + \mu_1 \int_\Omega u^k - \mu_1 \int_\Omega u^{k+1} - \mu_1 a_1 \int_\Omega u^k v \\ &\le \chi_1 \frac{k-1}{k}\int_\Omega \nabla u^k \cdot \nabla w + \mu_1 \int_\Omega u^k - \mu_1 \int_\Omega u^{k+1} - \mu_1 a_1 \int_\Omega u^k v \end{aligned}$$

$$
\begin{aligned}
&= -\chi_1 \frac{k-1}{k} \int_\Omega u^k \Delta w + \mu_1 \int_\Omega u^k - \mu_1 \int_\Omega u^{k+1} - \mu_1 a_1 \int_\Omega u^k v \\
&\le -\frac{\gamma(k+1)}{k} \int_\Omega u^k - \chi_1 \frac{k-1}{k} \int_\Omega u^k \Delta w \\
&\quad + \left(\mu_1 + \frac{\gamma(k+1)}{k}\right) \int_\Omega u^k - \mu_1 \int_\Omega u^{k+1}
\end{aligned}
\tag{3.2}
$$

for all $t \in (s_0, T_{\max})$. Then Young's inequality implies the following two inequalities for any $\varepsilon > 0$ (to be determined) and some constants c_1 and c_2:

$$
\left(\mu_1 + \frac{\gamma(k+1)}{k}\right) \int_\Omega u^k \le \varepsilon \int_\Omega u^{k+1} + c_1|\Omega| \tag{3.3}
$$

and

$$
-\chi_1 \frac{k-1}{k} \int_\Omega u^k \Delta w \le \chi_1 \int_\Omega u^k |\Delta w| \le \eta \int_\Omega u^{k+1} + c_2 \eta^{-k} \chi_1^{k+1} \int_\Omega |\Delta w|^{k+1}, \tag{3.4}
$$

where $c_1 = c_1(\mu_1, \varepsilon, k, \gamma) = \frac{1}{k}(1+\frac{1}{k})^{-(k+1)} \varepsilon^{-k} (\mu_1 + \frac{\gamma(k+1)}{k})^{k+1}$ and $c_2 = \sup_{k>1} \frac{1}{k}(1+\frac{1}{k})^{-(k+1)} < \infty$. By substituting (3.3) and (3.4) into (3.2), we find that

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{k}\int_\Omega u^k\right) &\le -\gamma(k+1)\left(\frac{1}{k}\int_\Omega u^k\right) - (\mu_1 - \varepsilon - \eta) \int_\Omega u^{k+1} \\
&\quad + c_2 \eta^{-k} \chi_1^{k+1} \int_\Omega |\Delta w|^{k+1} + c_1|\Omega|.
\end{aligned}
\tag{3.5}
$$

Similarly, for some constants c_3 and c_4, we have

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{k}\int_\Omega v^k\right) &\le -\gamma(k+1)\left(\frac{1}{k}\int_\Omega v^k\right) - (\mu_2 - \varepsilon - \eta) \int_\Omega v^{k+1} \\
&\quad + c_4 \eta^{-k} \chi_2^{k+1} \int_\Omega |\Delta w|^{k+1} + c_3|\Omega|.
\end{aligned}
\tag{3.6}
$$

Applying the variation-of-constants formula to the above inequalities shows that

$$
\begin{aligned}
\frac{1}{k}\int_\Omega u^k(\cdot,t) &\le e^{-\gamma(k+1)(t-s_0)} \frac{1}{k}\int_\Omega u^k(\cdot,s_0) - (\mu_1 - \varepsilon - \eta) \int_{s_0}^t e^{-\gamma(k+1)(t-s)} \int_\Omega u^{k+1} \\
&\quad + c_2 \eta^{-k} \chi_1^{k+1} \int_{s_0}^t e^{-\gamma(k+1)(t-s)} \int_\Omega |\Delta w|^{k+1} + c_1|\Omega| \int_{s_0}^t e^{-\gamma(k+1)(t-s)} \\
&\le -(\mu_1 - \varepsilon - \eta) e^{-\gamma(k+1)t} \int_{s_0}^t \int_\Omega e^{\gamma(k+1)s} u^{k+1} \\
&\quad + c_2 \eta^{-k} \chi_1^{k+1} e^{-\gamma(k+1)t} \int_{s_0}^t \int_\Omega e^{\gamma(k+1)s} |\Delta w|^{k+1} + c_5
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
\frac{1}{k}\int_\Omega v^k(\cdot,t) &\le -(\mu_2 - \varepsilon - \eta) e^{-\gamma(k+1)t} \int_{s_0}^t \int_\Omega e^{\gamma(k+1)s} v^{k+1} \\
&\quad + c_4 \eta^{-k} \chi_2^{k+1} e^{-\gamma(k+1)t} \int_{s_0}^t \int_\Omega e^{\gamma(k+1)s} |\Delta w|^{k+1} + c_6
\end{aligned}
\tag{3.8}
$$

for all $t \in (s_0, T_{\max})$, where

$$c_5 = c_1|\Omega| \int_{s_0}^{t} e^{-\gamma(k+1)(t-s)} + \frac{1}{k}\int_\Omega u^k(\cdot, s_0)$$

and

$$c_6 = c_3|\Omega| \int_{s_0}^{t} e^{-\gamma(k+1)(t-s)} + \frac{1}{k}\int_\Omega v^k(\cdot, s_0)$$

are independent of t. Now, we apply Lemma 2.2 to see that there is $C_k > 0$ such that

$$\begin{aligned}\int_{s_0}^{t}\int_\Omega e^{\gamma(k+1)s}|\Delta w|^{k+1} \le C_k \int_{s_0}^{t}\int_\Omega e^{\gamma(k+1)s}u^{k+1} + C_k \int_{s_0}^{t}\int_\Omega e^{\gamma(k+1)s}v^{k+1} \\ + C_k \big\|w(\cdot, s_0)\big\|_{W^{2,k+1}(\Omega)}^{k+1}. \end{aligned} \tag{3.9}$$

Put the inequalities (3.7) and (3.8) together and apply (3.9); then we arrive at

$$\begin{aligned}&\frac{1}{k}\left(\int_\Omega u^k(\cdot, t) + \int_\Omega v^k(\cdot, t)\right) \\ &\quad \le -\big(\mu_1 - \varepsilon - \eta - c_2\eta^{-k}\chi_1^{k+1}C_k\big)\int_{s_0}^{t}\int_\Omega e^{\gamma(k+1)s}u^{k+1} \\ &\qquad - \big(\mu_2 - \varepsilon - \eta - c_4\eta^{-k}\chi_2^{k+1}C_k\big)\int_{s_0}^{t}\int_\Omega e^{\gamma(k+1)s}v^{k+1} + c_7 \end{aligned} \tag{3.10}$$

for all $t \in (s_0, T_{\max})$, with the constant $c_7 > 0$ being independent of t.

Let $\mu_{k,\eta} = \max\{\eta + c_2\eta^{-k}\chi_1^{k+1}C_k, \eta + c_4\eta^{-k}\chi_2^{k+1}C_k\}$, which is independent of ε. We can choose $\varepsilon \in (0, \min\{\mu_1, \mu_2\} - \mu_{k,\eta})$ such that

$$\mu_1 - \varepsilon - \eta - c_2\eta^{-k}\chi_1^{k+1}C_k > 0, \qquad \mu_2 - \varepsilon - \eta - c_4\eta^{-k}\chi_2^{k+1}C_k > 0.$$

It entails

$$\frac{1}{k}\left(\int_\Omega u^k(\cdot, t) + \int_\Omega v^k(\cdot, t)\right) \le c_8 \tag{3.11}$$

for all $t \in (s_0, T_{\max})$, with the constant $c_8 = c_8(\mu_1, \varepsilon, \eta, k, \gamma, w(s_0))$ being independent of t. This completes the proof. □

In order to prove Theorem 1.1, we should give an estimation for $(u, v.w)$ when $t \in (0, s_0)$. We know by Lemma 2.1 that $u(\cdot, s_0), v(\cdot, s_0), w(\cdot, s_0) \in C^2(\bar{\Omega})$ with $\frac{\partial w(\cdot, s_0)}{\partial n} = 0$ on $\partial\Omega$, so that we can pick $M > 0$ such that

$$\begin{cases} \sup_{0\le t\le s_0}\|u(\cdot, t)\|_{L^\infty(\Omega)} \le M, & \sup_{0\le t\le s_0}\|v(\cdot, t)\|_{L^\infty(\Omega)} \le M, \\ \sup_{0\le t\le s_0}\|w(\cdot, t)\|_{L^\infty(\Omega)} \le M, & \sup_{0\le t\le s_0}\|\Delta w(\cdot, t)\|_{L^\infty(\Omega)} \le M. \end{cases} \tag{3.12}$$

Combining Lemma 3.1 with the estimates (3.12), we readily arrive at our main result.

Proof of Theorem 1.1 Let $\mu_0 = \inf_{\eta>0} \mu_{k_0,\eta}$. We know by Lemma 3.1 and (3.12) that (2.6) holds when $\min\{\mu_1, \mu_2\} > \mu_0$, and hence (2.7) is true. Lemma (2.1) shows that (u, v) is global. □

4 Conclusion

The paper considers a quasilinear fully parabolic two-species chemotaxis system of higher dimension. The existence of a unique global bounded classical solution of problem (1.1) is established under the assumption that the coefficients of the kinetic terms are large enough. We point out that the convexity of Ω and the assumption $\gamma \geq \frac{1}{2}$, which are required in [17], are unnecessary in our theorem due to the technique used here. We also notice that the result of Theorem 1.1 is independent of the value of p and q in (1.1), and thus extends the result for the semilinear case.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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