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Analysis of a class of nonlinear fractional differential models generated by impulsive effects

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Abstract

The purpose of this paper is to analyze a class of nonlinear fractional differential models generated by impulsive effects. Based on variational methods, some new results are obtained in regard to the existence of solutions. Moreover, our results enrich and extend some existing conclusions.

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Keywords: fractional applied differential model; impulsive effect; boundary value problem; p -Laplacian operator

1 Introduction

In this paper, we investigate the existence of solutions for an impulsive fractional Dirichlet boundary value problem with a p -Laplacian operator and a controlled parameter as follows:

$$\begin{cases} {}_t D_T^\alpha \phi_p({}_0 D_t^\alpha u(t)) + a(t)u(t) = f(t, u(t)) + \mu b(t)|u(t)|^{v-2}u(t), & \text{a.e. } t \in J, \\ \Delta({}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j))) = I_j(u(t_j)), & j = 1, 2, \dots, m, m \in \mathbb{N}, \\ u(0) = u(T) = 0, \end{cases} \quad (1.1)$$

where $a(t), b(t) \in C([0, T], \mathbb{R})$; $f(t, u) \in C([0, T] \times \mathbb{R}, \mathbb{R})$; $\mu \in (0, \infty)$, $p \in (1, \infty)$, $\alpha \in [\frac{1}{p}, 1)$, $v \in [1, p - 1)$; $t_0 = 0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$; $J = [0, T] \setminus \{t_1, t_2, \dots, t_m\}$; $\phi_p(s) = |s|^{p-2}s$ ($s \neq 0$), $\phi_p(0) = 0$; $I_j \in C(\mathbb{R}, \mathbb{R})$;

$$\begin{aligned} \Delta({}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j))) &= {}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j^+)) - {}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j^-)), \\ {}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j^+)) &= \lim_{t \rightarrow t_j^+} {}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j)), \\ {}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j^-)) &= \lim_{t \rightarrow t_j^-} {}_t I_T^{1-\alpha} \phi_p({}_0 D_t^\alpha u(t_j)). \end{aligned}$$

In the recent years, more and more fractional mathematical models have occurred in many application fields (see [1–4]). It should be pointed out that Leszczynski and Blaszczyk [5] took advantage of the fractional mathematical model to show that the height of granular

material of silo decreases over time as follows.

$${}_t^c D_T^\alpha ({}_0 D_t^\alpha h^*(t)) + \beta h^*(t) = 0, \quad t \in [0, T], \tag{1.2}$$

where $\alpha \in (0, 1)$, ${}_t^c D_T^\alpha$ represents the right Caputo fractional derivatives, ${}_0 D_t^\alpha$ denotes left Riemann-Liouville fractional derivatives, $h^*(t) = h_{\text{bed}} - h(t)$ in which $h(t)$ and h_{bed} stand for falling height of the granular bed relative to empty silo and the initial bed height, respectively.

Some classical nonlinear functional methods have been applied to investigate solvability of boundary value problems for fractional differential equations such as fixed point theorems (see [6–9]), coincidence degree theory (see [10–12]), etc. Recently, the operator ${}_t^c D_T^\alpha {}_0 D_t^\alpha$ with the classical Dirichlet boundary condition possessing variational structure has been presented in [13], in which the following fractional differential system was considered:

$$\begin{cases} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) = \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where $\alpha \in (0, 1]$; $F(t, u) : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the classical Carathéodory condition. It should be mentioned that ${}_t D_T^\alpha u = {}_t^c D_T^\alpha u$ and ${}_0 D_t^\alpha u = {}_0^c D_t^\alpha u$ when $u(0) = u(T) = 0$. Based on the mountain pass theorem and under $F(t, u)$ satisfying a class of superlinear growth conditions and the classical local Ambrosetti-Rabinowitz condition

$$0 < F(t, u) \leq \frac{1}{\theta} \langle u, \nabla F(t, u) \rangle, \quad \theta > 2, |u| \geq M \text{ or } u \in \mathbb{R} \setminus \{0\}, \tag{1.4}$$

where $\theta, M > 0$ are constants, the existence of weak nontrivial solutions was proved when $\alpha \in (\frac{1}{2}, 1]$. In fact, if $\alpha \in (\frac{1}{2}, 1]$, the so-called fractional Sobolev space $E_0^{\alpha, 2}$ is compactly embedded into $C([0, T], \mathbb{R})$. Thus, the proof is more clear. Torres [14] gave the further study on this type problem when $\mathbb{N} = 1$. Chen and Liu [15] extended the corresponding results on (1.3) to a p -Laplacian operator case where $1 < p < \infty$. Moreover, Bergounioux et al. [16], Idczak and Walczak [17] and Jin and Liu [18] devoted their works to perfecting a fractional-type Sobolev space. It should be pointed out that Bonanno et al. [19] investigated the following impulsive fractional Dirichlet boundary value problem:

$$\begin{cases} {}_t D_T^\alpha ({}_0^c D_t^\alpha u(t)) + a(t)u(t) = \lambda f(t, u(t)), & \text{a.e. } t \in J, \\ \Delta ({}_t I_T^{1-\alpha} ({}_0^c D_t^\alpha u(t_j))) = \mu I_j(u(t_j)), & j = 1, 2, \dots, m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.5}$$

where $a(t) \in C([0, T], (0, \infty))$, $f(t, u) \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and $\lambda, \mu \in (0, \infty)$, and proved the existence of three solutions. For application of variational methods in boundary value problems of integer or fractional differential equations with impulsive effects, please see [20–27] and the references therein.

After that, by the gene property, Ledesma and Nyamoradi [28] studied the following eigenvalue problem:

$$\begin{cases} {}_t D_T^\alpha \phi_p ({}_0 D_t^\alpha u(t)) = \lambda \phi_p(u), & t \in [0, T], \\ u(0) = u(T) = 0 \end{cases} \tag{1.6}$$

and found that $\Lambda := \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of (1.5)}\}$ is a nonempty infinite set, $\sup \Lambda = \infty$ and

$$\lambda_1 = \inf_{u \in E_0^{\alpha,p} \setminus \{0\}} \frac{\int_0^T |{}_0D_t^\alpha u(t)|^p dt}{\int_0^T |u(t)|^p dt} > 0.$$

Moreover, in [28], based on the above results, the authors took further study of the following impulsive fractional Dirichlet boundary value problem with p -Laplacian operator:

$$\begin{cases} {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) + a(t)u(t) = f(t, u(t)), & \text{a.e. } t \in J, \\ \Delta({}_tI_T^{1-\alpha} \phi_p({}_0D_t^\alpha u(t_j))) = I_j(u(t_j)), & j = 1, 2, \dots, m, \\ u(0) = u(T) = 0. \end{cases} \tag{1.7}$$

For stating the main results of [28], we firstly introduce the following assumptions with respect to f and I_j :

- (H0) $a(t) \in C([0, T], \mathbb{R})$ and $\text{ess inf}_{t \in [0, T]} a(t) > -\lambda_1$.
- (H1) $f(t, u) = o(|u|^{p-1})$ as $|u| \rightarrow 0$ uniformly for $t \in [0, T]$.
- (H2) There exist constants $D > 0, \theta > p$ such that, for $t \in [0, T]$,

$$0 < \theta F(t, u) \leq uf(t, u) \quad \text{for } |u| \geq D,$$

where $F(t, u) = \int_0^u f(t, s) ds$.

- (H3) There exist constants $d_j > 0$ and $\gamma_j \in (p - 1, \theta - 1)$,

$$|I_j(t)| \leq d_j |t|^{\gamma_j} \quad \text{for any } t \in \mathbb{R}.$$

- (H4) For t large enough, $I_j(t)$ satisfy

$$\theta \int_0^t I_j(s) ds \geq I_j(t)t.$$

- (H5) For any $t \in \mathbb{R}$, $I_j(t)$ satisfy

$$\int_0^t I_j(s) dt \geq 0.$$

- (H6) $I_j(u)$ and $f(t, u)$ are odd on u .

Theorem 1.1 (see [28]) *Assume that conditions (H0)-(H6) hold. Then problem (1.7) has infinitely many weak solutions.*

Motivated by the works mentioned above, we take further study on this topic with the concave-convex nonlinearity (1.1). For comparing our main result with Theorem 1.1, we present the following assumed conditions and our main result.

- (H2') There exist constants $D > 0, \theta > p$ such that, for $t \in [0, T]$,

$$\theta F(t, u) \leq uf(t, u) \quad \text{for } |u| \geq D$$

and

$$\inf_{|u|=D} F(t, u) > 0.$$

(H3') There exist constants $a_j, d_j > 0$ and $\gamma_j \in [0, \theta - 1]$

$$|I_j(t)| \leq a_j + d_j |t|^{\gamma_j} \quad \text{for any } t \in \mathbb{R}.$$

(H7) There exist constants $M, L > 0, \beta > \theta$ such that, for $t \in [0, T]$,

$$F(t, u) \leq M |u|^\beta \quad \text{for } |u| \geq L.$$

Now, we describe our main result.

Theorem 1.2 *If conditions (H0), (H1), (H2'), (H3') and (H4)-(H7) are satisfied, there exists a constant $\mu_* > 0$ such that problem (1.1) has infinitely many weak solutions for $\mu \in [0, \mu_*)$.*

Remark 1.3 It should be noted that (H2') occurred earlier in [29] and was used to deal with multiplicity of solutions of fourth-order elliptic equations. Clearly, (H2') and (H3') are weaker than (H2) and (H3), respectively. Moreover, if $\mu = 0$, (H7) can be removed. Thus, our conclusion extends Theorem 1.1. Furthermore, if γ_j is located in $\in [0, p - 1]$ in (H3'), (H4) can be removed. Noting that if $p = 2, \alpha = 1$, one has

$${}_t D_T^\alpha (|{}_0 D_t^\alpha u(t)|^{p-2} {}_0 D_t^\alpha u(t)) = {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) = -u''.$$

Therefore, our main result also generalizes the corresponding result of [30].

Remark 1.4 Here, $I_j(u)$ could be p -superlinear or p -sublinear growth. It becomes more general than the previous papers (see [28] and [30]).

Moreover, we also consider the nonlinearity satisfying p -sublinear growth. We need the following assumptions.

(H3'') There exist constants $a_j, d_j > 0$ and $\gamma_j \in [0, p - 1]$

$$|I_j(t)| \leq a_j + d_j |t|^{\gamma_j} \quad \text{for any } t \in \mathbb{R}.$$

(H8) There exists a function $v \in E_0^{\alpha,p}$ such that

$$\frac{1}{\mu} \int_0^T F(t, v(t)) dt + \frac{1}{v} \int_0^T b(t) |v(t)|^v dt > 0.$$

(H9)

$$\limsup_{|u| \rightarrow \infty} \frac{F(t, u)}{|u|^p} = 0, \quad \text{uniformly for } t \in [0, T].$$

Theorem 1.5 *If conditions (H0), (H2''), (H8) and (H9) are satisfied, there exists a constant $\mu^* > 0$ such that problem (1.1) has at least one weak solution for $\mu \in (\mu^*, \infty)$.*

2 Preliminaries

First, we show the basic definitions and propositions of fractional integral and derivative, fractional-type Sobolev space. Let $\|u\|_{L^p} = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$ and $\|u\|_{\infty} = \max_{t \in [0, T]} |u(t)|$ be the norms in $L^p([0, T], \mathbb{R})$ and $C([0, T], \mathbb{R})$, respectively. The constant c stands for a different constant in different sentences.

Definition 2.1 ([2, 3]) Let f be a function defined on $[a, b]$.

- (i) The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function f are defined by

$$\begin{aligned}
 {}_a I_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \\
 {}_t I_b^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \quad t \in [a, b],
 \end{aligned}$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma(\alpha)$ is the standard gamma function.

- (ii) If $\alpha = n, n \in \mathbb{N}$, they become the usual definitions

$$\begin{aligned}
 {}_a I_t^n f(t) &= \frac{1}{\Gamma(n)} \int_a^t (t-s)^{n-1} f(s) ds, \\
 {}_t I_b^n f(t) &= \frac{1}{\Gamma(n)} \int_t^b (s-t)^{n-1} f(s) ds, \quad t \in [a, b].
 \end{aligned}$$

Definition 2.2 ([2, 3]) Let f be a function defined on $[a, b]$.

- (i) The left and right Riemann-Liouville fractional derivatives of order α for a function f denoted by ${}_a D_t^\alpha f(t)$ and ${}_t D_b^\alpha f(t)$, respectively, are defined by

$${}_a D_t^\alpha f(t) = \frac{d^n}{dt^n} {}_a I_t^{n-\alpha} f(t), \quad {}_t D_b^\alpha f(t) = (-1)^n \frac{d^n}{dt^n} {}_t I_b^{n-\alpha} f(t),$$

where $t \in [a, b], n - 1 \leq \alpha < n$ and $n \in \mathbb{N}$.

- (ii) If $\alpha = n - 1, n \in \mathbb{N}$, they become the usual definitions

$${}_a D_t^{n-1} f(t) = f^{(n-1)}(t), \quad {}_t D_b^{n-1} f(t) = (-1)^n f^{(n-1)}(t), \quad t \in [a, b].$$

Definition 2.3 ([2, 3]) Let $\alpha \geq 0$ and $n \in \mathbb{N}$.

- (i) If $\alpha \in (n - 1, n)$ and $f \in AC^n([a, b], \mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order α for a function f denoted by ${}^c D_t^\alpha f(t)$ and ${}^c D_b^\alpha f(t)$, respectively, exist almost everywhere on $[a, b]$, ${}^c D_t^\alpha f(t)$ and ${}^c D_b^\alpha f(t)$ are represented by

$${}^c D_t^\alpha f(t) = {}_a I_t^{n-\alpha} \frac{d^n}{dt^n} f(t), \quad {}^c D_b^\alpha f(t) = (-1)^n {}_t I_b^{n-\alpha} \frac{d^n}{dt^n} f(t), \quad t \in [a, b].$$

- (ii) If $\alpha = n - 1$ and $f \in AC^{n-1}([a, b], \mathbb{R}^N)$, then ${}^c D_t^{n-1} f(t)$ and ${}^c D_b^{n-1} f(t)$ are represented by

$${}^c D_t^{n-1} f(t) = f^{(n-1)}(t), \quad {}^c D_b^{n-1} f(t) = (-1)^{(n-1)} f^{(n-1)}(t), \quad t \in [a, b].$$

Proposition 2.4 ([2, 3]) *Let $n \in \mathbb{N}$ and $n - 1 < \alpha < n$. If f is a function defined on $[a, b]$, for which the Caputo fractional derivatives ${}_a^c D_t^\alpha f(t)$ and ${}_t^c D_b^\alpha f(t)$ of order α exist together with the Riemann-Liouville fractional derivatives ${}_a D_t^\alpha f(t)$ and ${}_t D_b^\alpha f(t)$, then*

$${}_a^c D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(j - \alpha + 1)} (t - a)^{j-\alpha}, \quad t \in [a, b], \tag{2.1}$$

$${}_t^c D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(b)}{\Gamma(j - \alpha + 1)} (b - t)^{j-\alpha}, \quad t \in [a, b]. \tag{2.2}$$

Remark 2.5 In view of (2.1) and (2.2), it is easy to find that ${}_0^c D_t^\alpha u(t) = {}_0 D_t^\alpha u(t)$, ${}_t^c D_T^\alpha u(t) = {}_t D_T^\alpha u(t)$, $t \in [0, T]$ by $u(0) = u(T) = 0$.

Proposition 2.6 ([3]) *We have the following property of fractional integration:*

$$\int_a^b [{}_a I_t^\alpha f(t)]g(t) dt = \int_a^b [{}_t I_b^\alpha g(t)]f(t) dt, \quad \alpha > 0,$$

provided that $f \in L^p([a, b], \mathbb{R}^N)$, $g \in L^q([a, b], \mathbb{R}^N)$ and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \alpha$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \alpha$.

Definition 2.7 ([17]) Let $0 < \alpha \leq 1$ and $u, v \in L^1[0, T]$. For any $\varphi \in C_0^\infty([0, T], \mathbb{R})$, we have

$$\int_0^T v\varphi dt = \int_0^T u {}_t D_T^\alpha \varphi dt,$$

so v is said to be an α -weak fractional derivative for u , and there exists a left Riemann-Liouville derivative such that $v = {}_0 D_t^\alpha u$.

Define the following fractional-type Sobolev space:

$$E^{\alpha,p} = \{u \in L^p([0, T], \mathbb{R}) : {}_0 D_t^\alpha u \in L^p([0, T], \mathbb{R})\}$$

with the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |{}_0 D_t^\alpha u(t)|^p dt + \int_0^T |u(t)|^p dt \right)^{1/p}. \tag{2.3}$$

Based on [17], if $1 < p < \infty$, it is a reflexive and separable Banach space. Moreover, $E_0^{\alpha,p}$ represents the closure of $C_0^\infty([0, T], \mathbb{R})$ in the norm of $E^{\alpha,p}$. So, $E_0^{\alpha,p} \subset E^{\alpha,p}$ is also a reflexive and separable Banach space. Clearly, $E_0^{\alpha,p}$ becomes the well-known space $W_0^{1,p}$ when $\alpha = 1$.

Lemma 2.8 ([13]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|{}_0 D_t^\alpha u\|_{L^p}. \tag{2.4}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/q}}{\Gamma(\alpha)(q(\alpha - 1) + 1)^{1/q}} \|{}_0 D_t^\alpha u\|_{L^p}. \tag{2.5}$$

Based on (2.4), we can consider $E_0^{\alpha,p}$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |{}_0D_t^\alpha u(t)|^p dt \right)^{1/p} = \|{}_0D_t^\alpha u\|_{L^p}, \quad \forall u \in E_0^{\alpha,p}, \tag{2.6}$$

and (2.3) is equivalent to (2.6).

For our problem, we define a new norm in $E_0^{\alpha,p}$ by

$$\|u\| = \left(\int_0^T |{}_0D_t^\alpha u(t)|^p dt + \int_0^T a(t)|u(t)|^p dt \right)^{1/p}. \tag{2.7}$$

Since $\text{ess inf}_{t \in [0,T]} a(t) > -\lambda_1$, following [28], the norm $\|u\|$ is equivalent to $\|u\|_{\alpha,p}$.

Proposition 2.9 ([13]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence u_n converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_n \rightharpoonup u$. Then $u_n \rightarrow u$ in $C([0, T], \mathbb{R})$, i.e., $\|u_n - u\|_\infty \rightarrow 0$, $n \rightarrow \infty$.*

Based on the above proposition, there exists a constant $S > 0$ such that $\|u\|_\infty \leq S\|u\|$. For $v \in E_0^{\alpha,p}$, multiplying the two sides of equation (1.1) by v and integrating from 0 to T , one has

$$\begin{aligned} & \int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t))v(t) + \int_0^T a(t)\phi_p(u(t))u(t)v(t) dt \\ & - \int_0^T f(t, u(t))v(t) dt - \mu \int_0^T b(t)|u(t)|^{p-2}u(t)v(t) dt = 0. \end{aligned}$$

Based on Proposition 2.6, we have

$$\begin{aligned} & \int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t))v(t) dt \\ & = - \sum_{j=0}^m \int_{t_j}^{t_{j+1}} v(t) d[{}_tI_T^{1-\alpha} \phi_p({}_0D_t^\alpha u(t))] \\ & = - \sum_{j=0}^m {}_tI_T^{1-\alpha} \phi_p({}_0D_t^\alpha u(t))v(t)|_{t_j}^{t_{j+1}} + \sum_{j=0}^m \int_{t_j}^{t_{j+1}} \phi_p({}_0D_t^\alpha u(t)){}_0D_t^\alpha v(t) dt \\ & = \sum_{j=1}^m ({}_tI_T^{1-\alpha} \phi_p({}_0D_t^\alpha u(t_j^+))v(t_j) - {}_tI_T^{1-\alpha} \phi_p({}_0D_t^\alpha u(t_j^-))v(t_j)) \\ & \quad + \int_0^T \phi_p({}_0D_t^\alpha u(t)){}_0D_t^\alpha v(t) dt \\ & = I_j(u(t_j))v(t_j) + \int_0^T \phi_p({}_0D_t^\alpha u(t)){}_0D_t^\alpha v(t) dt. \end{aligned}$$

Now, we describe the definition of weak solution of (1.1).

Definition 2.10 Let $u \in E_0^{\alpha,p}$ be a weak solution of (1.3) if

$$\int_0^T \phi_p({}_0D_t^\alpha u(t)) {}_0D_t^\alpha v(t) dt + \int_0^T a(t)\phi_p(u(t))u(t)v(t) dt + \sum_{j=1}^m I_j(u(t_j))v(t_j) - \int_0^T f(t, u(t))v(t) dt - \mu \int_0^T b(t)|u(t)|^{v-2}u(t)v(t) dt = 0$$

holds for any $v \in E_0^{\alpha,p}$.

Define the functional $\Phi : E_0^{\alpha,p} \rightarrow \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p} \|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t) dt - \int_0^T F(t, u(t)) dt - \frac{\mu}{v} \int_0^T h(t)|u(t)|^v dt.$$

Based on the continuity of f and I_j , following [31], one has $\phi \in C^1(E_0^{\alpha,p}, \mathbb{R})$. For any $v \in E_0^{\alpha,p}$, we can get

$$\langle \Phi'(u), v \rangle = \int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t))v(t) + \int_0^T a(t)\phi_p(u(t))u(t)v(t) dt + \sum_{j=1}^m I_j(u(t_j))v(t_j) - \int_0^T f(t, u(t))v(t) dt - \mu \int_0^T b(t)|u(t)|^{v-2}u(t)v(t) dt.$$

Therefore, the weak solutions of problem (1.1) are the corresponding critical points of Φ . In order to obtain our main results, we introduce the following tools.

Definition 2.11 ([32]) Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$ satisfy $\Phi(-u) = \Phi(u)$, $u \in X$. Let

$$\Sigma = \{A \subset X - \{0\} \mid A \text{ is closed in } X \text{ and symmetric with respect to } 0\}.$$

The genus of A is defined by

$$\gamma(A) = \begin{cases} \inf\{n \in \mathbb{Z}_+ \mid \text{there exists an odd mapping } \varphi \in C(A, \mathbb{R}^n \setminus \{0\})\}, \\ 0, & A = \emptyset, \\ +\infty, & \text{when } \varphi \text{ is non-existent.} \end{cases}$$

Lemma 2.12 ([32]) Let $n, k \in \mathbb{N}$. The properties of the genus γ are as follows:

- (i) If $X = X_1 \oplus X_2$, $\dim X_1 = k$, $\gamma(A) > k$, then $A \cap X_2 \neq \emptyset$;
- (ii) If Ω is a symmetric bounded domain near zero in \mathbb{R}^n , there exists a mapping $h \in C(A, \partial\Omega)$ with odd homeomorphism such that, for $A \in \Sigma$, then $\gamma(A) = n$;
- (iii) If $\gamma(A) = k$, $0 \notin A$, then there exist at least k pairs of different points in A .

Lemma 2.13 ([32]) Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$ satisfy $\Phi(-u) = \Phi(u)$, $u \in X$. The pseudo-index is defined by

$$i^*(A) = \inf_{h \in \Lambda_*(\rho)} \gamma(A \cap h(\partial B_1)),$$

where

$$\begin{aligned} \Lambda_*(\rho) &\equiv \{h \in C(X, X) \mid h \text{ is an odd homeomorphism,} \\ &\quad \text{for some } \rho > 0, h(B_1) \subset \Phi^{-1}(0, \infty) \cup B_\rho\}, \\ A \in \Sigma^* &\equiv \{A \in \Sigma \mid A \text{ is compact}\}. \end{aligned}$$

Suppose that Φ satisfies the $(PS)_c$ -condition and

- (i) there are constants $\rho, \sigma > 0$ such that $\Phi|_{X_1^\perp \cap \partial B_\rho} \geq \sigma$ for the subspace $X_1 \subset X$, $\dim X_1 = m_1$;
- (ii) there is a subspace $X_2 \subset X$, $\dim X_2 = m_2 > m_1$, $l > 0$ such that $\Phi(u) \leq 0, \forall u \in X_2 \setminus B_l$.

Then Φ at least possesses $m_2 - m_1$ pairs of different critical points:

$$c_n^* = \inf_{i^*(A) \geq n} \sup_{u \in A} \Phi(u).$$

Definition 2.14 ([32, 33]) Let X be a real Banach space with its dual X^* and $\Phi \in C^1(X, \mathbb{R})$. For any $\{u_n\} \subset X$, $\{u_n\}$ has a convergent subsequence if $\Phi(u_n)$ is bounded or $\Phi(u_n) \rightarrow c, c \in \mathbb{R}$ and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then we say that $\Phi(u)$ satisfies the Palais-Smale condition or the Palais-Smale condition at the level c ((PS) -condition or $(PS)_c$ -condition for short).

Lemma 2.15 ([34]) Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy the (PS) -condition. If Φ is bounded from below, then $c = \inf_E \Phi$ is a critical point.

3 Main results

In order to prove the theorem, we need the following lemmas. Firstly, in $E_0^{\alpha,p}$, we can choose a completely orthonormal basis $\{e_i\}_{i=1}^\infty$. Set $Y_i = \mathbb{R}e_i, X_k = \bigoplus_{i=1}^k Y_i, X_{k+r} = \bigoplus_{i=1}^{k+r} Y_i$ and $X_k^\perp = \overline{\bigoplus_{i=k+1}^\infty Y_i}$, so $W_T^{1,2} = X_k^\perp \oplus X_k$ and $\dim X_{k+r} - \dim X_k = r$, where $r \in \mathbb{N}$. $b^\pm(t) = \max\{\pm b(t), 0\}$.

Lemma 3.1 If the assumptions of Theorem 1.2 hold, then Φ satisfies the $(PS)_c$ -condition for $\mu \in (0, \infty)$.

Proof Let $\{u_n\}_{n \in \mathbb{N}} \subset E_0^{\alpha,p}$ such that $\Phi(u_n) \rightarrow c$ and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, which tell us the fact that there exists a constant $C > 0$ such that $|\Phi(u_n)| \leq C, \|\Phi'(u_n)\|_{(E_0^{\alpha,p})^*} \leq C$. Next, our aim is to prove that $\{u_n\}$ is a bounded sequence in $E_0^{\alpha,p}$. If not, we assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. For any $u \in E_0^{\alpha,p} \setminus \{0\}$, let $v_n = \frac{u_n}{\|u_n\|}$, then v_n is bounded in $E_0^{\alpha,p}$. From the fact that $E_0^{\alpha,p}$ is a reflexive Banach space, we can find a subsequence of $\{v_n\}$ (called again $\{v_n\}$) such that $v_n \rightharpoonup v$ in $E_0^{\alpha,p}, v_n \rightarrow v$ uniformly in $C([0, T], \mathbb{R})$. By (H3'), we can get

$$\begin{aligned} \int_0^T F(t, u_n) dt &= \frac{1}{p} \|u_n\|^p + \sum_{j=1}^m \int_0^{u_n(t_j)} I_j(t) dt - \frac{\mu}{v} \int_0^T b(t) |u(t)|^v dt - \Phi(u_n) \\ &\leq \frac{1}{p} \|u_n\|^p + \sum_{j=1}^m a_j S \|u_n\| + \sum_{j=1}^m d_j S^{\gamma_j+1} \|u_n\|^{\gamma_j+1} \\ &\quad + \frac{\mu}{v} TS^v \|b\|_\infty \|u\|^v + C, \end{aligned}$$

which shows the fact that there exists $c > 0$ such that

$$\begin{cases} \lim_{n \rightarrow \infty} \int_0^T \frac{F(t, u_n)}{\|u_n\|^p} dt \leq c & \text{for } \gamma_j \in [0, p-1), \\ \lim_{n \rightarrow \infty} \int_0^T \frac{F(t, u_n)}{\|u_n\|^{\gamma_*+1}} dt \leq c & \text{for } \gamma_j \in [p-1, \theta-1), \end{cases} \tag{3.1}$$

where $\gamma_* = \max\{\gamma_j\}$. From (H1), for any $\varepsilon > 0$, there exists a constant $D_1 > 0$ such that

$$|f(t, u)| \leq \varepsilon |u|^{p-1}, \quad |u| \leq D_1. \tag{3.2}$$

Therefore, for $|u| \leq D_1$, there exists a constant $\varepsilon_1 > 0$ such that

$$|uf(t, u) - \theta F(t, u)| \leq \varepsilon_1(1 + \theta)u^p.$$

For $(t, u) \in [0, T] \times [D_1, D]$, we can find a constant $c > 0$ such that

$$|uf(t, u) - \theta F(t, u)| \leq c,$$

which together with (H2') yields that

$$uf(t, u) - \theta F(t, u) \geq -\varepsilon_1(1 + \theta)u^p - c.$$

Based on (H4), there exists a constant $c > 0$ such that

$$\theta \sum_{j=1}^m \int_0^{u_n(t_j)} I_j(t) dt - \sum_{j=1}^m I_j(u_n(t_j))u_n(t_j) \geq -c.$$

Therefore, we have

$$\begin{aligned} \theta C + C\|u_n\| &\geq \theta \Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \\ &= \left(\frac{\theta}{p} - 1\right)\|u_n\|^p + \theta \sum_{j=1}^m \int_0^{u_n(t_j)} I_j(t) dt - \sum_{j=1}^m I_j(u_n(t_j))u_n(t_j) \\ &\quad + \int_0^T (u_n(t)f(t, u_n(t)) - \theta F(t, u_n(t))) dt \\ &\quad - \mu \frac{\theta - v}{v} \int_0^T b(t)|u_n(t)|^v dt \\ &\geq \|u_n\|^p + \int_0^T (u_n(t)f(t, u_n(t)) - \theta F(t, u_n(t))) dt \\ &\quad - \mu \frac{\theta - v}{v} \int_0^T b(t)|u_n(t)|^v dt - c \\ &\geq \|u_n\|^p - \varepsilon_1(1 + \theta)T\|u_n\|_\infty^p - \mu \frac{\theta - v}{v} \|b^+\|_{L^1} \|u_n\|_\infty^v - c. \end{aligned}$$

This shows the fact that there exists a constant $c > 0$ such that

$$\lim_{n \rightarrow \infty} \|v_n\|_\infty = \lim_{n \rightarrow \infty} \frac{\|u_n\|_\infty}{\|u_n\|} \geq c > 0.$$

Hence, we obtain $v \neq 0$. From (H2'), for $s \in [D/|u|, 1]$, $|u| \geq D$, one has

$$\frac{d}{ds} \left(\frac{F(t, su)}{s^\theta} \right) = \frac{f(t, su)su - \theta F(t, su)}{s^{\theta+1}} \geq 0.$$

Thus, for $t \in [0, T]$, we have

$$F(t, u) \geq \frac{|u|^\theta}{D^\theta} \inf_{|u|=D} F(t, u), \tag{3.3}$$

which together with (H2') yields that

$$\begin{cases} \frac{F(t, u)}{|u|^p} \geq \frac{|u|^{\theta-p}}{D^\theta} \inf_{|u|=D} F(t, u) \rightarrow \infty & \text{as } |u| \rightarrow \infty \text{ for } \gamma_j \in [0, p-1), \\ \frac{F(t, u)}{|u|^{\gamma_*+1}} \geq \frac{|u|^{\theta-\gamma_*-1}}{D^\theta} \inf_{|u|=D} F(t, u) \rightarrow \infty & \text{as } |u| \rightarrow \infty \text{ for } \gamma_j \in [p-1, \theta-1). \end{cases} \tag{3.4}$$

Based on Fatou's lemma, one has

$$\begin{cases} \int_{v \neq 0} \frac{F(t, u_n)}{|u_n|^p} |v_n|^p dt \rightarrow \infty & \text{as } n \rightarrow \infty \text{ for } \gamma_j \in [0, p-1), \\ \int_{v \neq 0} \frac{F(t, u_n)}{|u_n|^{\gamma_*+1}} |v_n|^{\gamma_*+1} dt \rightarrow \infty & \text{as } n \rightarrow \infty \text{ for } \gamma_j \in [p-1, \theta-1). \end{cases}$$

In view of (3.3), we know $F(t, u) \geq 0$ for $|u| \geq D, t \in [0, T]$. From (H1), there exist constants $c, \varepsilon_2 > 0$ such that

$$F(t, u) \geq -\varepsilon_2 u^p - c, \quad u \in \mathbb{R}, t \in [0, T].$$

Let

$$\mathcal{O} := \text{meas}\{t \in [0, T] : v(t) = 0\}.$$

We have

$$\begin{cases} \int_{v=0} \frac{F(t, u_n)}{\|u_n\|^p} dt \geq -\varepsilon_2 S^p \mathcal{O} - \frac{c}{\|u_n\|^p} & \text{for } \gamma_j \in [0, p-1), \\ \int_{v=0} \frac{F(t, u_n)}{\|u_n\|^{\gamma_*+1}} dt \geq \frac{-\varepsilon_2 S^p \mathcal{O}}{\|u_n\|^{\gamma_*+1-p}} - \frac{c}{\|u_n\|^{\gamma_*+1}} & \text{for } \gamma_j \in [p-1, \theta-1). \end{cases}$$

This means

$$\begin{cases} \liminf_{n \rightarrow \infty} \int_{v=0} \frac{F(t, u_n)}{\|u_n\|^p} dt > -\infty & \text{for } \gamma_j \in [0, p-1), \\ \liminf_{n \rightarrow \infty} \int_{v=0} \frac{F(t, u_n)}{\|u_n\|^{\gamma_*+1}} dt > -\infty & \text{for } \gamma_j \in [p-1, \theta-1). \end{cases}$$

Thus, we can obtain

$$\begin{cases} \liminf_{n \rightarrow \infty} \int_0^T \frac{F(t, u_n)}{\|u_n\|^p} dt = \liminf_{n \rightarrow \infty} (\int_{v=0} + \int_{v \neq 0}) \frac{F(t, u_n)}{\|u_n\|^p} dt \\ \quad \rightarrow \infty & \text{for } \gamma_j \in [0, p-1), \\ \liminf_{n \rightarrow \infty} \int_0^T \frac{F(t, u_n)}{\|u_n\|^{\gamma_j+1}} dt = \liminf_{n \rightarrow \infty} (\int_{v=0} + \int_{v \neq 0}) \frac{F(t, u_n)}{\|u_n\|^{\gamma_j+1}} dt \\ \quad \rightarrow \infty & \text{for } \gamma_j \in [p-1, \theta-1). \end{cases}$$

This is a contradiction to (3.1). Hence, $\{u_n\}$ is bounded. By a similar standard argument as the one in [15], we can show $u_n \rightarrow u$ in $E_0^{\alpha,p}$. For the completeness, we state the proof as follows. Based on the fact that $E_0^{\alpha,p}$ is a reflexive Banach space, $\{u_n\}$ has a convergent subsequence (named again $\{u_n\}$) such that $u_n \rightarrow u$ in $E_0^{\alpha,p}$, so $u_n \rightarrow u$ uniformly in $C([0, T], \mathbb{R})$. Thus, we have

$$\int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t)) dt \rightarrow 0, \quad n \rightarrow \infty, \tag{3.5}$$

$$\sum_{j=1}^m (I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \rightarrow 0, \quad n \rightarrow \infty, \tag{3.6}$$

$$\int_0^T a(t)(\phi_p(u_n(t)) - \phi_p(u(t)))(u_n(t) - u(t)) dt \rightarrow 0, \quad n \rightarrow \infty, \tag{3.7}$$

$$\int_0^T b(t)(|u_n|^{v-2}u_n(t) - |u_n(t)|^{v-2}u_n(t))(u_n(t) - u(t)) dt \rightarrow 0, \quad n \rightarrow \infty. \tag{3.8}$$

By $\Phi'(u_n) \rightarrow 0$ and $u_n \rightarrow u$, we can obtain

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \rightarrow 0, \quad n \rightarrow \infty. \tag{3.9}$$

Thus, we have

$$\begin{aligned} & \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \\ &= \int_0^T (\phi_p({}_0D_t^\alpha u_n(t)) - \phi_p({}_0D_t^\alpha u(t)))({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)) dt \\ & \quad + \int_0^T a(t)(\phi_p(u_n(t)) - \phi_p(u(t)))(u_n(t) - u(t)) dt \\ & \quad + \sum_{j=1}^m (I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \\ & \quad - \int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t)) dt \\ & \quad - \mu \int_0^T b(t)(|u_n(t)|^{v-2}u_n(t) - |u_n(t)|^{v-2}u_n(t))(u_n(t) - u(t)) dt. \end{aligned}$$

Based on [35], we have

$$\begin{aligned} & \int_0^T (\phi_p({}_0D_t^\alpha u_n(t)) - \phi_p({}_0D_t^\alpha u(t)))({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)) dt \\ & \geq \begin{cases} c \int_0^T |{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)|^p dt, & p \geq 2, \\ c \int_0^T \frac{|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)|^2}{(|{}_0D_t^\alpha u_n(t)| + |{}_0D_t^\alpha u(t)|)^{2-p}} dt, & 1 < p < 2. \end{cases} \tag{3.10} \end{aligned}$$

If $p \geq 2$, following (3.5)-(3.10), $\|u_n - u\| \rightarrow 0, n \rightarrow \infty$. If $1 < p < 2$, by Hölder’s inequality, one has

$$\begin{aligned} & \int_0^T |{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)|^p dt \\ & \leq c \left(\int_0^T \frac{|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)|^2}{(|{}_0D_t^\alpha u_n(t)| + |{}_0D_t^\alpha u(t)|)^{2-p}} dt \right)^{\frac{p}{2}} (\|u_n\| + \|u\|)^{\frac{p(2-p)}{2}}. \end{aligned}$$

Thus, we can get

$$\begin{aligned} & \int_0^T (\phi_p({}_0D_t^\alpha u_n(t)) - \phi_p({}_0D_t^\alpha u(t)))({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)) dt \\ & \geq \frac{c}{(\|u_n\| + \|u\|)^{2-p}} \left(\int_0^T |{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)|^p dt \right)^{\frac{2}{p}}. \end{aligned}$$

Hence, one has $\|u_n - u\| \rightarrow 0, n \rightarrow \infty$. Therefore, $\Phi(u)$ satisfies the $(PS)_c$ -condition. \square

Lemma 3.2 *If the assumptions of Theorem 1.2 hold, then there exist constants $\rho, \sigma, \mu_* > 0$ such that $\Phi|_{X_k^\perp \cap \partial B_\rho} \geq \sigma$ for $\mu \in [0, \mu_*)$.*

Proof By (H1) and (H7), for any $\varepsilon > 0$, there exists C_ε such that, for $x \in \mathbb{R}, t \in [0, T]$,

$$F(t, u) \leq \varepsilon |u|^p + C_\varepsilon |u|^\beta, \tag{3.11}$$

which implies that

$$\begin{aligned} \int_0^T F(t, u) dt & \leq \varepsilon \int_0^T |u|^p dt + C_\varepsilon \int_0^T |u|^\beta dt \\ & \leq \varepsilon TS^p \|u\|^p + C_\varepsilon TS^\beta \|u\|^\beta. \end{aligned}$$

Thus, from (H5), we can obtain

$$\begin{aligned} \Phi(u) & = \frac{1}{p} \|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t) dt - \int_0^T F(t, u(t)) dt - \frac{\mu}{v} \int_0^T b(t) |u(t)|^v dt \\ & \geq \frac{1}{p} \|u\|^p - \int_0^T F(t, u(t)) dt - \frac{\mu}{v} \int_0^T b^+(t) |u(t)|^v dt \\ & \geq \frac{1}{p} \|u\|^p - \varepsilon TS^p \|u\|^p - C_\varepsilon TS^\beta \|u\|^\beta - \frac{\mu}{v} TS^v \|b^+\|_{L^1} \|u\|^v \\ & = \|u\|^v \left(\left(\frac{1}{p} - \varepsilon TS^p \right) \|u\|^{p-v} - C_\varepsilon TS^\beta \|u\|^{\beta-v} - \frac{\mu}{v} TS^v \|b^+\|_{L^1} \right). \end{aligned}$$

Choosing $\varepsilon = \frac{5}{6TpS^p}$, we have

$$\varphi(u) \geq \|u\|^v \left(\frac{1}{6p} \|u\|^{p-v} - C_\varepsilon TS^\beta \|u\|^{\beta-v} - \frac{\mu}{v} TS^v \|b^+\|_{L^1} \right).$$

Let

$$y(t) = \frac{1}{6p} t^{p-v} - C_\varepsilon TS^\beta t^{\beta-v}, \quad t \geq 0.$$

By a simple calculation, we can find

$$\rho = \left[\frac{p-v}{6pC_\varepsilon TS^\beta(\beta-v)} \right]^{\frac{1}{\beta-p}}$$

such that

$$y(\rho) = \max_{t \geq 0} y(t) = \frac{\beta-p}{6p(\beta-v)} \left[\frac{p-v}{6pC_\varepsilon TS^\beta(\beta-v)} \right]^{\frac{p-v}{\beta-p}} > 0.$$

Thus, there exists a constant

$$\mu_* = \frac{v(\beta-p)}{6pTS^v(\beta-v) \|b^+\|_{L^1}} \left[\frac{p-v}{6pC_\varepsilon TS^\beta(\beta-v)} \right]^{\frac{p-v}{\beta-p}}.$$

When $\mu \in [0, \mu_*)$, we can find a constant $\sigma > 0$ such that

$$\Phi|_{X_k^+ \cap \partial B_\rho} \geq \sigma. \quad \square$$

Lemma 3.3 *If the assumptions of Theorem 1.2 hold, then there exists a constant $l > 0$ such that $\Phi(u) \leq 0, \forall u \in X_{k+r} \setminus B_l$ for $\mu \in (0, \infty)$.*

Proof For $t \in [0, T]$, we know that

$$F(t, u) \geq \frac{|u|^\theta}{D^\theta} \inf_{|u|=D} F(t, u).$$

Thus, we have

$$F(t, u) \geq k|u|^\theta, \quad |u| \geq D, t \in [0, T], \tag{3.12}$$

where $k = D^{-\theta} \inf_{|u|=D} F(t, u) > 0$. By (H1), there exist constants $\varepsilon_3, c > 0$ such that

$$F(t, u) \geq -\varepsilon_3 u^p - c, \quad |u| \leq D, t \in [0, T]. \tag{3.13}$$

Hence, one has

$$F(t, u) \geq k|u|^\theta - lu^p - c, \quad u \in \mathbb{R}, t \in [0, T], \tag{3.14}$$

where $l = kD^{\theta-p} + \varepsilon_3$. Thus, for any $u \in X_{k+j}$, by the equivalence of the norms on the finite-dimensional space, there exist constants $l_1, l_2 > 0$ such that

$$\begin{aligned} \varphi(u) &\leq \frac{1}{p} \|u\|^p + \sum_{j=1}^m a_j S \|u\| + \sum_{j=1}^m d_j S^{\gamma_j+1} \|u\|^{\gamma_j+1} + l_1^p \|u\|^p \\ &\quad + \frac{\mu}{v} TS^v \|b\|_\infty \|u\|^v + cT - kl_2^\theta \|u\|^\theta, \end{aligned}$$

which implies $\Phi(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$ because of $\theta > \{p, \gamma_j + 1\}$. There exists a constant $l > 0$ such that $\Phi(u) \leq 0, \forall u \in X_{k+r} \setminus B_l$ for $\mu \in (0, \infty)$. \square

Next, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Based on the genus property and the definition of c_n^* , we have, for $r \in \mathbb{N}$,

$$\sigma \leq c_{k+s}^* < +\infty, \quad r \geq s \geq 1.$$

For all $A \in \Sigma^*$, we know that $i^*(A) \geq k + s$. Let $h_0 = \rho \cdot id$, then $h_0 \in \Lambda_*(\rho)$ and

$$\gamma(A \cap \partial B_\rho) = \gamma(A \cap h_0(\partial B_1)) \geq \inf_{h \in \Lambda_*(\rho)} \gamma(A \cap h(\partial B_1)) = i^*(A) > k.$$

From (i) of Lemma 2.12, one has $A \cap \partial B_\rho \cap X_k^\perp \neq \emptyset$. By Lemma 3.2, we have

$$\sup_{u \in A} \Phi(u) \geq \inf_{u \in \partial B_\rho \cap X_k^\perp} \Phi(u) \geq \sigma.$$

In view of the arbitrariness of $A \in \Sigma^*$, so $c_{k+s}^* \geq \sigma$. Based on Lemma 2.13, if $k + s \leq \dim X_{k+r}$, one has $c_{k+s}^* < +\infty$. Then we have

$$\sigma \leq c_{k+1}^* \leq c_{k+2}^* \leq \dots \leq c_{k+r}^* < +\infty.$$

From Lemmas 3.1, 3.2 and 3.3, we know that (i) and (ii) of Lemma 2.13 and the $(PS)_c$ -condition are satisfied. Moreover, $\Phi(u) = \Phi(-u)$ and $\Phi(0) = 0$. Thus, from Lemma 2.13, $\Phi(u)$ possesses at least r pairs of different critical points. Since r is arbitrary and $\lambda_r \rightarrow \infty, r \rightarrow \infty$, then problem (1.1) has infinitely many nontrivial weak solutions. \square

Lemma 3.4 *If the assumptions of Theorem 1.5 hold, then problem (1.1) has at least one weak solution for $\mu \in (0, \infty)$.*

Proof In view of (H9), for any $\xi > 0$, we can find a constant $K > 0$ such that, for $t \in [0, T]$,

$$F(t, u) \leq \xi |u|^p, \quad |u| \geq K.$$

Since f is continuous, there exists $c > 0$ such that

$$F(t, u) \leq \xi |u|^p + c \quad \text{for } (t, u) \in [0, T] \times \mathbb{R}.$$

From (H3''), we have

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t) dt - \int_0^T F(t, u(t)) dt - \frac{\mu}{v} \int_0^T b(t) |u(t)|^v dt \\ &\geq \frac{1}{p} \|u\|^p - \sum_{j=1}^m a_j S \|u\| - \sum_{j=1}^m d_j S^{\gamma_j+1} \|u\|^{\gamma_j+1} - \xi S^p T \|u\|^p - \frac{\mu}{v} \|b^+\|_{L^1} S^v \|u\|^v - cT. \end{aligned}$$

Setting $\xi = \frac{1}{2pTS^p}$, one has

$$\Phi(u) \geq \frac{1}{2p} \|u\|^p - \sum_{j=1}^m a_j S \|u\| - \sum_{j=1}^m d_j S^{\gamma_j+1} \|u\|^{\gamma_j+1} - \frac{\mu}{v} \|b^+\|_{L^1} S^v \|u\|^v - cT.$$

In view of $0 < \gamma_j < p - 1$, it is easy to get that $\Phi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$. Therefore, $\Phi(u)$ is coercive, which presents the fact that $\Phi(u)$ is bounded from below. Let $\{u_n\} \subset E_0^{\alpha,p}$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $\Phi(u)$ is coercive, so $\{u_n\}$ is bounded. By a similar way as Lemma 3.1, we can get that $\Phi(u)$ satisfies the (PS)-condition. Therefore, problem (1.1) has at least one weak solution for $\mu \in (0, \infty)$. \square

Next, we show the proof of Theorem 1.5.

Proof of Theorem 1.5 Let $u_* \in E_0^{\alpha,p}$ be a critical point of $\Phi(u)$ such that $\Phi(u_*) = \inf_{E_0^{\alpha,p}} \Phi(u)$. Based on (H8), we can find a function $v \in E_0^{\alpha,p}$ such that

$$\frac{1}{\mu} \int_0^T F(t, v(t)) dt + \frac{1}{v} \int_0^T b(t) |v(t)|^v dt > 0.$$

Thus, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \|v\|^p + \sum_{j=1}^m \int_0^{v(t_j)} I_j(t) dt - \int_0^T F(t, v(t)) dt - \frac{\mu}{v} \int_0^T b(t) |v(t)|^v dt \\ &\leq \frac{1}{p} \|v\|^p + \sum_{j=1}^m a_j S \|v\| + \sum_{j=1}^m d_j S^{\gamma_j+1} \|v\|^{\gamma_j+1} - \int_0^T F(t, v(t)) dt \\ &\quad - \frac{\mu}{v} \int_0^T b(t) |v(t)|^v dt. \end{aligned}$$

Therefore, there exists

$$\mu^* = \frac{\frac{1}{p} \|v\|^p + \sum_{j=1}^m a_j S \|v\| + \sum_{j=1}^m d_j S^{\gamma_j+1} \|v\|^{\gamma_j+1}}{\frac{1}{\mu} \int_0^T F(t, v(t)) dt + \frac{1}{v} \int_0^T b(t) |v(t)|^v dt} > 0$$

such that $\Phi(v) < 0$ if $\mu > \mu^*$. Thus, $\Phi(u_*) = c < 0$, which implies the fact that u_* is a non-trivial weak solution of problem (1.1). \square

4 Conclusion

In this paper, we are devoted to analyzing a class of nonlinear fractional differential models generated by impulsive effects. By variational methods, we can find the range of controlled parameters to ensure the existence of solutions for this type of differential model when nonlinearity f is p -suplinear growth or p -sublinear growth. It should be pointed out that if $\lambda = 0$ and the nonlinearity f is p -suplinear growth, our results enrich and extend some previous results.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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