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Infinite solutions having a prescribed number of nodes for a Schrödinger problem

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Abstract

In this paper, we are concerned with the multiplicity of solutions for a Schrödinger problem. A weaker super-quadratic assumption is required for the nonlinearity. Then we give a new proof for the infinite solutions to the problem, having a prescribed number of nodes. It turns out that the weaker condition of the nonlinearity suffices to guarantee the infinitely many solutions. At the same time, a global characterization of the critical values of the non-radial nodal solutions are given.

Keywords: Schrödinger equation; infinite solutions; prescribed number of nodes; super-quadratic condition

1 Introduction

The semilinear equation

$$-\Delta u + u = f(x,u), \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

originates from various problems in physics and mathematical physics, and is called Euclidean field equation in cosmology [1], and nonlinear Klein-Gordon or Schrödinger equations when one is looking for certain types of solitary waves [2]. More generally, (1.1) can be explained as the case of $p = 2$ in a more general problem:

$$-\Delta_p u + |u|^{p-2}u = f(x,u), \quad u \in W^{1,p}(\mathbb{R}^N). \tag{1.2}$$

Since (1.1) is invariant under rotations, it is natural to search for spherically symmetric solutions. The radial solutions of (1.1) are proved by Bartsch-Willem [3] and Liu-Wang [4]. The existence question of non-radial solutions to (1.1) or (1.2) was open for a long time [5], until it was proved by Bartsch-Willem [6] and Liu-Wang [4]. Fan [7] considered $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and nonperiodic perturbations being stationary at infinity, where the perturbations are carried out not only on the coefficients but also on the exponents. Using the concentration-compactness principle, Fan proved the existence of ground state solutions vanishing at infinity under appropriate assumptions. Later Alves-Liu [8] improved the result of Fan in [7], and obtained ground states of $p(x)$-Laplacian equations in $\mathbb{R}^N$. They also established a Bartsch-Wang type compact embedding theorem for variable exponent spaces. Ayoujil [9] was concerned with the existence and multiplicity of solutions to the $p(x)$-Laplacian Steklov problem without the well-known Ambrosetti-Rabinowitz type growth conditions. By means of critical point theorems with Cerami condition, he proved the existence and multiplicity results of the solutions under weaker conditions.

For the equation

$$-\Delta_p u = f(x,u), \quad u \in W_0^{1,p}(\Omega), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, Dinca-Jebelean-Mawhin [10] obtained the existence results under Dirichlet boundary condition. Bartsch-Liu [11] proved the existence of several solutions of (1.3), that is, a pair of subsolution and supersolution, a positive and a negative solution, and a sign-changing solution. Bonanno-Candito [12] established the existence of three solutions for the Neumann boundary condition of (1.3).

In this paper, we are concerned with the multiple solutions of (1.2), and require the following assumptions on the nonlinearity $f(x,u)$:

(f_1) $f(x,0) = 0, f(x,t) = o(|t|^{p-2}t)$, as $|t| \to 0$, uniformly in x.
(f_2) $f \in C(\mathbb{R}^N, \mathbb{R})$ and there exist $C > 0$ and $q \in (p,p^*)$ such that

$$\bigl|f(x,t)\bigr| \le C\bigl(1 + |t|^{q-1}\bigr),$$

where $p^* = Np/(N-p)$ if $N > p$, and $p^* = \infty$ if $N \le p$.
(f_3) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = +\infty$, where $F(x,t) = \int_0^t f(x,s)\,ds$.
(f_4) There exists $R > 0$ such that, for any x, $\frac{f(x,t)}{|t|^{p-2}t}$ is increasing in $t \ge R$, and decreasing in $t \le -R$.

Remark 1.1 The assumption (f_3) comes from the following condition:

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = +\infty.$$

In the case $p = 2$, (f_3) characterizes problem (1.2) as superlinear at infinity. It is an extension of a very natural super-quadratic condition (SQ condition for short), SQ condition: $\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty$.

Remark 1.2 The SQ condition is weaker than the famous Ambrosetti-Rabinowitz growth condition (AR condition for short). Since the work of Ambrosetti-Rabinowitz [13], the AR condition has most frequently appeared in the superlinear elliptic boundary value problem. AR condition: There exist $\mu > p$ and $R > 0$ such that

$$0 < \mu F(x,t) \le f(x,t)t, \quad \text{for } x \in D \text{ and } |t| \ge R.$$

It is important not only in establishing the mountain-pass geometry of the functional, but also in obtaining the bounds of PS sequences. In fact, the AR condition implies that, for some $C > 0$,

$$F(x,t) \ge C|t|^{\mu}, \quad \mu > p.$$

In recent years there were some articles trying to drop the AR condition in the study of the superlinear problems. For equation (1.1), Liu-Wang [4] first posed the SQ condition to get the bounds of a minimizing sequence on the Nehari manifold. Furthermore, under coercive conditions of a potential function $V(x)$, they proved the existence of three solutions of equation $-\Delta u + V(x)u = f(x,u)$ ($u \in H^1(\mathbb{R}^N)$), one positive, one negative and one sign-changing solution. Later Li-Wang-Zeng [14] gave a natural generalization of Liu-Wang's results [4] to two noncompact cases, which do not have compact embedding. We made use of a combination of the techniques in [4] and the concentration-compactness principle of Lions [15, 16]. Then we gave general conditions which ensure the existence of ground state solutions. Miyagaki-Souto [17] established the existence of a nontrivial solution of (1.1) by combining some arguments of Struwe-Tarantello in [18]. Then Liu [19] extended the results of Miyagaki-Souto [17], and obtained the existence and multiplicity results for superlinear p-Laplacian equations (1.2) without the AR condition. To overcome the difficulty that the Palais-Smale sequences of the Euler-Lagrange functional may be unbounded, they consider the Cerami sequences.

Tan-Fang [20] considered the $p(x)$-Laplacian equations on the bounded domain and expanded a recent result [21] of Gasinski-Papageorgiou. The nonlinearity is superlinear but does not satisfy the usual AR condition near infinity, or its dual version near zero. They obtained the existence and multiplicity results via Morse theory and modified functional methods. Ge [22] dealt with the superlinear elliptic problem without Ambrosetti-Rabinowitz type growth condition in a bounded domain with smooth boundaries. He obtained the existence results of nontrivial solutions for every parameter. Using variational arguments, Carvalho-Goncalves-Silva [23] established the existence of multiple solutions for quasilinear elliptic problems driven by the Φ-Laplacian operator.

The main result of this paper is as follows.

Theorem 1.3 *Under the assumptions* (f_1)-(f_4), *for every integer* $k > 0$, *there exist a pair* u_k^+ *and* u_k^- *of radial solutions of* (1.2) *with* $u_k^-(0) < 0 < u_k^+(0)$, *having exactly k nodes*; $0 < \rho_1^\pm < \cdots < \rho_k^\pm < \infty$.

Here a node $\rho > 0$ is defined such that $u(\rho) = 0$.

Theorem 1.4 *Under the assumptions* (f_1)-(f_4) *and if* $f(x,u)$ *is odd in* u, *there exist infinitely many non-radial nodal solutions of* (1.2).

Remark 1.5 It is also possible to replace the oddness of $f(x,u)$ in Theorem 1.4 by other conditions, we refer the reader to the work of Jones-Küpper [24].

We further assume that:

(f_5) $f \in C(\mathbb{R}^N, \mathbb{R})$ and for some $C > 0$,

$$\left|f_u'(x,t)\right| \le C\left(1 + |t|^{q-2}\right),$$

where $q = p^*$ if $N \ge 3$ and $q \in (p, p^*)$ if $N = 2$.

Corollary 1.6 *Assume* $N = 4$ *or* $N \ge 6$, *the assumptions* (f_2)-(f_5) *hold, and* f *is odd in* u. *Then equation* (1.2) *has an unbounded sequence of non-radial sign-changing solutions.*

In the present paper, we give a new proof for the infinite solutions to problem (1.2) having a prescribed number of nodes, and the results are proved under the weaker SQ condition. It turns out that the SQ condition on $f(x,u)$ suffice to guarantee infinitely many solutions. Our theorems generalize the results in [4] to the case of $p \neq 2$. At the same time, a global characterization of the critical values of the nodal radial solutions are given.

2 Preliminaries

In this section, we give some notations and some preliminary lemmas, which will be adopted in Section 3.

Solutions of (1.2) correspond to the critical points of the functional

$$J(u) := \int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p - F(x,u), \quad u \in W^{1,p}\big(\mathbb{R}^N\big),$$

where $W^{1,p}(\mathbb{R}^N)$ is endowed with the norm $\|u\| = (\int_{\mathbb{R}^N}(|\nabla u|^p + |u|^p))^{\frac{1}{p}}$.

Notation 2.1 We define the Nehari manifold

$$\mathcal{N}_1 = \big\{u \in X_1 : u \neq 0, \big\langle J'(u), u\big\rangle = 0\big\},$$

where $X_1 := \{u \in W^{1,p}(\mathbb{R}^N) : u(x) = u(|x|)\}$. And

$$\mathcal{N}_2 = \big\{u \in X_2 : u \neq 0, \big\langle J'(u), u\big\rangle = 0\big\},$$

where $X_2 := W^{1,p}(\mathbb{R}^N)$. For $0 \le \rho < \sigma \le \infty$, we define

$$\begin{aligned}
&\Omega(\rho,\sigma) := \operatorname{int}\big\{x \in \mathbb{R}^N : \rho \le |x| \le \sigma\big\},\\
&X_{\rho,\sigma} := \big\{u \in W^{1,p}\big(\Omega(\rho,\sigma)\big) : u(x) = u\big(|x|\big)\big\},\\
&\mathcal{N}_{\rho,\sigma} = \big\{u \in X_{\rho,\sigma} : u \neq 0, \big\langle J'(u), u\big\rangle = 0\big\}.
\end{aligned}$$

Define $u(x) = 0$ for $x \notin \Omega(\rho,\sigma)$ if $u \in X_1$. Obviously $X_{\rho,\sigma} \subset X_1$ and $\mathcal{N}_{\rho,\sigma} \subset \mathcal{N}_1$.

We fix k, and we define

$$\begin{aligned}
\mathcal{N}_k^+ = \big\{&u \in X_1 : \text{there exist } 0 = \rho_0 < \rho_1 < \cdots < \rho_k < \rho_{k+1} = \infty, \text{such that}\\
&(-1)^j u|_{\Omega(\rho_j,\rho_{j+1})} \ge 0 \text{ and } u|_{\Omega(\rho_j,\rho_{j+1})} \in \mathcal{N}_{\rho_j,\rho_{j+1}}, \text{for } j = 0,\ldots,k\big\}.
\end{aligned}$$

On $[0,\infty) \times \mathbb{R}$, we define

$$f^+(r,u) = \begin{cases} f(r,u), & \text{if } u \ge 0,\\ -f(r,-u), & \text{if } u < 0,\end{cases}$$

and $F^+(r,u) := \int_0^u f^+(r,s)\,ds$,

$$J^+(u) := \int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p - F^+(x,u).$$

Similarly, we define

$$f^-(r,u) = \begin{cases} f(r,u), & \text{if } u \leq 0, \\ -f(r,-u), & \text{if } u > 0, \end{cases}$$

and $F^-(r,u)$, $J^-(u)$.

The letters C will always denote various universal constants.

Lemma 2.2 *Under assumptions* (f_1)-(f_4), *the equation*

$$-\Delta_p u + |u|^{p-2}u = f(x,u), \quad u \in X_{\rho,\sigma}, \tag{2.1}$$

has a weak solution u *such that*

$$J(u) = \max_{t>0} J(tu) = \inf_{v\in X_{\rho,\sigma}\setminus\{0\}} \max_{t>0} J(tv) > 0.$$

Proof By the assumptions (f_1) and (f_2), J has a strict local minimum at 0. For any $u \neq 0$, $J(tu) \to -\infty$ as $t \to \infty$. Thus

$$c := \inf_{v\in X_{\rho,\sigma}\setminus\{0\}} \max_{t>0} J(tv) > J(0) = 0 \tag{2.2}$$

is well defined.

Let $\{u_n\}$ be a minimizing sequence of c such that

$$J(u_n) = \max_{t>0} J(tu_n) \to c$$

as $n \to \infty$.

First we want to prove that $\{u_n\}$ is bounded. If not, consider $v_n := u_n/\|u_n\|$, then $\|v_n\| = 1$. By passing to a subsequence, we may assume $v_n \to v$ weakly in $X_{\rho,\sigma}$ and strongly in $L^r(X_{\rho,\sigma})$ for any $r \in [p,p^*]$. Note that (f_1) and (f_2) imply $\int_{X_{\rho,\sigma}} F(x,u)$ is weakly continuous on $X_{\rho,\sigma}$.

If $v \neq 0$, we have

$$\frac{c + o(1)}{\|u_n\|^p} = \frac{1}{p} - \int_{X_{\rho,\sigma}} \frac{F(x,u_n)}{u_n^p} v_n^p.$$

By (2.2),

$$\frac{1}{p} > \int_{X_{\rho,\sigma}} \frac{F(x,u_n)}{u_n^p} v_n^p.$$

Then by (f_3) and Fadou's lemma, passing to the limit on both sides,

$$\frac{1}{p} > \int_{X_{\rho,\sigma}} \frac{F(x,u_n(x))}{u_n^p} v^p = \infty.$$

This gives a contradiction.

If $v = 0$, fixing an $R > \sqrt[p]{pc}$, by $\|v_n\| = 1$, we have

$$J(u_n) \geq J(Rv_n) = \frac{1}{p}R^p - \int_{X_{\rho,\sigma}} F(x, Rv_n).$$

$J(u_n)$ converges towards c, but $R^p/p - \int_{X_{\rho,\sigma}} F(x, Rv_n)$ tends to $R^p/p > c$, a contradiction. Thus $\{u_n\}$ is bounded.

Assume u_n weakly converges to u. As $n \to \infty$,

$$\int_{X_{\rho,\sigma}} u_n f(x, u_n) \to \int_{X_{\rho,\sigma}} uf(x, u).$$

Since, for some $\alpha > 0$, $\|u_n\|^p > \alpha$, and

$$\|u_n\|^p = \int_{X_{\rho,\sigma}} u_n f(x, u_n),$$

so $u \neq 0$.

There is $s > 0$ such that $J(su) = \max_{t>0} J(tu)$. Then

$$J(su) \leq \liminf_{n\to\infty} J(su_n) \leq \liminf_{n\to\infty} J(u_n) = c.$$

(f_4) implies that $\max_{t>0} J(tu)$ is achieved at only one point $t = s$. It is also the unique one such that $\langle J'(tu), u\rangle = 0$.

Next we claim that su is a critical point of J. Without loss of generality, we assume $s = 1$. If u is not a critical point, there is $v \in C_0^\infty(\Omega)$ such that $\langle J'(u), v\rangle = -2$. There is $\varepsilon_0 > 0$ such that, for $|t - 1| + |\varepsilon| \leq \varepsilon_0$, $\langle J'(tu + \varepsilon v), v\rangle \leq -1$.

If $\varepsilon > 0$ is small, let $t_\varepsilon > 0$ be the unique number such that

$$\max J(tu + sv) = J(t_\varepsilon u + \varepsilon v).$$

Then $t_\varepsilon \to 1$ as $\varepsilon \to 0$.

If ε is small such that $|t_\varepsilon - 1| + \varepsilon \leq \varepsilon_0$, then $J(t_\varepsilon u + \varepsilon v) \geq c$, but by the assumption that $\langle J'(tu + \varepsilon v), v\rangle \leq -1$, so

$$J(t_\varepsilon u + \varepsilon v) = J(t_\varepsilon u) + \int_0^1 \langle J'(t_\varepsilon u + s\varepsilon v), \varepsilon v\rangle \, ds \leq c - \varepsilon < c.$$

This is a contradiction. □

Lemma 2.3 *Under assumptions* (f_2)-(f_5), *if f is odd in u, equation* (2.1) *has infinitely many pairs of solutions.*

Proof It is clear that the solutions occur in pairs due to the oddness of $f(x, u)$. Under the assumptions, any critical point of J restricted on $\mathcal{N}_2$ is a critical point of J in X_2. To verify the PS condition it suffices to show that any PS sequence is bounded. This is similar to the proof of Lemma 2.2. We omit the details.

If the PS condition is satisfied on $\mathcal{N}_2$, then the standard Ljusternik-Schnirelmann theory gives rise to an unbounded sequence of critical values of J; see the details in [25]. □

3 Proof of theorems

In this section, we prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3 First by Lemma 2.2, the infimum

$$c^+(\rho,\sigma) := \inf_{\mathcal{N}_{\rho,\sigma}} J^+$$

is achieved. Since $|u|$ is also a minimizer, we assume the minimizer u is a positive solution of the problem

$$-\Delta_p u + |u|^{p-2}u = f(x,u), \quad u \in X_{\rho,\sigma}. \tag{3.1}$$

Similarly, the infimum

$$c^-(\rho,\sigma) := \inf_{\mathcal{N}_{\rho,\sigma}} J^-$$

is also achieved by negative minimizers which are negative solutions of (3.1).

Then we work on the Nehari manifold $\mathcal{N}_k^+$, and construct a $u_k^+ \in \mathcal{N}_k^+$ such that

$$c_k^+ := \inf_{\mathcal{N}_k^+} J$$

is achieved by some u_k^+, which gives the desired solutions in Theorem 1.3.

Let $\{u_n\}$ be a minimizing sequence of c_k^+. As the same arguments hold in the proof of Lemma 2.2, $\{u_n\}$ is bounded.

Since $u_n \in \mathcal{N}_k^+$, there exist $0 = \rho_0^n < \rho_1^n < \cdots < \rho_k^n < \rho_{k+1}^n = \infty$ such that $(-1)^j u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)} \geq 0$ and $u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)} \in \mathcal{N}_{\rho_j^n,\rho_{j+1}^n}$ for $j = 0,\dots,k$.

Note that

$$\|u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)}\|^p = \int_{\Omega(\rho_j^n,\rho_{j+1}^n)} u_n f(r,u_n).$$

By (f_1)-(f_2), 0 is a strict local minimizer of J, thus there is a $\delta > 0$ such that $\|u\| \geq \delta$ for $u \in \mathcal{N}_{\rho_j^n,\rho_{j+1}^n}$. Fix $q \in (p,p^*)$, and for any $\varepsilon > 0$, there is a constant $C > 0$ such that

$$\int_{\Omega(\rho_j^n,\rho_{j+1}^n)} u_n f(r,u_n) \leq \varepsilon \int_{\Omega(\rho_j^n,\rho_{j+1}^n)} |u_n|^p + C \int_{\Omega(\rho_j^n,\rho_{j+1}^n)} |u_n|^q,$$

where $q \in (p,p^*)$. Therefore, by choosing $\varepsilon > 0$ small we find a $C > 0$ such that

$$\delta^p \leq \|u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)}\|^p \leq C \int_{\Omega(\rho_j^n,\rho_{j+1}^n)} |u_n|^q. \tag{3.2}$$

Using (3.2), in a similar way as in [3], we see that $\{\rho_{k+1}^n\}_n$ is bounded away from ∞, $\{\rho_{j+1}^n - \rho_j^n\}_n$ is bounded away from 0 for each j, and there are $0 = \rho_0 < \rho_1 < \cdots < \rho_k < \rho_{k+1} = \infty$ such that $\rho_j^n \to \rho_j$ as $n \to \infty$, for $j = 1,\dots,k$.

Along a subsequence of $\{u_n\}$, we may assume that $u_n \to u$ weakly in X_1, and strongly in $L^r(X_1)$ for any $r \in [p, p^*]$. It follows that $u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)} \to u|_{\Omega(\rho_j,\rho_{j+1})}$ weakly in X_1, and strongly in $L^r(X_1)$ ($r \in [p, p^*)$). And $(-1)^j u|_{\Omega(\rho_j,\rho_{j+1})} \geq 0$, for $u \in \mathcal{N}_{\rho_j^n,\rho_{j+1}^n}$.

Let $n \to \infty$ in (3.2). It implies that $u|_{\Omega(\rho_j,\rho_{j+1})} \not\equiv 0$. Thus we can choose an $\alpha_j > 0$ such that $\alpha_j u|_{\Omega(\rho_j,\rho_{j+1})} \in \mathcal{N}_{(\rho_j,\rho_{j+1})}$ for $j = 1, \dots, k$. Define

$$u_k^+ := \sum_{j=0}^{k} \alpha_j u|_{\Omega(\rho_j,\rho_{j+1})}.$$

By the definition of u_k^+, we observe that $u_k^+ \in \mathcal{N}_k^+$.

Next we want to show

1. c_k^+ is archived by u_k^+, that is, $J(u_k^+) = c_k^+$,
2. u_k^+ is a radial function having nodes $0 < \rho_1 < \cdots < \rho_k < \infty$,
3. u_k^+ is a solution of (1.2).

The weak convergence of $u_n|_{\Omega_{(\rho_j^n,\rho_{j+1}^n)}}$ in X_1 and strong convergence in $L^r(X_1)$ ($p < r < p^*$) imply

$$c_k^+ \leq J\bigl(u_k^+\bigr) = \sum_{j=0}^{k} J(\alpha_j u|_{\Omega(\rho_j,\rho_{j+1})}) \leq \sum_{j=0}^{k} \liminf_{n\to\infty} J(\alpha_j u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)}). \tag{3.3}$$

Also

$$\sum_{j=0}^{k} \liminf_{n\to\infty} J(u_n|_{\Omega(\rho_j^n,\rho_{j+1}^n)}) = \liminf_{n\to\infty} J(u_n) = c_k^+. \tag{3.4}$$

So $J(u_k^+) = c_k^+$.

Then the equality in (3.3) implies that $\alpha_j u|_{\Omega(\rho_j^n,\rho_{j+1}^n)}$ is a minimizer of

$$\inf_{\mathcal{N}_{\rho_j^n,\rho_{j+1}^n} \cap P^+} J^+, \quad \text{if } j \text{ is even},$$

and a minimizer of

$$\inf_{\mathcal{N}_{\rho_j^n,\rho_{j+1}^n} \cap P^-} J^-, \quad \text{if } j \text{ is odd},$$

where $P^{\pm} := \{u \in X_1 : \pm u \geq 0\}$. At the same time, $\alpha_j u|_{\Omega(\rho_j,\rho_{j+1})}$ is a minimizer of $\inf_{\mathcal{N}_{\rho_j,\rho_{j+1}}} J^{\pm}$. When j is even, $\alpha_j u|_{\Omega(\rho_j^n,\rho_{j+1}^n)}$ is a positive solution of (1.2), and when j is odd, $\alpha_j u|_{\Omega(\rho_j^n,\rho_{j+1}^n)}$ is a negative solution. Then the strong maximum principle implies that $u_k^+(0) > 0$ and $(-1)^j u_k^+(x) > 0$, for $\rho_j < |x| < \rho_{j+1}$ ($j = 0, 1, \dots, k$), and

$$(-1)^j \lim_{|x|\uparrow\rho_j} \frac{\partial u_k^+(x)}{\partial |x|} > 0, \qquad (-1)^j \lim_{|x|\downarrow\rho_j} \frac{\partial u_k^+(x)}{\partial |x|} > 0, \quad \text{for } j = 1, \dots, k.$$

So u_k^+ has exactly k nodes.

In order to prove u_k^+ is a solution of (1.2), for simplicity we assume $\alpha_j = 1$ for all j. If u_k^+ is not a critical point of J, then there is a $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\bigl\langle J'\bigl(u_k^+\bigr), \varphi\bigr\rangle = -2.$$

Observe that there is a $\tau > 0$ such that, if $|s_j - 1| \le \tau$ ($j = 0, \dots, k$) and $0 \le \varepsilon \le \tau$, the function

$$g(s,\varepsilon) := \sum_{j=0}^{k} s_j u|_{\Omega(\rho_j,\rho_{j+1})} + \varepsilon\varphi,$$

where $s = (s_1, \dots, s_k)$, has exactly k nodes $0 < \rho_1(s,\varepsilon) < \cdots < \rho_k(s,\varepsilon) < \infty$. And $\rho_j(s,\varepsilon)$ is continuous in $(s,\varepsilon) \in D \times [0,\tau]$, where $D := \{(s_1, \dots, s_k) \in \mathbb{R}^k : |s_j - 1| \le \tau\}$, and

$$\langle J'(g(s,\varepsilon)), \varphi\rangle < -1. \tag{3.5}$$

In order to deduce a contradiction, we set $s \in D$, and

$$g_1(s) = \sum_{i=0}^{k} s_i u|_{\Omega(\rho_i,\rho_{i+1})} + \tau\eta(s)\varphi,$$

where $\eta(s) : D \to [0,1]$ ($s = (s_1, \dots, s_k)$) is a cut-off function such that

$$\eta(s_1, \dots, s_k) = \begin{cases} 1, & \text{if } |s_i - 1| \le \tau/4 \text{ for all } i, \\ 0, & \text{if } |s_i - 1| \ge \tau/2 \text{ for at least one } i. \end{cases}$$

Then, for each $s \in D$, $g_1(s) \in C(D,X)$ and $g_1(s)$ has exactly k nodes $0 < \rho_1(s) < \cdots < \rho_k(s) < \infty$, where $\rho_j(s)$ is continuous.

Further, we define for $j = 1, \dots, k$,

$$h_j(s) := \langle J'(g_1(s))|_{\Omega(\rho_j(s),\rho_{j+1}(s))}, g_1(s)|_{\Omega(\rho_j(s),\rho_{j+1}(s))}\rangle.$$

And we define $h : D \to \mathbb{R}^k$ as $h(s) := (h_1(s), \dots, h_k(s))$. Then $h(s) \in C(D, \mathbb{R}^k)$.

For a fixed j, if $|s_j - 1| = \tau$ then $\eta(s) = 0$ and $\rho_i(s) = \rho_i$ for all $i = 1, \dots, k$. So by the definition of $g_1(s)$,

$$h_j(s) = \langle J'(s_j u)|_{\Omega(\rho_j,\rho_{j+1})}, s_j u|_{\Omega(\rho_j,\rho_{j+1})}\rangle = \begin{cases} > 0, & \text{if } s_j = 1 - \tau, \\ < 0, & \text{if } s_j = 1 + \tau. \end{cases}$$

Therefore, the degree $\deg(h, \operatorname{int}(D), 0)$ is well defined and $\deg(h, \operatorname{int}(D), 0) = (-1)^k$. Thus there is an $s \in \operatorname{int}(D)$ such that $h(s) = 0$, that is, $g_1(s) \in \mathcal{N}_k^+$.

It is obvious that

$$J(g_1(s)) \ge c_k^+. \tag{3.6}$$

On the other hand, by (3.5),

$$\begin{aligned} J(g_1(s)) &= J\left(\sum_{j=0}^{k} s_j u|_{\Omega(\rho_j,\rho_{j+1})}\right) + \int_0^1 \left\langle J'\left(\sum_{j=0}^{k} s_j u|_{\Omega(\rho_j,\rho_{j+1})} + \theta\tau\eta(s)\varphi\right), \tau\eta(s)\varphi\right\rangle d\theta \\ &\le J\left(\sum_{j=0}^{k} s_j u|_{\Omega(\rho_j,\rho_{j+1})}\right) - \tau\eta(s). \end{aligned}$$

If $|s_j - 1| \leq \tau/2$ for each j, then by (3.4)

$$J\big(g_1(s)\big) < J\left(\sum_{j=0}^{k} s_j u|_{\Omega(\rho_j,\rho_{j+1})}\right) \leq \sum_{j=0}^{k} J(u|_{\Omega(\rho_j,\rho_{j+1})}) = c_k^+, \tag{3.7}$$

which contradicts (3.6).

If $|s_j - 1| > \tau/2$ for at least one j, by (3.4)

$$J\big(g_1(s)\big) \leq J\left(\sum_{j=0}^{k} s_j u|_{\Omega(\rho_j,\rho_{j+1})}\right) < \sum_{j=0}^{k} J(u|_{\Omega(\rho_j,\rho_{j+1})}) = c_k^+, \tag{3.8}$$

which is a contradiction with (3.6) too. The proof is finished. □

Proof of Theorem 1.4 Using a result of Lions in [26], it is possible to find a subspace E of X_2 consisting of functions which are not radial and such that the inclusion $E \hookrightarrow L^s$ is compact for $p < s < p^*$; see the details in Theorem IV.1 of [26] or the proof of Theorem 2.1 in [6]. By Proposition 3.2 in [6], the subspace E should be chosen to satisfy the compactness. Here we describe E briefly. Let G be a group acting on X_2 via orthogonal maps $\varrho(g): X_2 \to X_2$, such that the functional J is G-invariant, and the inclusion $X_2^G \hookrightarrow L^s$ is compact for $p < s < p^*$, where $X_2^G := \{u \in X_2, \varrho(g)u = u, \text{ for all } g \in G\}$. We set $E := X_2^G$. Then we follow the same steps in Lemma 2.2, and combine with Lemma 2.3 to get the infinitely many non-radial nodal solutions of (1.2). □

4 Conclusion

We are concerned with the multiplicity of solutions for a Schrödinger problem. Based on our work [14] about (1.1), which gave a natural generalization of Liu-Wang's results [4] to two noncompact cases, we give a new proof for the infinite solutions having a prescribed number of nodes to problem (1.2) in the present paper. It turns out that the weaker condition of the nonlinearity suffices to guarantee the infinitely many solutions.

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Competing interests
The author declares that she has no competing interests.

Authors' contributions
The whole work was carried out, read and approved by the author.

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