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An evaluation of powers of the negative spectrum of Schrödinger operator equation with a singularity at zero

Ilyas Hashimoglu*

*Correspondence:
i.hasimoglu@karabuk.edu.tr
Department of Business Administration, Faculty of Business, Karabuk University, Karabuk, Turkey

Abstract

In this study, we investigate the discreteness and finiteness of the negative spectrum of the differential operator L in the Hilbert space $L_2(H, [0, \infty))$, defined as $Ly = -\frac{d^2y}{dx^2} + \frac{A(A+I)}{x^2}y - Q(x)y$, under the boundary condition $y(0) = 0$.

In the case when the negative spectrum is finite, we obtain an evaluation for the sums of powers of the absolute values of negative eigenvalues. The obtained result is applied to a class of equations of mathematical physics.

MSC: 34K08; 47A10; 35J10; 34B09

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1 Introduction

1.1 Related work

The theory of operator-differential equations with unbounded operator coefficients is a common tool for studying infinite systems of ordinary differential equations, partial differential equations, and integro-differential equations. Numerous studies have been devoted to the spectral theory of such equations.

The first significant study in this direction belongs to Kostyuchenko and Levitan [1]. They studied the asymptotic behavior of the spectrum of the Sturm-Liouville operator with the operator coefficient. Later this area was the subject of research by Gorbachuk and Gorbachuk [2, 3], Otelbaev [4], Solomyak [5], Maksudov *et al.* [6], Vladimirov [7], Aslanova [8], Bayramoglu and Aslanova [9], Gesztesy *et al.* [10], and Hashimoglu [11].

In the paper [12], Birman and Solomyak studied the negative spectrum and obtained evaluations for the number of negative eigenvalues of an ordinary second-order differential equation given on the half-axis. Asymptotic formulas for negative eigenvalues of scalar differential equations were obtained by Skachek [13, 14], Rozenblyum [15], Birman and Solomyak [16], Laptev [17], Birman and Laptev [18], Laptev and Safronov [19], Laptev and Solomyak [20].

For operator-differential equations, the negative spectrum was studied in the papers of Gasymov, Zhikov and Levitan [21], Yafaev [22], Adigezalov *et al.* [23–25], and Aslanov and Gadirli [26].

1.2 Formulation of the problem

Consider the operator-differential expression

$$\mathcal{L} = -\frac{d^2}{dx^2} + \frac{A(A+I)}{x^2} - Q(x), \tag{1}$$

where I is the identity operator, and the operators A and $Q(x)$ act in the separable Hilbert space H, and satisfy the following conditions:

(a) $A = A^* \geq 0$ and $(A+I)^{-1} \in \sigma_\infty$.

(b) For almost all x, the operator $Q(x)$ is self-adjoint, nonnegative, and Bochner-integrable.

We denote by L_2 the Hilbert space of functions with values in H defined on the half-axis, which satisfy the condition

$$\|u\|_{L_2}^2 = \int_0^\infty \|u(x)\|_H^2\,dx.$$

It is assumed that the quadratic form

$$l[u,u] = \int_0^\infty \left[\|u'\|_H^2 + \left(\frac{A(A+I)}{x^2} u, u \right)_H \right] dx - \int_0^\infty (Qu,u)_H\,dx$$

defined on smooth functions that are finite near $x = 0$ and $x = \infty$, is lower semibounded and admits a closure in L_2. We denote by L the self-adjoint operator corresponding to the closure of the quadratic form $l[u,u]$. The operator L is generated by the differential expression (1), and the boundary condition

$$u(0) = 0. \tag{2}$$

The purpose of this paper is to study the discreteness and finiteness of the negative spectrum of the operator L. To obtain them, we start from the general theorems of Birman [12] on perturbations of quadratic forms.

In the case where the negative spectrum is finite, we evaluate the sums of the powers of negative eigenvalues and apply the results to the equations of mathematical physics.

We note that in this paper the negative spectrum is evaluated for the first time in the case of finiteness of the negative spectrum for operator-differential equations.

2 Main results

2.1 Discreteness and finiteness of the negative spectrum of the Schrödinger operator equation with singularity at zero

We introduce some notation. We denote by $L_2(Q)$ the set of H-valued functions for which the seminorm

$$\|u\|_{L_2(Q)}^2 = \int_0^\infty (Qu,u)_H\,dx$$

is finite. Next, we introduce the spaces $W_2^1(A)$ and $L_2^1(A)$, which are the closures of the set of smooth, finite near $x = 0$ and $x = \infty$ H-valued functions, respectively, with metrics

$$\|u\|^2_{W_2^1(A)} = \int_0^\infty \left[\|u'\|_H^2 + \|u\|_H^2 + \left(\frac{A(A+I)}{x^2} u, u \right)_H \right] dx,$$
$$\|u\|^2_{L_2^1(A)} = \int_0^\infty \left[\|u'\|_H^2 + \left(\frac{A(A+I)}{x^2} u, u \right)_H \right] dx.$$

For the interval $[0,N]$, we denote the corresponding spaces of H-valued functions by $L_2(0,N)$, $L_2(0,N;Q)$, $W_2^1(0,N;A)$ and $L_2^1(0,N;A)$.

To prove the discreteness and finiteness of the negative part of the spectrum of the operator L we start with providing two general theorems of Birman [12] on perturbations of quadratic forms.

Theorem 1 ([12], Theorem 1.3) *Let $A \geq 0$ and $B \geq 0$ be symmetric operators in H with the common domain $D(A)$ and $C^{(\alpha)} = A - \alpha B$ ($\alpha > 0$). Then, for the operator $C^{(\alpha)}$ to be lower semibounded and for its closure $\tilde{C}^{(\alpha)}$ to have only a discrete negative spectrum for all $\alpha > 0$, it is necessary and sufficient that the form $B[u,u]$ was completely continuous in $D_1[A]$. Here $D_1[A]$ denotes a complete Hilbert space with scalar product $(u,v)_{D_1[A]} = (Au,v) + (u,v)$.*

Theorem 2 ([12], Theorem 1.4) *Let A and B be symmetric nonnegative operators in H with the common domain $D(A)$, $A[u,u] > 0$ for $u \neq 0$, and the form $B[u,u]$ admits the closure $\overline{B}[u,u]$ in semi-space H_A. Then, for the operator $C^{(\alpha)} = A - \alpha B$ to be semibounded below and for the negative spectrum of the operator $\tilde{C}^{(\alpha)}$ to be finite for all $\alpha > 0$, it is necessary and sufficient that the form $B[u,u]$ was completely continuous in H_A.*

It follows from Theorem 1 that the operator has only a discrete negative spectrum when the embedding operator from $W_2^1(A)$ into $L_2(Q)$ is completely continuous. Similarly, in order to find the finiteness condition for the negative spectrum of the operator L, by Theorem 2 it is necessary to find the condition when the embedding operator from $L_2^1(A)$ into $L_2(Q)$ is completely continuous.

The following assertion follows from the Hausdorff theorem on finite ε-nets: In order that the set U of vector-valued functions $y(x)$ be compact in the metric of $L_2(Q)$, it is necessary and sufficient that the set U be compact in the metric $L_2(0,N;Q)$, for any N, and

$$\lim_{N\to\infty} \left[\sup_{u\in U} \int_N^\infty (Qu,u)_H \, dx \right] = 0. \tag{3}$$

Lemma 1 *If conditions* (a) *and* (b) *hold for the operators A and $Q(x)$, then the embedding operator from $W_2^1(0,N;A)$ into $L_2(0,N;Q)$ is completely continuous.*

Proof Let V be the embedding operator from $W_2^1(0,N;A)$ into $L_2(0,N)$, and let B be the operator of multiplication by $Q^{\frac{1}{2}}(x)$ in $L_2(0,N)$:

$$(Bu)(x) = Q^{\frac{1}{2}}(x)u(x).$$

For the proof of Lemma 1, it is sufficient to prove the continuity of the operator BV from $W_2^1(0,N;A)$ to $L_2(0,N)$. Let us prove that V is completely continuous.

If we can prove that, for the set $Y = \{u \in Y : \int_0^N [\|u'\|_H^2 + \|u\|_H^2 + (\frac{A(A+I)}{x^2}u, u)_H]\,dx \leq 1\}$ (that is, for the unit ball), for any $\varepsilon > 0$, there exists a compact ε-net in $L_2(0,N)$, then we can conclude the complete continuity of the operator V.

Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \ldots$ be eigenvalues of the operator A, and $e_1, e_2, \ldots, e_n, \ldots$ be the corresponding orthonormal eigenvectors.

Then $u(x) = \sum_{k=1}^{\infty} u_k(x)e_k$ and $\|u\|_H^2 = \sum_{k=1}^{\infty} |u_k(x)|^2$, where $u_k(x) = (u(x), e_k)_H$.

For any $\varepsilon > 0$ there exists a number $l(\varepsilon)$ such that for $l \geq l(\varepsilon)$: $\int_0^N (\sum_{k=l+1}^{\infty} |u_k(x)|^2)\,dx < \varepsilon$. Indeed, since $\gamma_l \to \infty$ as $l \to \infty$, then for any $\varepsilon > 0$ there is a number $l(\varepsilon)$ such that for $l \geq l(\varepsilon)$:

$$\frac{\gamma_l(\gamma_l + 1)}{N^2} > \frac{1}{\varepsilon}.$$

From this, for $l \geq l(\varepsilon)$, it follows that

$$\begin{aligned}
\int_0^N \left(\sum_{k=l+1}^{\infty} \left|u_k(x)\right|^2 \right) dx &= \frac{N^2}{\gamma_l(\gamma_l + 1)} \int_0^N \left(\sum_{k=l+1}^{\infty} \frac{\gamma_l(\gamma_l + 1)}{N^2} \left|u_k(x)\right|^2 \right) dx \\
&\leq \frac{N^2}{\gamma_l(\gamma_l + 1)} \int_0^{\infty} \left[\left\|u'\right\|_H^2 + \|u\|_H^2 + \left(\frac{A(A+I)}{x^2} u, u \right)_H \right] dx \\
&\leq \frac{N^2}{\gamma_l(\gamma_l + 1)} < \varepsilon.
\end{aligned}$$

Hence, the set $E_l = (u_1(x), u_2(x), \ldots, u_l(x))$ is an ε-net for the set Y, where $u_k(x) = (u(x), e_k)_H$.

For the functions $u_k(x)$ $(k = 1, 2, \ldots, l)$, the following inequalities are satisfied:

$$\begin{aligned}
&\int_0^N \left|u_k(x)\right|^2 dx \leq \int_0^N (u, u)\,dx \leq 1, \\
&\int_0^N \left|u_k(x + \eta) - u_k(x)\right|^2 = \int_0^{\eta} \int_0^{\eta} ds\,dt \int_0^N \left|u_k'(x + s)u_k(x + t)\right| dx \\
&\qquad \leq \int_0^{\eta} \int_0^{\eta} ds\,dt \int_0^{\infty} \left\|u'(x)\right\|_H^2 dx \leq \eta^2.
\end{aligned}$$

These inequalities show that the functions $u_1(x), u_2(x), \ldots, u_l(x)$ are uniformly bounded and equicontinuous. This proves the compactness of the set E_l, that is, for a set Y, for any $\varepsilon > 0$ there exists a compact ε-net. The complete continuity of the operator V is proved.

Now we prove that the operator BV is completely continuous. The operator-valued function $Q(x)$ is Bochner-integrable; therefore, for any number $\varepsilon > 0$, it can be approximated by a finite-valued operator function such that $\|Q_\varepsilon(x)\|_H < c$ and

$$\int_0^{\infty} \left\|Q(x) - Q_\varepsilon(x)\right\|_H^2 dx < \varepsilon. \tag{4}$$

It is obvious that the operator $Q_\varepsilon(x)V$ is completely continuous. Notice that

$$\|BVu\|_{L_2(0,N)}^2 = \int_0^N \left(Q^{\frac{1}{2}}u, Q^{\frac{1}{2}}u\right)_H dx = \int_0^N (Qu, u)_H\,dx \tag{5}$$

and

$$\|BVu\|^2_{L_2(0,N)} - \|B_\varepsilon Vu\|^2_{L_2(0,N)} \le \int_0^N \left\|Q(x) - Q_\varepsilon(x)\right\|^2_H \|u\|^2_H \,dx.$$

According to (4), and the well-known inequality

$$\max_x \left\|u(x)\right\|_H \le c\|u\|_{W_2^1(0,N)} \le c\|u\|_{W_2^1(0,N;A)}$$

we obtain

$$\int_0^N \left\|Q(x) - Q_\varepsilon(x)\right\|^2_H \|u\|^2_H \,dx \le \max_{x\epsilon[0,N]} \left\|u(x)\right\|^2_H.\varepsilon \le c\varepsilon\|u\|^2_{W_2^1(0,N;A)}. \tag{6}$$

From (5) and (6) we obtain

$$\|BVu\|^2_{L_2(0,N)} \le \|B_\varepsilon Vu\|^2_{L_2(0,N)} + \varepsilon\|u\|^2_{W_2^1(0,N;A)}. \tag{7}$$

From this inequality it follows that the operator BV is completely continuous. Indeed, for any sequence of functions $u_1(x), u_2(x), \dots, u_l(x), \dots$ weakly convergent to zero in the metric $W_2^1(0,N;A)$, and with $\|u\|_{W_2^1(0,N;A)} = 1$ we obtain

$$\lim_{n\to\infty} \|BVu_n\|^2_{L_2(0,N)} \le \lim_{n\to\infty} \|B_\varepsilon Vu_n\|^2_{L_2(0,N)} + \varepsilon.$$

Taking into account that

$$\lim_{n\to\infty} \|B_\varepsilon Vu_n\|^2_{L_2(0,N)} \le \lim_{n\to\infty} \|B_\varepsilon Vu_n\|^2_{W_2^1(0,N;A)} = 0$$

we obtain

$$\lim_{n\to\infty} \|BVu_n\|^2_{L_2(0,N)} \le \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we get

$$\lim_{n\to\infty} \|BVu_n\|_{L_2(0,N)} = 0,$$

that is, the operator BV is completely continuous. The lemma is proved. □

Lemma 2 *Under the conditions of Lemma* 1, *the embedding operator from* $L_2^1(0,N;A)$ *into* $L_2(0,N;Q)$ *is completely continuous.*

The *proof* is similar to the proof of Lemma 1.

It remains to find under what conditions for $Q(x)$, condition (3) is satisfied, where U denotes the unit ball in $W_2^1(A)$, or in $L_2^1(A)$.

Theorem 3 *In order that the embedding operator from* $W_2^1(A)$ *into* $L_2^1(Q)$ *be completely continuous, it is sufficient that*

$$\lim_{x\to\infty} \left\| \int_x^{x+1} Q(t)\,dt \right\|_H = 0. \tag{8}$$

Proof Suppose that, for any $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that $\| \int_x^{x+1} Q(t)\,dt\|_H < \varepsilon$ when $x \geq N$. We divide the semi-axis $[0,\infty)$ into segments $\Omega_p = (a_p, a_{p+1})$. We define $\Phi(x) = \Phi_p(x) = \int_x^{a_{p+1}} Q(t)\,dt \geq 0$. Since everywhere on $\Omega_p : \Phi'(x) \leq 0$, *i.e.* for any $f \in H$: $(\Phi'(x)f,f)_H = \frac{d}{dx}(\Phi(x)f,f)_H \leq 0$, it follows $(\Phi(x)f,f)_H \leq (\Phi(a_p)f,f)_H$ for any $x \epsilon \Omega_p$. Then $\|\Phi(x)\|_H \leq \|\Phi(a_p)\|_H \leq \| \int_{a_p}^{a_{p+1}} Q(t)\,dt\|_H$ for any $x \epsilon \Omega_p$.

Now let us evaluate the integral $\int_{a_p}^{a_{p+1}} (Q(x)y,y)_H\,dx$ from above.

We first integrate the quadratic form by parts

$$\begin{aligned}
\int_{a_p}^{a_{p+1}} \bigl(Q(x)y,y\bigr)_H\,dx &= -\int_{a_p}^{a_{p+1}} \bigl(\Phi'(x)y,y\bigr)_H\,dx \\
&= -\bigl(\Phi(x)f,f\bigr)_H\Big|_{a_p}^{a_{p+1}} + 2\operatorname{Re}\biggl(\int_{a_p}^{a_{p+1}} \bigl(\Phi'(x)y,y'\bigr)_H\,dx\biggr) \\
&\leq \bigl\|\Phi(a_p)\bigr\|_H \cdot \bigl\|y(a_p)\bigr\|_H^2 + \bigl\|\Phi(a_p)\bigr\|_H \cdot \int_{a_p}^{a_{p+1}} \|y\|_H \bigl\|y'\bigr\|_H\,dx \\
&\leq 4\bigl\|\Phi(a_p)\bigr\|_H \cdot \int_{a_p}^{a_{p+1}} \bigl[\bigl\|y'\bigr\|_H^2 + \|y\|_H^2\bigr]\,dx \\
&\leq 4\bigl\|\Phi(a_p)\bigr\|_H \cdot \int_{a_p}^{a_{p+1}} \biggl[\bigl\|y'\bigr\|_H^2 + \biggl(\frac{A(A+I)}{x^2}y,y\biggr)_H + \|y\|_H^2\biggr]\,dx.
\end{aligned}$$

Note that in the last evaluation we used the following well-known inequality:

$$\bigl\|y(a_p)\bigr\|_H^2 \leq 2\int_{a_p}^{a_{p+1}} \bigl[\bigl\|y'\bigr\|_H^2 + \|y\|_H^2\bigr]\,dx.$$

Thus, we have proved the main inequality:

$$\begin{aligned}
&\int_{a_p}^{a_{p+1}} \bigl(Q(x)y,y\bigr)_H\,dx \\
&\quad\leq 4\biggl\|\int_{a_p}^{a_{p+1}} Q(x)\,dx\biggr\|_H \int_{a_p}^{a_{p+1}} \biggl[\bigl\|y'\bigr\|_H^2 + \biggl(\frac{A(A+I)}{x^2}y,y\biggr)_H + \|y\|_H^2\biggr]\,dx \\
&\quad\leq 4\varepsilon \int_{a_p}^{a_{p+1}} \biggl[\bigl\|y'\bigr\|_H^2 + \biggl(\frac{A(A+I)}{x^2}y,y\biggr)_H + \|y\|_H^2\biggr]\,dx.
\end{aligned}$$

Summing over all Ω_p, we obtain

$$\int_N^\infty \bigl(Q(x)y,y\bigr)_H\,dx \leq 4\varepsilon \int_N^\infty \biggl[\bigl\|y'\bigr\|_H^2 + \biggl(\frac{A(A+I)}{x^2}y,y\biggr)_H + \|y\|_H^2\biggr]\,dx.$$

Since $y \in W_2^1(A)$ and $\|y\|_{W_2^1(A)} \leq 1$, for any $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that for $N > N(\varepsilon)$: $\int_N^\infty (Q(x)y,y)_H\,dx \leq 4\varepsilon$. Hence, we get $\lim_{N\to\infty} \int_N^\infty (Q(x)y,y)_H\,dx = 0$.

Taking Lemma 1 into account, we see that the embedding operator from $W_2^1(A)$ into $L_2(Q)$ is completely continuous. The theorem is proved. □

Theorem 4 *For an imbedding operator from $L_2^1(A)$ to $L_2(Q)$ to be completely continuous, it is sufficient that*

$$\lim_{x\to\infty} x\biggl\|\int_x^\infty Q(t)\,dt\biggr\|_H = 0. \tag{9}$$

Proof Let $\Phi(x) = \int_x^\infty Q(t)\,dt$. Then $\Phi'(x) = -Q(x) \leq 0$. We evaluate the integral $\int_N^\infty (Q(x)y,y)\,dx$ from above. We integrate the quadratic form by parts. Assuming $y(x)$ to be an N-finite H-valued function, we obtain

$$\begin{aligned}
\int_N^\infty \bigl(Q(x)y,y\bigr)_H\,dx &= -\int_N^\infty \bigl(\Phi'(x)y,y\bigr)_H\,dx \\
&= -\bigl(\Phi(x)y,y\bigr)_H\big|_N^\infty + 2\operatorname{Re}\biggl(\int_N^\infty \bigl(\Phi(x)y,y'\bigr)_H\,dx\biggr) \\
&\leq 2\int_N^\infty \bigl\|\Phi(x)\bigr\|_H \|y\|_H \bigl\|y'\bigr\|_H\,dx \\
&= 2N\int_N^\infty \bigl\|x\Phi(x)\bigr\|_H x^{-1}\|y\|_H\bigl\|y'\bigr\|_H\,dx \\
&\leq 2N\max_{x\geq N}\bigl[x\bigl\|\Phi(x)\bigr\|_H\bigr]\int_N^\infty x^{-1}\|y\|_H\bigl\|y'\bigr\|_H\,dx \\
&= 2N\biggl\|\int_N^\infty Q(t)\,dt\biggr\|_H\int_N^\infty x^{-1}\|y\|_H\bigl\|y'\bigr\|_H\,dx. \qquad (10)
\end{aligned}$$

As in the scalar case, it is easy to prove the inequality

$$\int_N^\infty x^{-1}\|y\|_H\bigl\|y'\bigr\|_H\,dx \leq c\int_N^\infty \bigl\|y'\bigr\|_H^2\,dx.$$

Taking this inequality into account, from (10) we obtain

$$\int_N^\infty \bigl(Q(x)y,y\bigr)_H\,dx \leq 2cN\biggl\|\int_N^\infty Q(t)\,dt\biggr\|_H\int_N^\infty\biggl[\bigl\|y'\bigr\|_H^2 + \biggl(\frac{A(A+I)}{x^2}y,y\biggr)_H\biggr]\,dx.$$

It follows from condition (9) that, for the set $\|y\|_{L_2^1(\mathrm{A})} \leq 1$,

$$\lim_{N\to\infty}\int_N^\infty \bigl(Q(x)y,y\bigr)_H\,dx = 0.$$

Taking Lemma 2 into account, we see that the embedding operator from $L_2^1(A)$ into $L_2(Q)$ is completely continuous. The theorem is proved. □

Theorems 3 and 4, and Theorems 1 and 2 from [12] imply the following theorems.

Theorem 5 *In order to the operator L to be lower semibounded and to have only a discrete negative spectrum, it is sufficient that condition* (8) *be satisfied.*

Theorem 6 *In order to the operator L to have only a finite negative spectrum, it is sufficient that condition* (9) *be satisfied.*

These theoretical results can be applied to the study of the negative spectra in certain problems of mathematical physics.

Example 1 We denote by $L_2(D)$ the Hilbert space of measurable functions $f(x)$ such that $\int_D |f(x)|^2\,dx < \infty$.

We will assume that D is the angular region of the plane with the center at the origin and with the angular value α.

In the space $L_2(D)$ we consider the operator L generated by the differential expression

$$\mathcal{L} = -\Delta - q(x) \quad \text{for } x \in D$$

and the boundary condition

$$u\mid_{\partial D} = 0.$$

Here we assume that $q(x)$ is a positive function of x.

Let us investigate the discrete spectrum of the operator L in the space $L_2(D)$. We will do this as in [27] by reducing the equation $Lu = \lambda^2 u$ to an infinite system of ordinary differential equations.

Denote $\Phi_n(\varphi) = \sqrt{\frac{2}{\alpha}} \sin \frac{n\pi}{\alpha}\varphi$. It is easy to see that the system of functions $\{\Phi_n(\varphi)\}_{n=1}^{\infty}$ forms a complete orthonormal system in the space $L_2(0,\alpha)$. We seek the solution of the equation $Lu = \lambda^2 u$ in the form

$$u(x,\lambda) = \sum_{l=1}^{\infty} \frac{1}{\sqrt{r}} u_l(r,\lambda)\Phi_l(\varphi).$$

Substituting this value into the equation

$$\mathcal{L}u = -\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) - q(x)u = \lambda^2 u$$

we obtain

$$\begin{aligned} &-\sum_{l=1}^{\infty} u_l''(r)\Phi_l(\varphi) + \sum_{l=1}^{\infty} \frac{(\frac{\pi l}{\alpha})^2 - \frac{1}{4}}{r^2} u_l(r)\Phi_l(\varphi) - \sum_{l=1}^{\infty} q(r,\varphi)u_l(r)\Phi_l(\varphi) \\ &\quad = \lambda^2 \sum_{l=1}^{\infty} u_l(r)\Phi_l(\varphi). \end{aligned}$$

We multiply both sides of this equation by $\Phi_{l_1}(\varphi)$, and integrate over φ from 0 to α. Then

$$-u_{l_1}'' + \frac{(\frac{\pi l}{\alpha})^2 - \frac{1}{4}}{r^2} u_{l_1} - \sum_{l=1}^{\infty} u_l(r) \int_0^{\alpha} q(r,\varphi)\Phi_{l_1}(\varphi)\Phi_l(\varphi)\,d\varphi = \lambda^2 u_{l_1}(r)$$

or

$$-u_{l_1}^{II} + \frac{(\frac{\pi l}{\alpha})^2 - \frac{1}{4}}{r^2} u_{l_1} - \sum_{l=1}^{\infty} u_l(r) q_{l_1 l}(r,\alpha) = \lambda^2 u_{l_1}(r), \qquad l_1 = 1, 2, \ldots \tag{11}$$

Let us denote

$$A = \begin{bmatrix} \frac{\pi}{\alpha} - \frac{1}{2} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{2\pi}{\alpha} - \frac{1}{2} & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{n\pi}{\alpha} - \frac{1}{2} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

$$Q(r) = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} & \cdots \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & q_{n3} & \cdots & q_{nn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \qquad y = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \\ \cdots \end{bmatrix}.$$

Then the system of equations (11) takes the form

$$\begin{cases} -y'' + \frac{A(A+I)}{r^2} y - Q(r)y = \lambda^2 y, \\ y(0) = 0. \end{cases} \tag{12}$$

We introduce the Hilbert space $L_2(l_2; [0,\infty))$ consisting of all functions $f(r) = (f_1(r), f_2(r), \ldots)$ such that $\int_0^\infty \|f(r)\|_{l_2}^2 \, dr < \infty$.

The scalar product in $L_2(l_2; [0,\infty))$ is defined as

$$(f,g)_{L_2(l_2;[0,\infty))} = \int_0^\infty \bigl(f(r), g(r)\bigr)_{l_2} \, dr.$$

Let $f(x) \in L_2(D)$. We expand this function to the orthonormal system $\{\Phi_n(\varphi)\}_{n=1}^\infty$:

$$f(x) = \sum_{n=1}^\infty \frac{1}{\sqrt{r}} f_n(r) \Phi_n(\varphi),$$

where

$$\sum_{n=1}^\infty \frac{1}{\sqrt{r}} f_n(r) = \int_0^\alpha f(x) \Phi_n(\varphi) \, d\varphi.$$

It is clear that

$$\begin{aligned} \|f(x)\|_{L_2(D)}^2 &= \int_D |f(x)|^2 \, dx \\ &= \int_0^\infty r \, dr \int_0^\alpha \frac{1}{r} \left(\sum_{n=1}^\infty f_n(r) \Phi_n(\varphi), \sum_{n=1}^\infty f_n(r) \Phi_n(\varphi) \right) d\varphi \\ &= \int_0^\infty \left(\sum_{n=1}^\infty |f_n(r)|^2 \right) dr. \end{aligned}$$

From this, it can be seen that the spaces $L_2(D)$ and $L_2(l_2; [0,\infty))$ are isomorphic. Consequently, the spectra of the operators $-\Delta - q(x)$ and $-\frac{d^2}{dr^2} + \frac{A(A+I)}{r^2} - Q(r)$ coincide. Therefore, we need to investigate only the spectrum of the last operator.

Now let us find under what conditions the operator L has only a discrete negative spectrum and under what conditions it has a finite number of negative eigenvalues.

Since $f = \sum_{n=1}^{\infty} f_n \Phi_n(\varphi)$,

$$\begin{aligned}\left(\int Q(r)\,dr\right)f &= \sum_{k=1}^{\infty} f_k \int Q(r)\Phi_k(\varphi)\,dr \\ &= \sum_{k=1}^{\infty} f_k \int q(r,\varphi)\Phi_k(\varphi)\,dr \\ &= \int q(r,\varphi)\,dr \sum_{k=1}^{\infty} f_k \Phi_k(\varphi) = \left(\int q(r,\varphi)\,dr\right)\cdot f.\end{aligned}$$

From this we get

$$\begin{aligned}\left\|\int Q(r)\,dr\right\| &= \sup_{\|f\|_{L_2(0,\alpha)}=1}\left(\int Q(r)\,dr f, f\right) \\ &= \sup_{\|f\|_{L_2(0,\alpha)}=1}\int q(r,\varphi)\,dr \|f\|^2 \\ &= \sup_{0\le\varphi\le\alpha}\int q(r,\varphi)\,dr.\end{aligned}$$

Hence, by Theorem 5, the negative spectrum of L is discrete, if the following condition is satisfied:

$$\lim_{x\to\infty}\sup_{0\le\varphi\le\alpha}\int_x^{x+1} q(r,\varphi)\,dr = 0.$$

When the condition

$$\lim_{x\to\infty}\sup_{0\le\varphi\le\alpha} x\int_x^{\infty} q(r,\varphi)\,dr = 0$$

is satisfied, from Theorem 6 it follows that the negative spectrum of the operator L is finite.

2.2 Evaluation of the sums of the powers of negative eigenvalues of the Schrödinger operator equation with singularity at zero

Consider the same differential operator as in Section 1.1:

$$\begin{cases} Ly = -\frac{d^2y}{dx^2} + \frac{A(A+I)}{x^2}y - Q(x)y, \\ y(0) = 0. \end{cases}$$

We have already shown that under conditions (a)-(b) (Section 1.2) the negative part of the spectrum of the operator L is finite, if $\lim_{x\to\infty} x\|\int_x^{\infty} Q(t)\,dt\|_H = 0$.

Suppose that the following conditions are also satisfied:

(c) $\int_0^{\infty} x\|Q(x)\|\,dx < \infty$,

(d) $\int_0^{\infty} x\|Q(x)\|^{\gamma+1}\,dx < \infty$ when $\gamma > 0$.

Under conditions (a)-(c), we evaluate the number of negative eigenvalues of the operator L.

Let N be the number of negative eigenvalues of the operator L. Then this number is less than the number of eigenvalues of the form $\int_0^\infty \|Q(x)\|(y,y)\,dx$ lying to the right side of 1 in the space $L_2^1(A)$, or the number of eigenvalues μ_k of an infinite system of differential equations

$$-\frac{d^2y_k}{dx^2}+\frac{\gamma_k(\gamma_k+1)}{x^2}y_k=\mu\left\|Q(x)\right\|y_k,\quad y_k(0)=0,k=1,2,\dots$$

lying to the left side of 1, where $\gamma_1\le\gamma_2\le\cdots\le\gamma_k\le\dots$ are the eigenvalues of the operator A.

Let N_k be the number of eigenvalues of the differential equation

$$-\frac{d^2y}{dx^2}+\frac{\gamma_k(\gamma_k+1)}{x^2}y=\mu\left\|Q(x)\right\|y,\quad y(0)=0, \tag{13}$$

lying to the left side of 1, in the space $L_2[0,\infty)$. Then

$$N\le\sum_k N_k. \tag{14}$$

Let us evaluate each number N_k. Equation (13) can be reduced to the integral equation

$$\frac{1}{\mu}y(x)=\frac{1}{2\gamma_k+1}\int_0^\infty\left\|Q(t)\right\|G_k(x,t)y(t)\,dt,$$

where

$$G_k(x,t)=\begin{cases}x^{\gamma_k+1}t^{\gamma_k}, & \text{when } 0\le x<t,\\ x^{-\gamma_k}t^{\gamma_k+1}, & \text{when } 0\le t\le x.\end{cases}$$

The number N_k is less than the number of eigenvalues of the operator $M_ky=\frac{1}{2\gamma_k+1}\int_0^\infty\|Q(t)\|G_k(x,t)y(t)\,dt$, which are greater than 1. Therefore

$$N_k\le\sum_{\mu_k\le1}1=\sum_{\frac{1}{\mu_k}=\alpha_k\ge1}1=\sum_{\alpha_k\ge1}\alpha_k=\sum_{k=1}^{\infty}\alpha_k=\operatorname{tr}(M_k)=\frac{1}{2\gamma_k+1}\int_0^\infty x\left\|Q(x)\right\|dx. \tag{15}$$

Let τ be a number such that

$$2\gamma_{\tau-1}+1\le\int_0^\infty x\left\|Q(x)\right\|dx\le2\gamma_\tau+1. \tag{16}$$

Then for $k>\tau$ we get $N_k\le\operatorname{tr}(M_k)<1$.

Therefore,

$$N_k=0\quad\text{for } k>\tau. \tag{17}$$

Taking into account (14), (15), (16), and (17), we obtain

$$N \leq \sum_{k=1}^{\tau} N_k \leq \sum_{k=1}^{\tau} \operatorname{tr}(M_k) \leq \sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \| Q(x) \| \, dx.$$

This proves the following theorem.

Theorem 7 *Under conditions* (a)-(c), *the number of the negative spectrum of the operator L satisfies the following inequality*:

$$N \leq \sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \| Q(x) \| \, dx, \tag{18}$$

where the number τ satisfies the inequality (16).

To evaluate the sum of the powers of negative eigenvalues, we need to prove the following lemma.

Lemma 3 *Let B be an operator with a finite negative spectrum. We denote by N_λ the number of negative eigenvalues of the operator B smaller than $-\lambda$ ($\lambda > 0$), and by $S_\gamma(B)$ ($\gamma > 0$), the sum of the numbers $|\lambda_i|^\gamma$ taken over all negative eigenvalues of the operator B. Then*

$$S_\gamma(B) = \sum_{\lambda_i < 0} |\lambda_i|^\gamma = \gamma \int_0^\infty \lambda^{\gamma-1} N_\lambda \, d\lambda. \tag{19}$$

Proof Let the number of negative eigenvalues of B be equal to M. Then

$$N_\lambda = \begin{cases} 0, & \text{if } -\lambda \leq \lambda_1, \\ i, & \text{if } \lambda_i < -\lambda \leq \lambda_{i+1}, \\ M, & \text{if } \lambda_M < -\lambda \leq 0. \end{cases}$$

Taking into account the value of N_λ, we obtain

$$\begin{aligned} \gamma \int_0^\infty \lambda^{\gamma-1} N_\lambda \, d\lambda &= \gamma \int_0^{|\lambda_1|} \lambda^{\gamma-1} N_\lambda \, d\lambda \\ &= \gamma \sum_{i=1}^{M-1} i \int_{|\lambda_{i+1}|}^{|\lambda_i|} \lambda^{\gamma-1} \, d\lambda + \gamma M \int_0^{|\lambda_M|} \lambda^{\gamma-1} \, d\lambda \\ &= \sum_{i=1}^{M-1} i \bigl(|\lambda_i|^\gamma - |\lambda_{i+1}|^\gamma \bigr) + M |\lambda_M|^\gamma \\ &= \bigl(|\lambda_1|^\gamma - |\lambda_2|^\gamma \bigr) + 2 \bigl(|\lambda_2|^\gamma - |\lambda_3|^\gamma \bigr) + \cdots \\ &\quad + (M-1) \bigl(|\lambda_{M-1}|^\gamma - |\lambda_M|^\gamma \bigr) + M |\lambda_M|^\gamma \\ &= \sum_{i=1}^{M} |\lambda_i|^\gamma = S_\gamma(B). \end{aligned}$$

The lemma is proved. □

By Theorem 7 and Lemma 3, we evaluate $S_\gamma(L)$. Let N_λ be the number of negative eigenvalues of L less than $-\lambda$ ($\lambda > 0$). Then

$$N_\lambda \le M, \tag{20}$$

where M is the number of negative eigenvalues of the operator

$$\begin{cases} Ky = -\frac{d^2y}{dx^2} + \frac{A(A+I)}{x^2}y - (\|Q(x)\| - \lambda)_+ Iy, \\ y(0) = 0. \end{cases}$$

$(\|Q(x)\| - \lambda)_+$ is the positive part of the function $\|Q(x)\| - \lambda$.

Taking into account (18), (19), and (20), we obtain

$$\begin{aligned} S_\gamma(L) &\le \gamma \int_0^\infty \lambda^{\gamma-1} M \, d\lambda \\ &\le \gamma \int_0^\infty \lambda^{\gamma-1} \left(\sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \bigl(\| Q(x) \| - \lambda \bigr)_+ dx \right) d\lambda \\ &= \gamma \int_0^\infty \sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \bigl(\| Q(x) \| - \lambda \bigr)_+ \lambda^{\gamma-1} \, d\lambda \, dx \\ &= \gamma \sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \, dx \int_0^{\|Q(x)\|} \bigl(Q(x) - \lambda \bigr) \lambda^{\gamma-1} \, d\lambda \\ &= \frac{1}{\gamma + 1} \sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \bigl\| Q(x) \bigr\|^{\gamma+1} dx. \end{aligned}$$

Thus we have proved the following theorem.

Theorem 8 *Under conditions* (a)-(d) *the sum of the numbers* $|\lambda_i|^\gamma$, *taken over all negative eigenvalues of the operator* L, *satisfies the following inequality*:

$$S_\gamma(L) \le \frac{1}{\gamma + 1} \sum_{k=1}^{\tau} \frac{1}{2\gamma_k + 1} \int_0^\infty x \bigl\| Q(x) \bigr\|^{\gamma+1} dx, \quad \gamma > 0,$$

where the number τ *satisfies the inequality* (16).

Remark The sum of the numbers $|\lambda_i|^\gamma$, taken over all negative eigenvalues of the operator in Example 1, satisfies the inequality

$$S_\gamma(L) = \frac{\alpha}{2\pi(\gamma + 1)} \sum_{k=1}^{\tau} \frac{1}{k} \int_0^\infty r \Bigl(\sup_{0 \le \varphi \le \alpha} q(r, \varphi) \Bigr)^{\gamma+1} dr,$$

where the number τ satisfies the following inequality:

$$\tau - 1 \le \frac{\alpha}{2\pi} \int_0^\infty r \Bigl(\sup_{0 \le \varphi \le \alpha} q(r, \varphi) \Bigr) dr < \tau.$$

3 Conclusion

In this paper, we established the discreteness and finiteness of the negative spectrum for the Schrödinger operator-differential equation with a singularity at zero. Investigating the negative part of the spectrum is important for the following reasons. The study of negative eigenvalues is interesting because every negative eigenvalue of the Schrödinger operator generates a soliton solution. In addition, evaluations of the number of negative eigenvalues play an important role both in quantum mechanics and in the spectral theory of differential operators.

The results obtained in this paper can be interpreted as a generalization of the results of [22], where the Schrödinger operator-differential equation without singularities is considered. In the case where the negative spectrum is finite, estimates were first obtained for the sums of the powers of absolute values of the negative eigenvalues. The obtained results were applied to a class of equations of mathematical physics. In the future, the proposed derivations can be developed for the case where the boundary condition contains the spectral parameter.

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