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Existence of weak solutions for a class of (p,q)-Laplacian systems

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Abstract
The existence of weak solutions for a class of (p,q)-Laplacian systems is obtained by using a new linking theorem on the product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

MSC: 35D30; 35J50; 35J92

Keywords: (p,q)-Laplacian systems; compactness condition; linking theorem; nonquadraticity condition

1 Introduction and main results

In this paper, we study the existence of weak solutions for the following (p,q)-Laplacian system:

$$\begin{cases} -\Delta_p u = \frac{1}{\alpha+1} G_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \frac{1}{\beta+1} G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary in R^N ($N \geq 3$), $1 < p,q < N$, $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, the nonlinearity $G \in C(\bar{\Omega} \times R^2, R)$ has the continuous derivatives $G_s(x,s,t), G_t(x,s,t)$ with respect to s and t for any $x \in \Omega$, and there exist $p < p_1 < p^*$, $q < q_1 < q^*$, and $c_0 > 0$ such that

$$\left|G_s(x,s,t)\right| \leq c_0\left(1 + |s|^{p_1-1} + |t|^{q_1(p_1-1)/p_1}\right), \tag{2}$$

$$\left|G_t(x,s,t)\right| \leq c_0\left(1 + |s|^{p_1(q_1-1)/q_1} + |t|^{q_1-1}\right), \tag{3}$$

for any $(x,s,t) \in \Omega \times R^2$, where $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent of p, as is the case for q^*.

Let $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be the product space with the norm

$$\left\|(u,v)\right\| = \|u\|_p + \|v\|_q \quad \text{for any } (u,v) \in W,$$

where $W_0^{1,p}(\Omega)$ is the usual Banach space with the norm $\|u\|_p = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$ for any $u \in W_0^{1,p}(\Omega)$. From Sobolev's embedding theorem, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ is

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continuous and compact for any $\theta \in (1, p^*)$ and there is a constant $C = C(N, \theta, \Omega) > 0$ such that

$$\|u\|_{L^\theta} \leq C\|u\|_p \quad \forall u \in W_0^{1,p}(\Omega), \tag{4}$$

where $\|\cdot\|_{L^\theta}$ denotes the norm of $L^\theta(\Omega)$. Throughout this paper, C always denotes an embedding constant with relation to (4).

In the past decades, many authors have considered the existence and multiplicity of weak solutions for the elliptic equations and the elliptic systems by the variational method (see [1–3] for the semilinear elliptic equations, [4, 5] for the semilinear elliptic systems, [6–9] for p-Laplacian equations, [10, 11] for p-Laplacian systems, [12–16] for (p,q)-elliptic systems, and references therein). The well known mountain pass theorem, the saddle point theorem, and the linking theorem by Rabinowitz (see [17]) are the three very important abstract critical point theorems to study the existence of weak solutions for a class of semilinear elliptic equations and elliptic systems with variational structure. However, because $-\Delta_p$ is no longer a linear operator and $W_0^{1,p}(\Omega)$ is a Banach space, the saddle point theorem and the linking theorem fail for the p-Laplacian equations and p-Laplacian systems. In 2007, with the aid of the Z_2-cohomological index of Fadell and Rabinowitz, Degiovanni and Lancelotti [18] established new linking structures over cones, corresponding to the saddle point theorem and the linking theorem by Rabinowitz, which have been widely applied to investigate the existence of weak solutions for p-Laplacian equations and systems, where a set E of $W_0^{1,p}(\Omega)$ is said to be a cone, if $tu \in E$ for any $u \in E$ and $t > 0$. The problem becomes more complicated for (p,q)-Laplacian systems, but there are many papers to study the existence and multiplicity of nontrivial solutions for (p,q)-Laplacian systems (see, for example, [15, 16] for the resonance case, [12, 13] for the superquadratic case, and [14] for the critical case). Especially, in [4], under the asymptotic noncrossing conditions and the nonquadraticity conditions, Costa proved that there was at least a weak solution for the semilinear elliptic systems by using the saddle point theorem and they proved there was at least a nontrivial solution for the semilinear elliptic systems under the crossing condition by using the linking theorem. In [13], with the aid of the mountain pass theorem, Boccardo and Guedes De Figueiredo proved that there exists a nontrivial solution of system (1) with $\alpha = \beta = 0$ when the crossing condition happens at the first eigenvalue of some kind of the eigenvalue problem with the weights. In [19], Ou and Tang also proved that there was a nontrivial solution of system (1) where the nonlinearity crossed two eigenvalues of the corresponding eigenvalue problem.

Inspired by [4, 13, 18], in this paper, we will extend the linking structures over cones to the product space $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and then study the existence of weak solutions for system (1) under the asymptotic noncrossing condition and the crossing condition, respectively.

We first recall the following nonlinear eigenvalue problem (the details can be found in [17]):

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + \lambda|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ -\Delta_q v = \lambda|v|^{q-2}v + \lambda|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\alpha \geq 0$, $\beta \geq 0$ satisfy

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

Define the functionals ϕ, φ on W as follows:

$$\phi(u,v) = \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q\,dx,$$
$$\varphi(u,v) = \frac{\alpha+1}{p}\int_\Omega |u|^p\,dx + \frac{\beta+1}{q}\int_\Omega |v|^q\,dx + \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx.$$

Define the manifold

$$\Sigma = \{(u,v) \in W : \varphi(u,v) = 1\}.$$

We can easily prove that $\phi(u,v), \varphi(u,v)$ are (p,q)-homogeneous, *i.e.*,

$$\phi(t^{1/p}u, t^{1/q}v) = t\phi(u,v), \qquad \varphi(t^{1/p}u, t^{1/q}v) = t\varphi(u,v) \quad \text{for any } t > 0 \text{ and } (u,v) \in W,$$

and Σ is a symmetric nonempty manifold in W. Denote by i the Z_2-cohomological index of Fadell and Rabinowitz (see [20]) and $\mathcal{A} = \{A \subset \Sigma : A \text{ is a compact symmetric set}\}$. Define

$$\Sigma_k = \{A \in \mathcal{A} : i(A) \geq k\}.$$

The eigenvalues of problem (5) can be variationally characterized as follows:

$$\lambda_k = \inf_{A \in \Sigma_k} \sup_{(u,v)\in A} \phi(u,v),$$

where $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, $\lambda_k \to \infty$ as $k \to \infty$. It is not clear whether the sequence $\{\lambda_k\}_{k\in N}$ contains all the eigenvalues of problem (5), but we will refer to $\{\lambda_k\}_{k\in N}$ as the variational eigenvalues of problem (5). Since $\phi(u,v), \varphi(u,v)$ are (p,q)-homogeneous, the eigenfunction space corresponding to λ_k

$$E_{\lambda_k} := \{(u,v) \in W : \phi(u,v) = \lambda_k \varphi(u,v)\}$$

is not the cone, but is a (p,q)-set, that is,

$$\left(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v\right) \in E_{\lambda_k} \quad \text{for any } t \geq 0 \text{ and } (u,v) \in E_{\lambda_k}.$$

Let

$$F(s,t) := \frac{\alpha+1}{p}|s|^p + \frac{\beta+1}{q}|t|^q + |s|^{\alpha+1}|t|^{\beta+1} \quad \text{for any } (s,t) \in R^2,$$
$$H(x,s,t) := \frac{1}{p}G_s(x,s,t)s + \frac{1}{q}G_t(x,s,t)t - G(x,s,t) \quad \text{for any } (x,s,t) \in \Omega \times R^2.$$

The main results of this paper are the following theorems.

Theorem 1 *Assume* (2) *and* (3) *and suppose* $\lambda_k < \lambda_{k+1}$ *are two consecutive eigenvalues of* (5). *If the following conditions hold*:

$$\lim_{|(s,t)|\to\infty} H(x,s,t) = +\infty \quad \text{uniformly for } x \in \Omega, \tag{6}$$

$$\lambda_k < \liminf_{|(s,t)|\to\infty} \frac{G(x,s,t)}{F(s,t)} \le \limsup_{|(s,t)|\to\infty} \frac{G(x,s,t)}{F(s,t)} \le \lambda_{k+1} \quad \text{uniformly for } x \in \Omega, \tag{7}$$

then system (1) *has at least a weak solution.*

Theorem 2 *Assume* (2) *and* (3) *and suppose* $\lambda_k < \lambda_{k+1}$ *are two consecutive eigenvalues of* (5). *If the nonlinearity G satisfies the following conditions*:

$$\lim_{|(s,t)|\to\infty} H(x,s,t) = -\infty \quad \text{uniformly for } x \in \Omega, \tag{8}$$

$$\lambda_k \le \liminf_{|(s,t)|\to\infty} \frac{G(x,s,t)}{F(s,t)} \le \limsup_{|(s,t)|\to\infty} \frac{G(x,s,t)}{F(s,t)} < \lambda_{k+1} \quad \text{uniformly for } x \in \Omega, \tag{9}$$

then system (1) *has at least a weak solution.*

Let $\mu_0 = \max\{\frac{p^*(p_1-1)}{p^*-1}, \frac{(q_1-1)p_1q^*}{q_1(q^*-1)}\}$, $\nu_0 = \max\{\frac{q^*(q_1-1)}{q^*-1}, \frac{(p_1-1)q_1p^*}{p_1(p^*-1)}\}$. We assume

$$\mu > \begin{cases} \max\{\frac{(q_1-1)p_1}{q_1(p-1)}, \mu_0\} & \text{if } p_1 \le q_1, \\ \max\{\frac{p_1-1}{p-1}, \mu_0\} & \text{if } p_1 < q_1, \end{cases} \qquad \nu > \begin{cases} \max\{\frac{q_1-1}{p-1}, \nu_0\} & \text{if } p_1 \le q_1, \\ \max\{\frac{(p_1-1)q_1}{p_1(p-1)} \nu_0\} & \text{if } p_1 < q_1, \end{cases} \tag{10}$$

for the case $p \le q$, and

$$\mu > \begin{cases} \max\{\frac{(q_1-1)p_1}{q_1(q-1)}, \mu_0\} & \text{if } p_1 \le q_1, \\ \max\{\frac{p_1-1}{q-1}, \mu_0\} & \text{if } p_1 < q_1, \end{cases} \qquad \nu > \begin{cases} \max\{\frac{q_1-1}{q-1}, \nu_0\} & \text{if } p_1 \le q_1, \\ \max\{\frac{(p_1-1)q_1}{p_1(q-1)}, \nu_0\} & \text{if } p_1 < q_1, \end{cases} \tag{11}$$

for the case $p > q$.

Theorem 3 *Assume that* $\lambda_k < \lambda_{k+1}$ *are two consecutive eigenvalues of* (5) *and suppose the nonlinearity G satisfies* (2), (3), *and the crossing conditions*

$$G(x,s,t) \ge \lambda_k F(s,t) \quad \forall (x,s,t) \in \bar{\Omega} \times R^2, \tag{12}$$

$$\limsup_{|(s,t)|\to 0} \frac{G(x,s,t)}{F(s,t)} \le \alpha_0 < \lambda_{k+1} < \beta_0 \le \liminf_{|(s,t)|\to\infty} \frac{G(x,s,t)}{F(s,t)} \quad \text{uniformly for } x \in \Omega, \tag{13}$$

where α_0, β_0 *are the two constants. If one of the following conditions holds*:

$$\liminf_{|(s,t)|\to\infty} \frac{H(x,s,t)}{|s|^\mu + |t|^\nu} \ge a > 0 \quad \text{uniformly for } x \in \Omega, \tag{14}$$

or

$$\limsup_{|(s,t)|\to\infty} \frac{H(x,s,t)}{|s|^\mu + |t|^\nu} \le -a < 0 \quad \text{uniformly for } x \in \Omega, \tag{15}$$

where a is a positive constant and μ, ν *satisfy* (10) *or* (11), *then system* (1) *has at least a nontrivial weak solution in* W.

Remark The conditions (6), (8), (14), and (15) are the generalizations of the nonquadraticity conditions from the semilinear elliptic equation to the (p,q)-elliptic systems, (7) and (9) are the asymptotic noncrossing conditions, and (13) is the crossing condition, introduced by Costa and Magalhaés in [2] and used to study the existence of weak solutions for the elliptic equations and the elliptic systems (see [4, 6–9] and the references therein).

Costa and Magalhaés [2] have proved the existence of a weak solution for the semilinear elliptic equation under the nonquadraticity conditions and the asymptotic noncrossing conditions by using the saddle point theorem by Rabinowitz (see [17]) and the existence of a nontrivial solution under the nonquadraticity conditions and the crossing condition by using the linking theorem by Rabinowitz (see [17]). As for the quasilinear elliptic equation, by using an abstract critical point theory, El Amrouss and Moussaoui in [6] proved the same result with Theorem 2 with $k = 1$. Ou and Li in [8] obtained the same result with Theorem 1 by using the G-linking theorem, where the eigenvalues of $-\Delta_p$ are defined by the cogenus. In [9] Yuan and Ou proved the same conclusions as in [2] by using the linking theorem over cones by Degiovanni and Lancelotti (see [18]), where the eigenvalues of $-\Delta_p$ are defined by the Z_2-cohomological index. However, as for the (p,q)-elliptic systems, because the functional $\phi(u,v)$, $\varphi(u,v)$ are (p,q)-homogeneous, the eigenfunction space corresponding to the eigenvalue λ_k $(k \geq 1)$ of problem (5) is not a cone. Hence the linking theorem over cones cannot be applied to prove our theorems and we must prove a new linking theorem on the product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ (see Lemma 3).

As is well known, Perera and Schechter in [11, 21, 22] also proved similar results with Theorem 1 and Theorem 2 by using the notion of sandwich pairs, in [21, 22] for p-Laplacian problems and in [11] for p-Laplacian systems. However, our results are different from Theorem 4.3 of [11]. Condition (7) is about the growth of $G(x,s,t)$ when $|(s,t)| \to \infty$, while condition (4.4) of Theorem 4.3 in [11] is a global condition, that is,

$$\lambda_k J(x,u) - W(x) \leq F(x,u) \leq \lambda_{k+1} J(x,u) + W(x) \quad \forall (x,u) \in \Omega \times R^m$$

for some $W(x) \in L^1(\Omega)$. On the other hand, our conditions (6) and (8) are weaker than conditions (i) and (ii) of Lemma 4.2 of [11], which result in the $(Ce)_c$ condition and the $(PS)_c$ condition, respectively.

2 Proofs of theorems

We define the functional $J : W \to R$ as follows:

$$J(u,v) = \frac{\alpha+1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{\beta+1}{q} \int_\Omega |\nabla v|^q \, dx - \int_\Omega G(x,u,v) \, dx. \tag{16}$$

From (2) and (3), by a standard argument, the functional J is well defined and $J \in C^1(W,R)$. From the variational point of view, a weak solution of system (1) corresponds to a critical point of the functional J in W. Theorem 2 is parallel to Theorem 1, hence we will only prove Theorem 1 and Theorem 3. In the following, we will introduce an abstract critical point theorem, which is based on a compactness condition - the (PS) condition or the (Ce) condition - and on a linking structure.

Definition 1 Let X be a real Banach space. The functional I satisfies the $(PS)_c$ condition at the level $c \in R$, if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c$, $I'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence. The functional I satisfies the (PS) condition if I satisfies the $(PS)_c$ condition at any $c \in R$.

Definition 2 The functional I satisfies the $(Ce)_c$ condition at the level $c \in R$, if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c$, $(1 + \|u_n\|)\|I'(u_n)\|_{X^*} \to 0$ as $n \to \infty$ has a convergent subsequence. The functional I satisfies the (Ce) condition if I satisfies the $(Ce)_c$ condition at any $c \in R$.

The (Ce) condition was introduced by Cerami [1] and it is a weaker version of the (PS) condition. Next, we will introduce the notions of the relative homotopical linking and the relative cohomotopical linking.

Definition 3 (see [3]) Let X be a metric space and $S \subset P$ and $B \subset A$ be four subsets of X. We say that (P, S) links (A, B), if $S \cap A = B \cap P = \emptyset$ and for every deformation $\eta : P \times [0,1] \to X \backslash B$ with $\eta(S \times [0,1]) \cap A = \emptyset$ we have $\eta(P \times \{1\}) \cap A \neq \emptyset$.

Definition 4 (see [18]) Let X be a metric space and $S \subset P$ and $B \subset A$ be four subsets of X. Let m be a nonnegative integer and $\mathcal{K}$ be a field. We say that (P, S) links (A, B) cohomologically in dimension m over $\mathcal{K}$, if $S \cap A = B \cap P = \emptyset$ and the restriction homomorphism $H^m(X \backslash B, X \backslash A, \mathcal{K}) \to H^m(P, S, \mathcal{K})$ is not identically zero, where H^* denotes Alexander-Spanier cohomology (see [23]).

Following from [18], if (P, S) links (A, B) cohomologically, then (P, S) links (A, B).

Theorem A (see [3]) *Let X be a Banach space and $f \in C^1(X, R)$. Let A, B, P, S be four subsets of X with $S \subset P$ and $B \subset A$ such that (P, S) links (A, B) and*

$$\sup_S f \le \inf_A f, \qquad \sup_P f \le \inf_B f,$$

where we agree that $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$. *Define*

$$c = \inf_{\eta \in \mathcal{N}} \sup f\big(\eta\big(P \times \{1\}\big)\big),$$

where $\mathcal{N}$ is the set of deformations $\eta : P \times [0,1] \to X \backslash B$ with $\eta(S \times [0,1]) \cap A = \emptyset$. Then we have

$$\inf_A f \le c \le \sup_P f.$$

Moreover, if f satisfies the $(PS)_c$ condition or the $(Ce)_c$ condition, then c is a critical value of f.

In the following, we will introduce the examples of the relative cohomological linking on the product space $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ which are used to prove our theorems. The main ideas come from [18].

For $\lambda_k < \lambda_{k+1} (k \geq 1)$, let

$$C_{\Sigma,-} = \{(u,v) \in \Sigma : \phi(u,v) \leq \lambda_k\}, \qquad C_{\Sigma,+} = \{(u,v) \in \Sigma : \phi(u,v) \geq \lambda_{k+1}\},$$
$$C_- = \{(u,v) \in W : \phi(u,v) \leq \lambda_k \varphi(u,v)\}, \qquad C_+ = \{(u,v) \in W : \phi(u,v) \geq \lambda_{k+1} \varphi(u,v)\}.$$

It is easy to see that $C_{\Sigma,-}$, $C_{\Sigma,+}$, C_-, C_+ are the symmetric subsets such that $C_- \cap C_+ = \{(0,0)\}$ and C_-, C_+ are the (p,q)-sets. From Theorem 10.10 of [24] and Theorem 2.7 of [18], we have the following conclusion.

Lemma 1 *Let $\lambda_k < \lambda_{k+1}$ for some $k \geq 1$. Then*

$$i(C_{\Sigma,-}) = i(C_{\Sigma,+}) = k,$$
$$i(C_- \backslash \{(0,0)\}) = i(\{(u,v) \in W : \phi(u,v) < \lambda_{k+1} \varphi(u,v)\}) = i(W \setminus C_+) = k.$$

Moreover, $(W, C_- \backslash \{(0,0)\})$ links C_+ cohomologically in dimension $k = i\ (C_- \backslash \{(0,0)\})$ over Z_2.

For the sake of convenience of the reader, we introduce the following the so-called five lemma (see [23]).

Lemma 2 (Five lemma) *Given a commutative diagram of Abelian groups and homomorphisms*

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{\alpha_1} & G_2 & \xrightarrow{\alpha_2} & G_3 & \xrightarrow{\alpha_3} & G_4 & \xrightarrow{\alpha_4} & G_5 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_1 & \xrightarrow{\beta_1} & H_2 & \xrightarrow{\beta_2} & H_3 & \xrightarrow{\beta_3} & H_4 & \xrightarrow{\beta_4} & H_5 \end{array}$$

in which each row is exact and $\gamma_i : G_i \to H_i$ are isomorphisms $(i = 1,2,4,5)$, $\gamma_3 : G_3 \to H_3$ is an isomorphism.

Lemma 3 *For $r_- > 0$, let*

$$D_- = \{(u,v) \in C_- : \phi(u,v) \leq r_-\}, \qquad S_- = \{(u,v) \in C_- : \phi(u,v) = r_-\}.$$

Then (D_-, S_-) links C_+ cohomologically in dimension $k = i(C_- \backslash \{(0,0)\})$ over Z_2. Moreover, for some $(e_1, e_2) \in W \backslash C_-$, let $r_- > r_+ > 0$ and

$$Q = \{(u + te_1, v + te_2) : (u,v) \in C_-, t \geq 0, \phi(u + te_1, v + te_2) \leq r_-\},$$
$$H = \{(u + te_1, v + te_2) : (u,v) \in C_-, t \geq 0, \phi(u + te_1, v + te_2) = r_-\},$$
$$D_+ = \{(u,v) \in C_+ : \phi(u,v) \leq r_+\}, \qquad S_+ = \{(u,v) \in C_+ : \phi(u,v) = r_+\}.$$

Then $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $k = i(C_- \backslash \{(0,0)\}) + 1$ over Z_2.

Proof For the sake of convenience, we will miss the coefficient field Z_2.

(a) Since S_-, $C_-\backslash\{(0,0)\}$, D_-, and W are homotopy equivalences, from Lemma 2, the restriction homomorphism

$$H^k\big(W, C_-\backslash\{(0,0)\}\big) \to H^k(D_-, S_-)$$

is an isomorphism. Following from Lemma 1, (D_-, S_-) links C_+ cohomologically in dimension $k = i(C_-\backslash\{(0,0)\})$.

(b) Let

$$E_+ = \big\{(u,v) \in C_+ : \phi(u,v) \geq r_+\big\}.$$

Since $W\backslash E_+$ is homeomorphic with a star shaped subset of W with respect to the origin, $H^*(W, W\backslash E_+)$ is trivial. From the exact sequence of triple $(W, W\backslash E_+, W\backslash C_+)$, the restriction homomorphism

$$H^k(W, W\backslash C_+) \to H^k(W\backslash E_+, W\backslash C_+)$$

is an isomorphism. Hence, it follows from (a) that the restriction homomorphism

$$H^k(W\backslash E_+, W\backslash C_+) \to H^k(D_-, S_-) \tag{17}$$

is not identically zero.

On the other hand, since $E_+ \cap (W\backslash S_+)$ is a closed subset of $W\backslash S_+$ contained in the open set $W\backslash D_+$, we have the excision isomorphism $H^k(W\backslash S_+, W\backslash D_+) \to H^k(W\backslash E_+, W\backslash C_+)$. Following from (17), (D_-, S_-) links (D_+, S_+) cohomologically in dimension $k = i(C_-\backslash\{(0,0)\})$. Consider the diagram

$$\begin{array}{ccccc}
H^k(W, W\backslash D_+) & \longrightarrow & H^k(W\backslash S_+, W\backslash D_+) & \longrightarrow & H^{k+1}(W, W\backslash S_+) \\
\downarrow & & \downarrow & & \downarrow \\
H^k(Q,H) & \longrightarrow & H^k(D_- \cup H, H) & \longrightarrow & H^{k+1}(Q, D_- \cup H) \\
& & \downarrow & & \\
& & H^k(D_-, S_-) & &
\end{array}$$

where vertical rows are restriction homomorphisms and horizontal rows come from exact sequences of the triples $(W, W\backslash S_+, W\backslash D_+)$ and $(Q, D_- \cup H, H)$. The restriction homomorphism

$$H^k(W\backslash S_+, W\backslash D_+) \to H^k(D_- \cup H, H)$$

does the same. Let $(e_1, e_2) \notin C_-$, $(\tilde{u}, \tilde{v}) = ((1-t)u + te_1, (1-t)v + te_2)$ and define a contraction $\psi : H \times [0,1] \to H$ by

$$\psi(u,v,t) = \left(\left(\frac{r_-}{\phi(\tilde{u},\tilde{v})}\right)^{1/p} \tilde{u}, \left(\frac{r_-}{\phi(\tilde{u},\tilde{v})}\right)^{1/q} \tilde{v}\right).$$

Since Q is also contractible in itself, $H^k(Q,H)$ is trivial. Consequently, from the exactness of the second row, the map

$$H^k(D_- \cup H, H) \to H^{k+1}(Q, D_- \cup H)$$

is injective. Therefore, from the commutativity of the right square, the restriction homomorphism $H^{k+1}(W, W\backslash S_+) \to H^{k+1}(Q, D_- \cup H)$ is not identically zero. $\square$

Proof of Theorem 1

(1) The functional J satisfies the (Ce) condition. Let (u_n, v_n) be a (Ce) sequence for the functional J, *i.e.*, there is a positive constant M_0 such that

$$\left|J(u_n, v_n)\right| \le M_0 \quad \text{and} \quad \left\|J'(u_n, v_n)\right\| \left(1 + \|u_n\|_p + \|v_n\|_q\right) \to 0 \quad \text{as } n \to \infty. \tag{18}$$

Since the nonlinearity G satisfies the subcritical growth conditions (2) and (3), if (u_n, v_n) is bounded in W, by a standard argument, it follows that (u_n, v_n) converges strongly in W. Hence we only prove that (u_n, v_n) is bounded in W. Arguing by contradiction, we assume $\|(u_n, v_n)\| = \|u_n\|_p + \|v_n\|_q \to \infty$ as $n \to \infty$. Let $K_n := \|u_n\|_p^p + \|v_n\|_q^q$ and it is easy to see that $K_n \to \infty$ as $n \to \infty$. Let $\bar{u}_n = u_n \setminus K_n^{1/p}, \bar{v}_n = v_n \setminus K_n^{1/q}$. Then $(\bar{u}_n, \bar{v}_n)$ is bounded in W, *i.e.*,

$$\|\bar{u}_n\|_p^p + \|\bar{v}_n\|_q^q = 1 \quad \text{for all } n.$$

We can choose a subsequence of (u_n, v_n) if necessary, also denoted by (u_n, v_n), and there exists $(\bar{u}, \bar{v}) \in W$ such that

$$(\bar{u}_n, \bar{v}_n) \rightharpoonup (\bar{u}, \bar{v}) \quad \text{weakly in } W, \tag{19}$$

$$(\bar{u}_n, \bar{v}_n) \to (\bar{u}, \bar{v}) \quad \text{strongly in } L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega), \tag{20}$$

$$\left(\bar{u}_n(x), \bar{v}_n(x)\right) \to \left(\bar{u}(x), \bar{v}(x)\right) \quad \text{for a.e. } x \in \Omega, \tag{21}$$

where $\theta_1 \in (1, p^*)$, $\theta_2 \in (1, q^*)$. Following from (18), there is a constant $M_1 > 0$ such that

$$\begin{aligned} M_1 &\ge \liminf_{n\to\infty} J(u_n, v_n) - \left\langle J'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle \\ &= \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{p}G_s(x, u_n, v_n)u_n + \frac{1}{q}G_t(x, u_n, v_n)v_n - G(x, u_n, v_n)\right) dx \\ &= \liminf_{n\to\infty} \int_\Omega H(x, u_n, v_n)\, dx. \end{aligned} \tag{22}$$

In view of the right side of (7) and the continuity of G, for any $\varepsilon > 0$, there is $M_2 := M_2(\varepsilon) > 0$ such that

$$\left|G(x, s, t)\right| \le (\lambda_{k+1} + \varepsilon)F(s, t) + M_2 \quad \forall (x, s, t) \in \bar{\Omega} \times R^2.$$

Hence, from (16) and the Young inequality, we obtain

$$\begin{aligned} &\min\left\{\frac{\alpha+1}{p}, \frac{\beta+1}{q}\right\}\left(\|u_n\|_p^p + \|v_n\|_q^q\right) \\ &\quad \le \frac{\alpha+1}{p}\|u_n\|_p^p + \frac{\beta+1}{q}\|v_n\|_q^q \end{aligned}$$

$$\begin{aligned}
&\leq J(u_n,v_n)+\int_\Omega G(x,u_n,v_n)\,dx\\
&\leq M_3+(\lambda_{k+1}+\varepsilon)\int_\Omega\left(\frac{\alpha+1}{p}|u_n|^p+\frac{\beta+1}{q}|v_n|^q+|u_n|^{\alpha+1}|v_n|^{\beta|+1}\right)dx\\
&\leq M_3+M_4(\lambda_{k+1}+\varepsilon)\bigl(\|u_n\|_{L^p}^p+\|v_n\|_{L^q}^q\bigr),
\end{aligned}$$

where $M_3 = M_0 + M_2|\Omega|$, $M_4 = 2\max\{\frac{\alpha+1}{p},\frac{\beta+1}{q}\}$. Dividing the above inequality by K_n and letting $n\to\infty$, it follows from (20) and (21) that

$$\min\left\{\frac{\alpha+1}{p},\frac{\beta+1}{q}\right\}\leq M_4(\lambda_{k+1}+\varepsilon)\bigl(\|\bar{u}\|_{L^p}^p+\|\bar{v}\|_{L^q}^q\bigr).$$

Therefore, there exists a subset $\tilde{\Omega}$ of Ω with positive measure, such that $\bar{u}(x)\neq 0$ or $\bar{v}(x)\neq 0$ for all $x\in\tilde{\Omega}$. From the definitions of $\bar{u}_n$ and $\bar{v}_n$, we have $|(u_n(x),v_n(x))|\to\infty$ as $n\to\infty$ for any $x\in\tilde{\Omega}$. From (6), there exists a positive constant M_5 such that

$$H(x,s,t)\geq M_5\quad\text{for any }(x,s,t)\in\bar{\Omega}\times R^2.$$

From Fatou's lemma, (6), and the above inequality, we have

$$\liminf_{n\to\infty}\int_\Omega H(x,u_n,v_n)\,dx\geq\int_{\tilde{\Omega}}\liminf_{n\to\infty}H(x,u_n,v_n)\,dx+M_5|\Omega\setminus\tilde{\Omega}|=+\infty,$$

which is a contradiction to (22). Therefore, we have proved that (u_n,v_n) is bounded in W.

(2) There is a positive constant r_- such that

$$\sup_{(u,v)\in S_-}J(u,v)<\inf_{(u,v)\in C_+}J(u,v)\quad\text{and}\quad\sup_{(u,v)\in D_-}J(u,v)<+\infty,$$

where $D_-=\{(u,v)\in C_-:\phi(u,v)\leq r_-\}$, $S_-=\{(u,v)\in C_-:\phi(u,v)=r_-\}$, $C_+=\{(u,v)\in W:\phi(u,v)\geq\lambda_{k+1}\varphi(u,v)\}$. From the left side of (7) and the continuity of G, for any $\varepsilon>0$, there exists $M_6=M_6(\varepsilon)>0$ such that

$$G(x,s,t)\geq(\lambda_k+\varepsilon)F(s,t)-M_6\quad\text{for any }(x,s,t)\in\bar{\Omega}\times R^2.$$

Hence, for any $(u,v)\in C_-$, we obtain

$$\begin{aligned}
J(u,v)&\leq\frac{\alpha+1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla v|^q\,dx\\
&\quad-(\lambda_k+\varepsilon)\int_\Omega F(u,v)\,dx+M_6|\Omega|\\
&\leq-\frac{\varepsilon}{\lambda_k}\left(\frac{\alpha+1}{p}\|u\|_p^p+\frac{\beta+1}{q}\|v\|_q^q\right)+M_6|\Omega|.
\end{aligned}\tag{23}$$

On the other hand, define

$$L(x,s,t)=G(x,s,t)-\lambda_{k+1}F(s,t).$$

By a simple calculation, we have

$$\begin{aligned} H(x,s,t) &= \frac{1}{p}G_s(x,s,t)s + \frac{1}{q}G_t(x,s,t)t - G(x,s,t) \\ &= \frac{1}{p}L_s(x,s,t)s + \frac{1}{q}L_t(x,s,t)t - L(x,s,t). \end{aligned}$$

From (6), for any $M_7 > 0$, there exists a positive constant L_0 such that

$$H(x,s,t) = \frac{1}{p}G_s(x,s,t)s + \frac{1}{q}G_t(x,s,t)t - G(x,s,t) \geq M_7, \tag{24}$$

for any $x \in \Omega$ and $|(s,t)| \geq L_0$. Moreover, for any $(\tilde{s},\tilde{t}) \in R^2$ with $|(\tilde{s},\tilde{t})| = 1$, from the right side of (7), we have

$$\limsup_{\tau\to+\infty} \frac{L(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t})}{\tau} \leq 0 \quad \text{uniformly for } x \in \Omega. \tag{25}$$

Hence, for $\tau \geq L_0$, we have

$$\begin{aligned} &\frac{d}{d\tau}\left(\frac{L(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t})}{\tau}\right) \\ &\quad = \frac{1}{\tau^2}\left(\frac{1}{p}L_s\big(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t}\big)\tau^{\frac{1}{p}}\tilde{s} + \frac{1}{q}L_t\big(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t}\big)\tau^{\frac{1}{q}}\tilde{t} - L\big(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t}\big)\right) \\ &\quad \geq \frac{M_7}{\tau^2}. \end{aligned}$$

Integrating the above inequality over the interval $[T_1,T_2] \subset [L_0,+\infty)$, we obtain

$$\frac{L(x,T_2^{\frac{1}{p}}\tilde{s},T_2^{\frac{1}{q}}\tilde{t})}{T_2} - \frac{L(x,T_1^{\frac{1}{p}}\tilde{s},T_1^{\frac{1}{q}}\tilde{t})}{T_1} \geq M_7\left(\frac{1}{T_1} - \frac{1}{T_2}\right).$$

Letting $T_2 \to +\infty$, from (25), we obtain

$$L\big(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t}\big) \leq -\frac{M_7}{2} \quad \text{for any } x \in \Omega \text{ and } \tau > L_0.$$

Therefore, it follows that

$$\lim_{\tau\to+\infty} L\big(x,\tau^{\frac{1}{p}}\tilde{s},\tau^{\frac{1}{q}}\tilde{t}\big) = -\infty \quad \text{uniformly for any } x \in \Omega.$$

Picking $\tau > 0$, it follows from the above expression that, for any $(u,v) \in C_{\Sigma,+}$,

$$\begin{aligned} &J\big(\tau^{\frac{1}{p}}u,\tau^{\frac{1}{q}}v\big) \\ &\quad = \frac{\tau(\alpha+1)}{p}\int_\Omega |\nabla u|^p\,dx + \frac{\tau(\beta+1)}{q}\int_\Omega |\nabla v|^q\,dx - \int_\Omega G\big(x,\tau^{\frac{1}{p}}u,\tau^{\frac{1}{q}}v\big)\,dx \\ &\quad = \frac{\tau(\alpha+1)}{p}\int_\Omega |\nabla u|^p\,dx + \frac{\tau(\beta+1)}{q}\int_\Omega |\nabla v|^q\,dx - \lambda_{k+1}\tau\int_\Omega F(u,v)\,dx \end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \bigl(G\bigl(x,\tau^{\frac{1}{p}}u,\tau^{\frac{1}{q}}v\bigr)-\lambda_{k+1}\tau F(u,v)\bigr)\,dx\\
&\geq -\int_\Omega L\bigl(x,\tau^{\frac{1}{p}}u,\tau^{\frac{1}{p}}v\bigr)\,dx\\
&\to +\infty,
\end{aligned}
$$

as $\tau \to +\infty$. From (23) and the above inequality, for fixed $\varepsilon > 0$, there exists a positive constant r_- such that

$$\max_{(u,v)\in S_-} J(u,v) < \inf_{(u,v)\in C_+} J(u,v),$$

which together with Theorem A and Lemma 3 implies that the functional J has a critical point. It follows that Theorem 1 is proved. □

Proof of Theorem 3

(1) The functional J satisfies the (PS) condition. Without loss of generality, we consider the case (14). Let (u_n, v_n) be a (PS) sequence of the functional J, that is,

$$J(u_n,v_n)\to c\in R \quad \text{and} \quad J'(u_n,v_n)\to 0 \quad \text{as } n\to\infty. \tag{26}$$

Similar to Theorem 1, we only prove that (u_n, v_n) is bounded in W. First of all, from (14), there is a positive constant M_8 such that

$$H(x,s,t)\geq a\bigl(|s|^\mu+|t|^\nu\bigr)-M_8 \quad \forall (x,s,t)\in\Omega\times R^2.$$

Hence, it follows that

$$
\begin{aligned}
J(u_n,v_n)-\left\langle J'(u_n,v_n),\left(\frac{1}{p}u_n,\frac{1}{q}v_n\right)\right\rangle &= \int_\Omega H(x,u_n,v_n)\,dx\\
&\geq a\int_\Omega \bigl(|u_n|^\mu+|v_n|^\nu\bigr)\,dx - M_8|\Omega|.
\end{aligned}
$$

Combining (26) and the above inequality, we obtain

$$\frac{\int_\Omega (|u_n|^\mu\,dx+|v_n|^\nu)\,dx}{\|u_n\|_p+\|v_n\|_q}\to 0 \quad \text{as } n\to\infty. \tag{27}$$

From the Hölder inequality, (2), (3), and (26), we have

$$
\begin{aligned}
&\bigl\langle J'(u_n,v_n),(u_n,v_n)\bigr\rangle\\
&\quad= (\alpha+1)\|u_n\|_p^p+(\beta+1)\|v_n\|_q^q-\int_\Omega \bigl(G_s(x,u_n,v_n)u_n+G_t(x,u_n,v_n)v_n\bigr)\,dx\\
&\quad\geq (\alpha+1)\|u_n\|_p^p+(\beta+1)\|v_n\|_q^q-\left|\int_\Omega G_s(x,u_n,v_n)\cdot u_n\,dx\right|\\
&\qquad-\left|\int_\Omega G_t(x,u_n,v_n)\cdot v_n\,dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&\geq (\alpha+1)\|u_n\|_p^p + (\beta+1)\|v_n\|_q^q - c_0\int_\Omega \bigl(|u_n| + |u_n|^{p_1-1}|u_n| + |u_n||v_n|^{\frac{(p_1-1)q_1}{p_1}}\bigr)\,dx \\
&\quad - c_0\int_\Omega \bigl(|v_n| + |v_n|^{q_1-1}|v_n| + |v_n||u_n|^{\frac{(q_1-1)p_1}{q_1}}\bigr)\,dx \\
&\geq (\alpha+1)\|u_n\|_p^p + (\beta+1)\|v_n\|_q^q - c_0\|u_n\|_{L^1} \\
&\quad - c_0\biggl(\int_\Omega |u_n|^{(p_1-1)\cdot\frac{\mu}{p_1-1}}\,dx\biggr)^{\frac{p_1-1}{\mu}}\cdot\biggl(\int_\Omega |u_n|^{\frac{\mu}{\mu+1-p_1}}\,dx\biggr)^{\frac{\mu+1-p_1}{\mu}} \\
&\quad - c_0\biggl(\int_\Omega |v_n|^{\frac{(p_1-1)q_1}{p_1}\cdot\frac{p_1\nu}{(p_1-1)q_1}}\,dx\biggr)^{\frac{(p_1-1)q_1}{p_1\nu}}\cdot\biggl(\int_\Omega |u_n|^{\frac{p_1\nu}{p_1\nu+(1-p_1)q_1}}\,dx\biggr)^{\frac{p_1\nu+(1-p_1)q_1}{p_1\nu}} \\
&\quad - c_0\|v_n\|_{L^1} - c_0\biggl(\int_\Omega |v_n|^{(q_1-1)\cdot\frac{\nu}{q_1-1}}\,dx\biggr)^{\frac{q_1-1}{\nu}}\cdot\biggl(\int_\Omega |v_n|^{\frac{\nu}{\nu+1-q_1}}\,dx\biggr)^{\frac{\nu+1-q_1}{\nu}} \\
&\quad - c_0\biggl(\int_\Omega |u_n|^{\frac{(q_1-1)p_1}{q_1}\cdot\frac{q_1\mu}{(q_1-1)p_1}}\,dx\biggr)^{\frac{(q_1-1)p_1}{q_1\mu}}\cdot\biggl(\int_\Omega |v_n|^{\frac{q_1\mu}{q_1\mu+(1-q_1)p_1}}\,dx\biggr)^{\frac{q_1\mu+(1-q_1)p_1}{q_1\mu}} \\
&\geq (\alpha+1)\|u_n\|_p^p + (\beta+1)\|v_n\|_q^q - c_0C\bigl(\|u_n\|_p + \|v_n\|_q\bigr) \\
&\quad - c_0C\|u_n\|_p\bigl(\|u_n\|_{L^\mu}^{p_1-1} + \|v_n\|_{L^\nu}^{\frac{(p_1-1)q_1}{p_1}}\bigr) \\
&\quad - c_0C\|v_n\|_q\bigl(\|u_n\|_{L^\mu}^{\frac{(q_1-1)p_1}{q_1}} + \|v_n\|_{L^\nu}^{q_1-1}\bigr),
\end{aligned}
\tag{28}
$$

for all n. Here we consider the case $p < q$ and $p_1 \leq q_1$. The other cases can be proved similarly. By a simple computation, we have

$$
p_1 - 1 < \frac{(q_1-1)p_1}{q_1}, \qquad \frac{(p_1-1)q_1}{p_1} < q_1 - 1,
$$

and we assume that

$$
\|u_n\|_{L^\mu}^{p_1-1} < \|u_n\|_{L^\mu}^{\frac{(q_1-1)p_1}{q_1}}, \qquad \|v_n\|_{L^\nu}^{\frac{(p_1-1)q_1}{p_1}} < \|v_n\|_{L^\nu}^{q_1-1}.
$$

Then in view of $\mu > \frac{(q_1-1)p_1}{q_1}\frac{1}{p-1}$ and (27), we have

$$
\begin{aligned}
&\frac{\|u_n\|_p\|u_n\|_{L^\mu}^{p_1-1} + \|v_n\|_q\|u_n\|_{L^\mu}^{\frac{(q_1-1)p_1}{q_1}}}{\|u_n\|_p^p + \|v_n\|_q^q} \\
&\quad\leq \frac{2^{p-1}}{2^{p-1}}\frac{(\|u_n\|_p + \|v_n\|_q)\|u_n\|_{L^\mu}^{\frac{(q_1-1)p_1}{q_1}}}{\|u_n\|_p^p + \|v_n\|_q^p} \\
&\quad= 2^{p-1}\frac{\|u_n\|_{L^\mu}^{\frac{(q_1-1)p_1}{q_1}}}{(\|u_n\|_p + \|v_n\|_q)^{p-1}} \\
&\quad\to 0 \quad \text{as } n\to\infty.
\end{aligned}
\tag{29}
$$

Similarly, we have

$$\frac{\|u_n\|_p \|v_n\|_{L^\nu}^{\frac{(p_1-1)q_1}{p_1}} + \|v_n\|_q \|v_n\|_{L^\nu}^{q_1-1}}{\|u_n\|_p^p + \|v_n\|_q^q} \to 0 \quad \text{as } n \to \infty.$$

Hence, following from (28), (29), and the above limit, we see that $\{(u_n, v_n)\}$ is bounded in W.

(2) For some $(e_1, e_2) \in W \setminus C_-$, there are two positive constants $r_- > r_+$ such that

$$\sup_{(u,v)\in D_-\cup H} J(u,v) < \inf_{(u,v)\in S_+} J(u,v)$$

and

$$\sup_{(u,v)\in Q} J(u,v) < +\infty,$$

where $Q = \{(u + te_1, v + te_2) : (u, v) \in C_-, t \geq 0, \phi(u + te_1, v + te_2) \leq r_-\}$, $D_- \cup H = \{(u, v) \in C_- : \phi(u, v) \leq r_-\} \cup \{(u + te_1, v + te_2) : (u, v) \in C_-, t > 0, \phi(u + te_1, v + te_2) = r_-\}$, $S_+ = \{(u, v) \in C_+ : \phi(u, v) = r_+\}$.

From (12) and (16), for any $(u, v) \in C_-$, we obtain

$$J(u,v) \leq \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q\,dx - \lambda_k \int_\Omega F(u,v)\,dx \leq 0. \tag{30}$$

Picking a constant $\hat{\beta}$ such that $\lambda_{k+1} < \hat{\beta} < \beta_0$, it follows from the right side of (13) that there exists a positive constant M_9 such that

$$G(x,s,t) \geq \hat{\beta}F(s,t) - M_9, \quad \forall (x,s,t) \in \Omega \times R^2.$$

Letting $(e_1, e_2) \in W\setminus C_-$, it follows from (16) and the above inequality that, for all $(u, v) \in \{(u + te_1, v + te_2) : (u, v) \in C_-, t > 0\}$,

$$\begin{aligned} J(u,v) &\leq \frac{\alpha+1}{p}\|u\|_p^p + \frac{\beta+1}{q}\|u\|_q^q - \hat{\beta}\int_\Omega F(u,v)\,dx + M_9|\Omega| \\ &\leq \frac{\lambda_{k+1} - \hat{\beta}}{\lambda_{k+1}}\left(\frac{\alpha+1}{p}\|u\|_p^p + \frac{\beta+1}{q}\|u\|_q^q\right) + M_9|\Omega|, \end{aligned}$$

which together with (30) shows there is $r_- > 0$ such that

$$J(u,v) \leq 0,$$

for any $(u, v) \in D_- \cup H = \{(u, v) \in C_- : \phi(u, v) \leq r_-\} \cup \{(u + te_1, v + te_2) : (u, v) \in C_-, t > 0, \phi(u + te_1, v + te_2) = r_-\}$.

On the other hand, choosing a constant $\hat{\alpha}$ such that $\alpha_0 < \hat{\alpha} < \lambda_{k+1}$, it follows from the left side of (13) that there is $\delta > 0$ such that

$$G(x,s,t) \leq \hat{\alpha}F(s,t), \quad \forall x \in \Omega, \left|(s,t)\right| \leq \delta. \tag{31}$$

From (2), (3), and the Young inequality, for any $(s,t)\in R^2$, we have

$$
\begin{aligned}
\bigl|G(x,s,t)\bigr| &= \left|\int_0^1 \bigl(G_s(x,rs,rt)s + G_t(x,rs,rt)t\bigr)\,dr\right| \\
&\le \int_0^1 \bigl(\bigl|G_s(x,rs,rt)\bigr||s| + G_t(x,rs,rt)||t|\bigr)\,dr \\
&\le c_0\int_0^1 \bigl(|s| + |s|^{p_1}r^{p_1-1} + |rt|^{\frac{q_1(p_1-1)}{p_1}}|s| \\
&\quad + |t| + |t|^{q_1}r^{q_1-1} + |rs|^{\frac{p_1(q_1-1)}{q_1}}|t|\bigr)\,dr \\
&\le c_0\biggl(|s| + |t| + \frac{1}{p_1}|s|^{p_1} + \frac{1}{q_1}|t|^{q_1} \\
&\quad + \frac{p_1}{p_1+q_1(p_1-1)}|t|^{\frac{q_1(p_1-1)}{p_1}}|s| + \frac{q_1}{p_1(q_1-1)+q_1}|s|^{\frac{p_1(q_1-1)}{q_1}}|t|\biggr) \\
&\le M_{10}\bigl(|s| + |t| + |s|^{p_1} + |t|^{q_1}\bigr),
\end{aligned}
$$

where M_{10} is a positive constant independent of (s,t). From (31) and the above inequality, there is a positive constant M_{11} such that

$$G(x,s,t)\le \hat{\alpha}F(s,t) + M_{11}\bigl(|s|^{p_1} + |t|^{q_1}\bigr) \quad \text{for all } (x,s,t)\in\Omega\times R^2.$$

Hence, from the above inequality, for any $(u,v)\in C_+$, we have

$$
\begin{aligned}
J(u,v) &\ge \frac{\alpha+1}{p}\|u\|_p^p + \frac{\beta+1}{q}\|v\|_q^q - \hat{\alpha}\int_\Omega F(u,v)\,dx - M_{11}\bigl(\|u\|_{L^{p_1}}^{p_1} + \|v\|_{L^{q_1}}^{q_1}\bigr) \\
&\ge \frac{\lambda_{k+1}-\hat{\alpha}}{\lambda_{k+1}}\left(\frac{\alpha+1}{p}\|u\|_p^p + \frac{\beta+1}{q}\|u\|_q^q\right) - CM_{11}\bigl(\|u\|_p^{p_1} + \|v\|_q^{q_1}\bigr).
\end{aligned}
$$

In view of $\lambda_{k+1} > \hat{\alpha}$ and $p < p_1$, $q < q_1$, there is a positive constant $r_- > r_+ > 0$ such that

$$J(u,v) > 0$$

for any $(u,v)\in S_+ = \{(u,v)\in C_+ : \phi(u,v) = r_+\}$.

Finally, from Theorem A and Lemma 3, Theorem 3 is proved. □

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