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Existence and general decay estimate for a nonlinear plate problem

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Abstract

In this work, we study a plate equation modelling a suspension bridge with weak damping and hanger restoring force. We prove the well-posedness and establish an explicit and general decay result without putting restrictive growth conditions on the frictional damping term.

Keywords: existence; decay rates; plate equation; suspension bridge

1 Introduction

The study of plate problems has been widely investigated by mathematicians and other scientists. Plate problems have a lot of applications in different areas of science and engineering such as material engineering, mechanical engineering, nuclear physics and optics. In order to describe the structural behaviour and the stability of large structures in our societies, plate models have been extensively used. For instance, the Kirchhoff theory of plates [1] establishes a two-dimensional mathematical model that is used to determine the stresses and deformations in thin plates subjected to forces and moments. The stability of Kirchhoff plates in the presence of a linear or nonlinear source has been studied by many authors. See, for instance, the results obtained in Komornik [2], Lagnese [3] and Lasiecka [4, 5]. Al-Gharabli and Messaoudi [6] studied the following nonlinear plate problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u + h(u_t) = ku \ln|u|u, & \text{in } \Omega \times (0, +\infty), \\ u = \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

and established decay of solutions. Lu [7] investigated the nonautonomous plate-type evolutionary problem

$$\begin{cases} u_{tt} + a(x)u_t + \Delta^2 u + \lambda u + f(u) = g(x,t), & \text{in } \Omega \times (0, T), \\ u = \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega \times [0, T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

and proved the existence of a uniform attractor. Ji and Lasiecka [8] considered a semilinear Kirchhoff plate with a nonlinear dissipation acting via moments

$$\begin{cases} w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w + f(w) = 0, & \text{in } \Omega \times (0, +\infty), \\ w = 0, \qquad \Delta w = -g(\frac{\partial w_t}{\partial \eta}), & \text{in } \partial\Omega \times [0, +\infty), \\ w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), & \text{in } \Omega, \end{cases} \tag{1.3}$$

and proved that the plate is uniformly stabilizable with uniform energy decay rates with respect to the parameter γ which represents rotational force. Moreover, they showed that as $\gamma \longrightarrow 0$, the solutions of the Kirchhoff plate equation converge to the solutions of the semilinear Euler-Bernoulli plate, which is also uniformly stable in finite energy norm.

Recently, plate models have also been of great importance in studying the structural behaviour and instability of suspension bridges. The first attempt to model a suspension bridge through a plate is due to Ferrero and Gazzola [9], where the following hyperbolic problem was introduced:

$$\begin{cases} u_{tt} + \eta u_t + \Delta^2 u + h(x,y,u) = f(x,y,t), & \text{in } \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for } (y,t) \in (-\ell,\ell) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for } (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for } (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma) u_{xxy}(x,\pm\ell,t) = 0, & \text{for } (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \qquad u_t(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $\Omega = (0,\pi) \times (-\ell,\ell) \subset \mathbb{R}^2$ is a planar rectangular plate, σ is the Poisson ratio, η is the damping coefficient, h is the nonlinear restoring force of the hangers and f is an external force. The well-posedness and long-time behaviour of this problem were proved in [9] under suitable assumptions on h. A quasilinear stationary variant of this equation was as well suggested in [10]. Wang [11] considered the following fourth-order equation:

$$u_{tt} + \delta u_t + \Delta^2 u + au = |u|^{m-2} u, \tag{1.5}$$

with the same boundary conditions and initial data as in [10]. He proved the existence and uniqueness of local solution and a finite time blow up result. Messaoudi and co-authors [12–16] have carried out extensive analysis of the suspension bridge plate model (1.4), where existence, decay and global attractor results have been established. For more related results, see Gazzola and Wang [17], Berchio et al. [18] and the book [19] on mathematical models for suspension bridges by Gazzola.

In this paper, we consider the following fourth-order plate equation:

$$u_{tt} + \Delta^2 u + \beta(t) g(u_t) + h(u) = 0, \quad \text{in } \Omega \times (0,T) \tag{1.6}$$

with the same boundary and initial conditions as in (1.4), where g is a nonlinear function to be specified later, β is the damping coefficient and u represents the downward displacement of a vibrating suspension bridge under the effect of weak frictional damping.

The main aim is to discuss the well-posedness of problem (1.6) and the decay rate of the associated energy functional without any restrictive growth condition on the damping term g. For the well-posedness, we reformulate (1.6) into a semigroup setting and apply the semigroup theory (see Pazy [20]). For the decay rate, we exploit some convexity properties used by Mustafa and Messaoudi [21]. The rest of this work is organised as follows. In Section 2, we present preliminary materials which will be helpful in obtaining our results. In Section 3, we discuss the well-posedness of problem (1.6). In Section 4, we study the decay rate of the energy functional associated with problem (1.6).

2 Preliminaries

In this section, we state some preliminary material that will be helpful in achieving our result. We assume that the functions β, g and h satisfy the following assumptions:

(A_1) $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function.

(A_2) $h : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz nondecreasing function such that $h(0) = 0$, and denote $H(s) = \int_0^s h(\tau)\,d\tau$, which is positive, such that

$$sh(s) - H(s) \geq 0, \quad \forall s \in \mathbb{R}.$$

(A_3) $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz nondecreasing C^1-function such that there exist $\epsilon, c_1, c_2 > 0$ and an increasing function $M \in C^1([0, +\infty))$ with M linear or $M(0) = M'(0) = 0$ is a strictly convex C^2-function on $[0, \epsilon)$ such that

$$\begin{cases} c_1|s| \leq |g(s)| \leq c_2|s|, & \text{if } |s| \geq \epsilon, \\ s^2 + g^2(s) \leq M^{-1}(sg(s)), & \text{if } |s| \leq \epsilon. \end{cases} \tag{2.1}$$

Remark 2.1

1. We obtain from assumption (A_3) that $sg(s) > 0$ for $s \neq 0$.
2. Assumption (A_3) with $\epsilon = 1$ was first introduced by Lasiecka and Tataru [22], where decay estimates for a second-order nonlinear wave equation with nonlinear boundary damping were established.
3. To achieve our decay result, we borrow the techniques used by Mustafa and Messaoudi in [21] to prove decay estimates for a second-order wave equation with Dirichlet boundary conditions.

As in [9], let us introduce the space

$$H_*{}^2(\Omega) := \{w \in H^2(\Omega) : w(0, y) = w(\pi, y) = 0, \forall y \in (-\ell, \ell)\}, \tag{2.2}$$

together with the inner product

$$(u, v)_{H_*^2} = \int_\Omega \left[\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\right] dx\,dy. \tag{2.3}$$

For the completeness of the space $H_*{}^2(\Omega)$, we have the following results by Ferrero and Gazzola [9].

Lemma 2.1 ([9]) *Assume that* $0 < \sigma < \frac{1}{2}$. *Then the usual* $H^2(\Omega)$*-norm and the norm defined by* $(\cdot,\cdot)_{H_*^2} = \|\cdot\|^2_{H_*^2(\Omega)}$ *are equivalent. Moreover,* $H_*^2(\Omega)$ *is a Hilbert space when endowed with the scalar product* $(\cdot,\cdot)_{H_*^2}$.

Theorem 2.1 ([9]) *Assume that* $0 < \sigma < \frac{1}{2}$ *and* $f \in L^2(\Omega)$. *Then there exists a unique* $u \in H_*^2(\Omega)$ *such that*

$$(u,v)_{H_*^2} = \int_\Omega fv, \quad \forall v \in H_*^2(\Omega). \tag{2.4}$$

Remark 2.2 The function $u \in H_*^2(\Omega)$ satisfying (2.4) is called the weak solution of the stationary problem

$$\begin{cases} \Delta^2 u = f, \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, \\ u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = u_{yyy}(x,\pm l) + (2-\sigma)u_{xxy}(x,\pm l) = 0. \end{cases} \tag{2.5}$$

Theorem 2.2 ([9]) *The weak solution* $u \in H_*^2(\Omega)$ *of* (2.4) *is in* $H^4(\Omega)$, *and there exists a constant* $C = C(l,\sigma) > 0$ *such that*

$$\|u\|_{H^4(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.6}$$

In addition, if $u \in C^4(\bar{\Omega})$, *then u is a classical solution of* (2.5).

Lemma 2.2 ([9]) *Let* $u \in H_*^2(\Omega)$ *and assume that* $1 \le p < +\infty$. *Then there exists a constant* $C_* = C_*(\Omega,p) > 0$ *such that*

$$\|u\|_{L^p(\Omega)} \le C_*\|u\|_{H_*^2(\Omega)}.$$

3 Well-posedness

In this section, we discuss the well-posedness of problem (1.6). We begin with the definition of a weak solution of problem (1.6).

Definition 3.1 We say that a function

$$u \in C\big([0,T], H_*^2(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big) \tag{3.1}$$

is a weak solution of (1.6) if

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t w + (u,w)_{H_*^2(\Omega)} + \beta(t)\int_\Omega g(u_t)w + \int_\Omega h(u)w = 0, \quad \forall w \in H_*^2(\Omega), \\ u(0) = u_0, \qquad u_t(0) = u_1, \\ \quad \text{for a.e } t \in (0,T). \end{cases} \tag{3.2}$$

Now, we reformulate problem (1.6) into a semigroup setting. Let $u_t = v$, then problem (1.6) becomes

$$\begin{cases} U_t + AU = F(t,U), \\ U(0) = U_0, \end{cases} \tag{3.3}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad AU = \begin{pmatrix} -v \\ \Delta^2 u \end{pmatrix}, \qquad F(t,U) = \begin{pmatrix} 0 \\ -h(u) - \beta(t)g(v) \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

We introduce the Hilbert space

$$\mathcal{H} = H_*^2(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$(U,V)_{\mathcal{H}} = (u,\tilde{u})_{H_*^2(\Omega)} + (v,\tilde{v})_{L^2(\Omega)}, \tag{3.4}$$

where

$$U = (u,v)^T, \qquad V = (\tilde{u},\tilde{v})^T \in \mathcal{H}.$$

Next, we consider the following stationary boundary conditions:

$$\begin{cases} u_{xx}(0,y) = u_{xx}(\pi,y) = 0, \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0. \end{cases} \tag{3.5}$$

The domain of the operator A is defined as

$$D(A) = \bigl\{(u,v) \in \mathcal{H} : u \in H^4(\Omega) \text{ satisfying (3.5) and } v \in H_*^2(\Omega)\bigr\}.$$

We have the following existence and uniqueness result for problem (3.3).

Theorem 3.1 *Let $U_0 \in \mathcal{H}$ be given. Assume that* (A_1)-(A_3) *hold. Then problem* (3.3) *has a unique global weak solution*

$$U \in C\bigl([0,T],\mathcal{H}\bigr)).$$

Proof To achieve this result, we show that the operator A is maximal monotone and F is locally Lipschitz continuous. For the monotonicity and maximality, see [15] for a complete detail proof.

Local lipschitzness: Let $U, V \in \mathbf{B}_R = \{(u,v) \in D(A) : \|(u,v)\|_{\mathcal{H}} \leq R\}$. By using Lemma 2.2, the local lipschitzness of h and g, and the boundedness of β, we get

$$\begin{aligned} \bigl\|F(t,U) - F(t,V)\bigr\|_{\mathcal{H}}^2 &= \left\| \begin{pmatrix} 0 \\ -h(u) - \beta(t)g(v) \end{pmatrix} - \begin{pmatrix} 0 \\ -h(\tilde{u}) - \beta(t)g(\tilde{v}) \end{pmatrix} \right\|_{\mathcal{H}}^2 \\ &= \int_\Omega \bigl|\bigl(h(\tilde{u}) - h(u)\bigr) + \beta(t)\bigl(g(\tilde{v}) - g(v)\bigr)\bigr|^2 \\ &\leq 2C_R\|u - \tilde{u}\|_{L^2(\Omega)}^2 + 2C_R\beta^2(0)\|v - \tilde{v}\|_{L^2(\Omega)}^2 \end{aligned}$$

$$\begin{aligned} &\leq 2C_R C_* \|u-\tilde{u}\|^2_{H^2_*(\Omega)} + 2C_R\beta^2(0)\|v-\tilde{v}\|^2_{L^2(\Omega)} \\ &\leq C_R\big(\|u-\tilde{u}\|^2_{H^2_*(\Omega)} + \|v-\tilde{v}\|^2_{L^2(\Omega)}\big) \\ &= C_R\|U-V\|^2_{\mathcal{H}}. \end{aligned} \tag{3.6}$$

So, F is locally Lipschitz. Thus, by the semigroup theory (see Pazy [20]), we obtain a local unique solution

$$U \in C\big([0,T_m),\mathcal{H}\big) \quad \text{for some } T_m > 0.$$

To obtain a global unique solution, it suffices to show that $\|U(t)\|_{\mathcal{H}}$ is bounded independently of t. To this end, we multiply $(1.6)_1$ by u_t and integrate over Ω to get

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\|u\|^2_{H^2_*} + \int_\Omega H(u)\right) = -\beta(t)\int_\Omega u_t g(u_t) \leq 0. \tag{3.7}$$

On the account of assumption (A_2) and remark number $(2.1)_1$, we obtain

$$\big\|U(t)\big\|^2_{\mathcal{H}} = \|u_t\|^2_{L^2} + \|u\|^2_{H^2_*} \leq E(t) \leq E(0),$$

where

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\|u\|^2_{H^2_*} + \int_\Omega H(u).$$

This completes the proof. □

4 Decay of the energy

In this section, we discuss the decay rates of the energy functional associated with problem (1.6). To achieve this, we state and prove several lemmas that will be fundamental in establishing the main result.

4.1 Technical lemmas

The energy functional associated with problem (1.6) is given by

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} + \int_\Omega H(u). \tag{4.1}$$

Lemma 4.1 *The energy functional defined in* (4.1) *satisfies*

$$\frac{dE(t)}{dt} = -\beta(t)\int_\Omega u_t g(u_t) \leq 0. \tag{4.2}$$

Proof Multiplying (1.6) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u_t|^2 + \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} + \int_\Omega H(u)\right) + \beta(t)\int_\Omega u_t g(u_t) = 0.$$

From (A_3) we get that $sg(s) > 0$ for all $s \neq 0$. Thus, by using (A_1), we obtain

$$\frac{dE(t)}{dt} = -\beta(t)\int_\Omega u_t g(u_t) \leq 0. \tag{4.3}$$

We note here that the calculations are justified for regular solutions. However, the result in (4.3) remains true for a weak solution by a density argument. □

Define the functional

$$F(t) = mE(t) + \int_\Omega uu_t, \tag{4.4}$$

where m is a positive constant to be specified later.

Lemma 4.2 *Assume that* (A_1)-(A_3) *hold. Then the functional F satisfies, along the solution of* (1.6), *the estimates*

$$F'(t) \leq -E(t) + C\int_\Omega \bigl(u_t^2 + \bigl|ug(u_t)\bigr|\bigr)$$

and

$$F \sim E,$$

where C is a positive constant.

Proof By using (1.6), definition (3.2), Lemma 4.1 and exploiting assumptions (A_1) and (A_2), direct differentiation gives

$$\begin{aligned} F'(t) &= mE'(t) + \int_\Omega u_t^2 + \int_\Omega uu_{tt} \\ &= -m\beta(t)\int_\Omega u_t g(u_t) + \int_\Omega u_t^2 - \|u\|^2_{H^2_*(\Omega)} - \beta(t)\int_\Omega ug(u_t) - \int_\Omega uh(u) \\ &\leq \int_\Omega u_t^2 - \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \int_\Omega H(u) - \beta(t)\int_\Omega ug(u_t) + \int_\Omega \bigl(H(u) - uh(u)\bigr) \\ &\leq -E(t) + \frac{3}{2}\int_\Omega u_t^2 + \beta(t)\int_\Omega \bigl|ug(u_t)\bigr| \\ &\leq -E(t) + C\int_\Omega \bigl(u_t^2 + \bigl|ug(u_t)\bigr|\bigr). \end{aligned} \tag{4.5}$$

Next, we show that $F \sim E$. Using Young's inequality and Lemma 2.2, we have

$$\begin{aligned} F(t) &\leq mE(t) + \frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\ &\leq mE(t) + \frac{1}{2}\int_\Omega u_t^2 + \frac{C_*}{2}\|u\|^2_{H^2_*(\Omega)} \leq \lambda_2 E(t). \end{aligned} \tag{4.6}$$

Also,

$$\begin{aligned} F(t) &\geq mE(t) - \frac{1}{2}\int_\Omega u_t^2 - \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\ &\geq mE(t) - \frac{1}{2}\int_\Omega u_t^2 - \frac{C_*}{2}\|u\|^2_{H^2_*(\Omega)} \\ &= \frac{(m-1)}{2}\int_\Omega u_t^2 + \frac{(m - C_*)}{2}\|u\|^2_{H^2_*(\Omega)} + m\int_\Omega H(u). \end{aligned}$$

We choose $m > 0$ large enough so that $(m-1), (m-C_*) > 0$ and arrive at

$$F(t) \geq \lambda_1 E(t). \tag{4.7}$$

Thus, we get from (4.6) and (4.7) that

$$\lambda_1 E(t) \leq F(t) \leq \lambda_2 E(t).$$

This completes the proof. □

Next, we choose $0 < \epsilon_1 \leq \epsilon$ so that

$$sg(s) \leq \min\{\epsilon, M(\epsilon)\}, \quad \forall |s| \leq \epsilon_1. \tag{4.8}$$

Then, for $|s| \geq \epsilon_1$, the function $s \longmapsto \frac{|g(s)|}{|s|}$ is continuous on compact intervals and thus attains its extrema. Thus, it follows from assumption (A_3) that

$$\begin{cases} c_1'|s| \leq |g(s)| \leq c_2'|s|, & \text{if } |s| \geq \epsilon_1, \\ s^2 + g^2(s) \leq M^{-1}(sg(s)), & \text{if } |s| \leq \epsilon_1. \end{cases} \tag{4.9}$$

As in [23], let us partition Ω as follows:

$$\Omega_1 = \{(x,y) \in \Omega : |u_t| \leq \epsilon_1\} \quad \text{and} \quad \Omega_2 = \{(x,y) \in \Omega : |u_t| > \epsilon_1\}.$$

Lemma 4.3 *The following inequalities hold for any $\epsilon > 0$ along the solution of* (1.6):

$$\int_{\Omega_1} \left(u_t^2 + \left|ug(u_t)\right|\right) \leq \int_{\Omega_1} u_t^2 + C_*\epsilon E(t) + C_\epsilon \int_{\Omega_1} \left|g(u_t)\right|^2 \tag{4.10}$$

and

$$\int_{\Omega_2} \left(u_t^2 + \left|ug(u_t)\right|\right) \leq C\epsilon E(t) - C_\epsilon E'(t), \tag{4.11}$$

where C_* is the embedding constant defined in Lemma 2.2 *and C_ϵ is a generic positive constant depending on ϵ.*

Proof For the first inequality (4.10), we use Young's inequality and Lemma 2.2 to get

$$\begin{aligned} \int_{\Omega_1} \left(u_t^2 + \left|ug(u_t)\right|\right) &\leq \int_{\Omega_1} u_t^2 + \epsilon \int_{\Omega_1} |u|^2 + C_\epsilon \int_{\Omega_1} \left|g(u_t)\right|^2 \\ &\leq \int_{\Omega_1} u_t^2 + C_*\epsilon \|u\|^2_{H^2_*(\Omega)} + C_\epsilon \int_{\Omega_1} \left|g(u_t)\right|^2 \\ &\leq \int_{\Omega_1} u_t^2 + C_*\epsilon E(t) + C_\epsilon \int_{\Omega_1} \left|g(u_t)\right|^2. \end{aligned} \tag{4.12}$$

For the second inequality (4.11), we use Lemma 2.2 and Hölder's inequality to obtain

$$\begin{aligned}\int_{\Omega_2}\left|ug(u_t)\right| &\le \left(\int_{\Omega_2}|u|^2\right)^{\frac{1}{2}}\left(\int_{\Omega_2}\left|g(u_t)\right|^2\right)^{\frac{1}{2}} \\ &\le \|u\|_{L^2(\Omega)}\left(\int_{\Omega_2}\left|g(u_t)\right|^2\right)^{\frac{1}{2}} \\ &\le C_*\|u\|_{H_*^2(\Omega)}\left(\int_{\Omega_2}\left|g(u_t)\right|^2\right)^{\frac{1}{2}}. \end{aligned}\tag{4.13}$$

Now, from $(4.9)_1$ we observe that

$$|s|^2 \le c_1'' sg(s) \quad \text{and} \quad \left|g(s)\right|^2 \le c_2'' sg(s) \quad \text{for some positive constants } c_1'', c_2''.$$

Thus, with this in mind and Young's inequality, we obtain

$$\begin{aligned}\int_{\Omega_2}\left(u_t^2 + \left|ug(u_t)\right|\right) &\le C\int_{\Omega_2} u_t g(u_t) + C\left(\|u\|^2_{H_*^2(\Omega)}\right)^{\frac{1}{2}}\left(\int_{\Omega_2} u_t g(u_t)\right)^{\frac{1}{2}} \\ &\le -CE'(t) + C\left(E(t)\right)^{\frac{1}{2}}\left(-E'(t)\right)^{\frac{1}{2}} \\ &\le -CE'(t) + C\left(\epsilon\left(E(t)\right) - C_\epsilon E'(t)\right) \\ &= C\epsilon E(t) - C_\epsilon E'(t). \end{aligned}\tag{4.14}$$

□

Lemma 4.4 *For ϵ small enough and two positive constants d, C, the functional defined by*

$$L(t) = F_1(t) + C_\epsilon E(t), \quad \text{where } F_1(t) = F(t) + C_\epsilon E(t)$$

satisfies, along the solution of (1.6), *the estimate*

$$L'(t) \le -dE(t) + C\int_{\Omega_1}\left(u_t^2 + \left|g(u_t)\right|^2\right) \tag{4.15}$$

and

$$L \sim E.$$

Proof Using Lemmas 4.2 and 4.3, direct computations give

$$\begin{aligned}F_1'(t) &= F'(t) + C_\epsilon E'(t) \\ &\le -E(t) + C\int_{\Omega_1}\left(u_t^2 + \left|ug(u_t)\right|\right) + C\int_{\Omega_2}\left(u_t^2 + \left|ug(u_t)\right|\right) \\ &\le -E(t) + C\int_{\Omega_1} u_t^2 + CC_e\epsilon E(t) + C_\epsilon\int_{\Omega_1}\left|g(u_t)\right|^2 + C\epsilon E(t) - C_\epsilon E'(t) \\ &\le -(1 - C\epsilon)E(t) + C_\epsilon\int_{\Omega_1}\left(u_t^2 + \left|g(u_t)\right|^2\right) - C_\epsilon E'(t). \end{aligned}$$

That is,

$$\big(F_1(t) + C_\epsilon E(t)\big)' \le -(1 - C\epsilon)E(t) + C_\epsilon \int_{\Omega_1} \big(u_t^2 + \big|g(u_t)\big|^2\big). \tag{4.16}$$

This implies

$$L'(t) \le -(1 - C\epsilon)E(t) + C_\epsilon \int_{\Omega_1} \big(u_t^2 + \big|g(u_t)\big|^2\big). \tag{4.17}$$

We then choose ϵ small enough so that $(1 - C\epsilon) > 0$ and obtain the result. It is easy to see that $L \sim E$ since $F \sim E$. This completes the proof. □

4.2 Main decay result

Now, we state and prove our main decay result.

Theorem 4.1 *Assume that* (A_1)-(A_3) *hold. Then there exist positive constants* $k_1, k_2, k_3, \epsilon_0$ *such that the solution of* (1.6) *satisfies*

$$E(t) \le k_3 M_1^{-1}\left(k_1 \int_0^t \beta(s)\,ds + k_2\right), \quad \forall t \ge 0, \tag{4.18}$$

where

$$M_1(t) = \int_t^1 \frac{1}{M_2(s)}\,ds, \qquad M_2(t) = tM'(\epsilon_0 t) \tag{4.19}$$

and M_1 *is strictly decreasing on* $(0,1]$ *and* $\lim_{t\to 0} M_1(t) = +\infty$.

Proof We have two cases as follows.

Case I. M is linear on $(0,\epsilon]$: Multiplying (4.15) by $\beta(t)$ and using $(4.9)_2$, we deduce that

$$\begin{aligned}
\beta(t)L'(t) &\le -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1}\big(u_t^2 + \big|g(u_t)\big|^2\big)\\
&\le -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1} M^{-1}\big(u_t g(u_t)\big)\\
&= -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1} u_t g(u_t)\\
&\le -d\beta(t)E(t) + C\beta(t)\int_{\Omega} u_t g(u_t)\\
&= -d\beta(t)E(t) - CE'(t).
\end{aligned}$$

By using (A_1), we obtain

$$\big(\beta(t)L(t) + CE(t)\big)' \le -d\beta(t)E(t). \tag{4.20}$$

Let $J_1 = \beta L + CE$. Then $J_1 \sim E$ since $L \sim E$, and we get from (4.20)

$$J_1'(t) \le -k_1\beta(t)J_1(t). \tag{4.21}$$

Simple integration of (4.21) over $(0,t)$ and using the fact that $J_1 \sim E$ give

$$E(t) \leq k_2 e^{-k_1 \int_0^t \beta(s)\,ds} = cM_1^{-1}\left(c\int_0^t \beta(s)\,ds\right).$$

Case II. M is nonlinear on $(0,\epsilon]$. In this case, we consider the functional $I(t)$ defined by

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u_t g(u_t).$$

We know that M is convex, so M^{-1} is concave. Thus, Jensen's inequality yields

$$M^{-1}\big(I(t)\big) \geq \frac{1}{|\Omega_1|}\int_{\Omega_1} M^{-1}\big(u_t g(u_t)\big). \tag{4.22}$$

By using $(4.9)_2$, we obtain

$$\beta(t)\int_{\Omega_1}\big(u_t^2 + \big|g(u_t)\big|^2\big) \leq \beta(t)\int_{\Omega_1} M^{-1}\big(u_t g(u_t)\big) \leq C\beta(t)M^{-1}\big(I(t)\big). \tag{4.23}$$

We multiply (4.15) by $\beta(t)$ and use (4.23) to arrive at

$$\begin{aligned}\beta(t)L'(t) &\leq -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1}\big(u_t^2 + \big|g(u_t)\big|^2\big)\\ &\leq -d\beta(t)E(t) + C\beta(t)M^{-1}\big(I(t)\big).\end{aligned} \tag{4.24}$$

This implies

$$\beta(t)L'(t) + E'(t) \leq -d\beta(t)E(t) + C\beta(t)M^{-1}\big(I(t)\big)$$

since $E' \leq 0$. Using (A_1), we obtain

$$R_0'(t) \leq -d\beta(t)E(t) + C\beta(t)M^{-1}\big(I(t)\big),$$

where

$$R_0 = \beta L + E \sim E. \tag{4.25}$$

Let $\epsilon_0 < \epsilon$, $C_0 > 0$ and define the functional

$$R_1(t) = M'\left(\epsilon_0\frac{E(t)}{E(0)}\right)R_0(t) + C_0 E(t). \tag{4.26}$$

Let us note here that $E(0) > 0$, otherwise $E(t) = 0$, $\forall t \in \mathbb{R}^+$, and thus the theorem is verified since $E'(t) \leq 0$. Now, since $R_0 \sim E$ and $E' \leq 0$, $M' > 0$ (M is increasing), $M'' > 0$ (M is convex) on $(0,\epsilon]$, then R_1 satisfies the following:

$$\alpha_1 R_1(t) \leq E(t) \leq \alpha_2 R_1(t) \quad \text{for some } \alpha_1,\alpha_2 > 0 \tag{4.27}$$

and it follows from (4.26) that

$$\begin{aligned} R_1'(t) &= \epsilon_0\left(\frac{E'(t)}{E(0)}\right)M''\left(\epsilon_0\frac{E(t)}{E(0)}\right)R_0(t) + M'\left(\epsilon_0\frac{E(t)}{E(0)}\right)R_0'(t) + C_0E'(t) \\ &\leq M'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\left[-d\beta(t)E(t) + C\beta(t)M^{-1}\big(I(t)\big)\right] + C_0E'(t) \\ &= -d\beta(t)E(t)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) \\ &\quad + C\beta(t)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right)M^{-1}\big(I(t)\big) + C_0E'(t). \end{aligned} \tag{4.28}$$

Now, let M^* be the convex conjugate of M in the sense of Young. Then

$$M^*(s) = s\big(M'\big)^{-1}(s) - M\big(\big(M'\big)^{-1}(s)\big), \quad \text{if } s \in \big(0, M'(\epsilon)\big) \tag{4.29}$$

and M^* satisfies the generalised Young's inequality

$$XY \leq M^*(X) + M(Y), \quad \text{if } X \in \big(0, M'(\epsilon)\big), Y \in (0, \epsilon). \tag{4.30}$$

Next, we set $X = M'(\epsilon_0\frac{E(t)}{E(0)})$ and $Y = M^{-1}(I(t))$. By using Lemma 4.1, the fact that $sg(s) \leq \min\{\epsilon, G(\epsilon)\}$, if $|s| \leq \epsilon_1$ and (4.28)-(4.30), we obtain

$$\begin{aligned} R_1'(t) &\leq -d\beta(t)E(t)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + C_0E'(t) \\ &\quad + C\beta(t)\left[M^*\left(M'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right) + M\big(M^{-1}\big(I(t)\big)\big)\right] \\ &= -d\beta(t)E(t)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + C_0E'(t) \\ &\quad + C\beta(t)M^*\left(M'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right) + C\beta(t)I(t) \\ &= -d\beta(t)E(t)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + C_0E'(t) \\ &\quad + C\epsilon_0\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) - C\beta(t)M\left(\epsilon_0\frac{E(t)}{E(0)}\right) + C\beta(t)I(t) \\ &\leq -E(0)d\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + C\epsilon_0\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) \\ &\quad + C\beta(t)\int_\Omega u_tg(u_t) + C_0E'(t) \\ &\leq -E(0)d\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + C\epsilon_0\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) \\ &\quad - CE'(t) + C_0E'(t). \end{aligned}$$

We choose C_0 large enough and ϵ_0 small enough such that

$$C - C_0 < 0, \qquad E(0)d - C\epsilon_0 > 0,$$

and arrive at

$$R_1'(t) \leq -k\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0\frac{E(t)}{E(0)}\right) = -k\beta(t)M_2\left(\epsilon_0\frac{E(t)}{E(0)}\right), \tag{4.31}$$

where $M_2(t) = tM'(\epsilon_0 t)$. We have that

$$M_2'(t) = M'(\epsilon_0 t) + \epsilon_0 tM''(\epsilon_0 t).$$

Thus, using the strict convexity of M on $(0,\epsilon]$, we get that $M_2, M_2' > 0$ on $(0,1]$. It follows from (4.27) and (4.31) that the functional

$$R_2(t) = \alpha_1\frac{R_1(t)}{E(0)}$$

satisfies

$$R_2 \sim E \tag{4.32}$$

and

$$R_2'(t) \leq -k_1\beta(t)M_2\big(R_2(t)\big) \quad \text{for some } k_1 > 0. \tag{4.33}$$

Inequality (4.33) implies that

$$\big(M_1\big(R_2(t)\big)\big)' \geq k_1\beta(t),$$

where

$$M_1(\tau) = \int_\tau^1 \frac{1}{M_2(s)}\,ds, \quad \tau \in (0,1].$$

Thus, integrating (4.33) over $(0,t)$ and noting that M_1 is strictly decreasing on $(0,1]$ give

$$R_2(t) \leq M_1^{-1}\left(k_1\int_0^t \beta(s)\,ds + k_2\right) \quad \text{for some } k_2 > 0. \tag{4.34}$$

Combining (4.32) and (4.34), we get the result. This completes the proof. □

5 Examples

In this section, we illustrate our result with some examples. As in [21], let $g_0 \in C^2([0,+\infty))$ be a strictly increasing function such that $g_0(0) = 0$, and for some positive constants c_1, c_2 and ϵ, the function g satisfies

$$\begin{cases} c_1|s| \leq |g(s)| \leq c_2|s|, & \forall |s| \geq \epsilon, \\ g_0(|s|) \leq |g(s)| \leq g_0^{-1}(|s|), & \forall |s| \leq \epsilon. \end{cases} \tag{5.1}$$

Define the function

$$M(s) = \left(\sqrt{\frac{s}{2}}\right)g_0\left(\sqrt{\frac{s}{2}}\right). \tag{5.2}$$

Then M is a C^2-strictly convex function on $(0,\epsilon]$ when g_0 is nonlinear and thus satisfies assumption (A_3). We give some examples of g_0 such that g satisfies (5.1) near 0.

(1) Let $g_0(s) = ks$, where k is a positive constant, then $M(s) = ks$ satisfies (A_3), and we get

$$E(t) \leq ke^{-k_1 \int_0^t \beta(s)\,ds}, \quad \forall t \geq 0.$$

(2) Let $g_0(s) = \frac{1}{s}e^{-\frac{1}{s^2}}$, then $M(s) = e^{-\frac{2}{s}}$ satisfies (A_3) near 0 and

$$E(t) \leq k\left(\ln\left(k_1 \int_0^t \beta(s)\,ds + k_2\right)\right)^{-1}, \quad \forall t \geq 0.$$

(3) Let $g_0(s) = e^{-\frac{1}{s}}$, then $M(s) = \sqrt{\frac{s}{2}}e^{-\sqrt{\frac{2}{s}}}$ satisfies (A_3) near 0, and we obtain

$$E(t) \leq k\left(\ln\left(k_1 \int_0^t \beta(s)\,ds + k_2\right)\right)^{-2}, \quad \forall t \geq 0.$$

6 Conclusion

This paper has been able to establish the well-posedness and decay estimate for a nonlinear plate equation with a partially hinged boundary condition. We also illustrated our result with some examples. This result is new for these types of problems, and it generalises many related problems in the literature.

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