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Multiple positive solutions to singular positone and semipositone m-point boundary value problems of nonlinear fractional differential equations

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Abstract

In this paper, we consider the properties of Green's function for the nonlinear fractional differential equation boundary value problem

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha}u(t)+f\big(t,u(t)\big)=0, \quad 0<t<1,\\ &u(0)=0, \qquad u(1)=\sum_{i=1}^{m-2}\beta_i u(\eta_i), \end{aligned}$$

where $1<\alpha<2$, $0<\beta_i<1$, $i=1,2,\dots,m-2$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1}<1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. Here our nonlinearity f may be singular at $u=0$. As an application of Green's function, we give some multiple positive solutions for singular positone and semipositone boundary value problems by means of the Leray–Schauder nonlinear alternative and a fixed point theorem on cones.

MSC: 34B15

Keywords: Multiple positive solutions; Singular fractional differential equation; Semipositone; m-point boundary value problem

1 Introduction

In this paper, we consider the existence and multiplicity of positive solutions of the nonlinear fractional differential equation semipositone boundary value problem:

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha}u(t)+f\big(t,u(t)\big)=0, \quad 0<t<1,\\ &u(0)=0, \qquad u(1)=\sum_{i=1}^{m-2}\beta_i u(\eta_i), \end{aligned} \tag{1.1}$$

where $1<\alpha<2$, $0<\beta_i<1$, $i=1,2,\dots,m-2$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1}<1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. Here our nonlinearity f may be singular at $u = 0$. The nonlinear fractional differential equation for the multi-point boundary value problem has been studied extensively. For details, see [1–14] and the references therein.

For $m = 3$, Bai [15] investigated the existence and uniqueness of positive solutions for a nonlocal boundary value problem of the fractional differential equation

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = \beta u(\eta), \end{aligned} \tag{1.2}$$

via the contraction map principle and fixed point index theory, where $1 < \alpha \leq 2, 0 < \beta\eta^{\alpha-1} < 1, 0 < \eta < 1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. The function f is continuous on $[0,1] \times [0,\infty)$. Wang, Xiang and Liu [16] investigated the existence and uniqueness of a positive solution to nonzero three-point boundary values problem for a coupled system of fractional differential equations. Ahmad and Nieto [17] considered the three point boundary value problems of the fractional order differential equation. By using some fixed point theorems, they obtained the existence and multiplicity result of positive solution to this problem. They considered the case when f has no singularities. Xu, Jiang et al. [18] deduced some new properties of Green's function of (1.2). By using some fixed point theorems, they obtained the existence, uniqueness and multiplicity of positive solutions to singular positone and semipositone problems. Hussein A.H. Salem [1] investigated the existence of pseudo-solutions for the nonlinear m-point boundary value problem of the fractional case,

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} x(t) + q(t) f\big(t, x(t)\big) = 0, \quad \text{a.e. on } [0,1], \alpha \in (n-1, n], n \geq 2, \\ &x(0) = x'(0) = \cdots = x^{(n-1)} = 0, \qquad x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i), \end{aligned} \tag{1.3}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \zeta_i > 0$ with $\sum_{i=1}^{m-2} \zeta_i \eta_i^{\alpha-1} < 1$. It is assumed that q is a real-valued continuous function and f is a nonlinear Pettis integrable function.

However, no paper to date has discussed the multiplicity for the semipositone singular problem. This paper attempts to fill this gap in the literature, and as a corollary, we give a result for singular positone problems.

2 Background materials

For convenience of the reader, we present here the necessary definitions from fractional calculus theory.

Definition 2.1 ([19]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([19]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to R$ is given by

$$\mathbf{D}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

From the definition of Riemann–Liouville's derivative, one has the following results.

Lemma 2.1 ([19]) *Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, *where* N *is the smallest integer greater than or equal to* α.

Lemma 2.2 ([20]) *Let* X *be a Banach space, and let* $P \subset X$ *be a cone in* X. *Assume* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, *and let* $S : P \to P$ *be a completely continuous operator such that either*

1. $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$, *or*
2. $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \le \|w\|$ $w \in P \cap \partial\Omega_2$.

Then S *has a fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Semipositone and positone singular problem

In this section, we consider the existence and multiplicity of positive solutions of the nonlinear fractional differential equation semipositone and positone boundary value problem

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{aligned} \tag{3.1}$$

where $1 < \alpha < 2, 0 < \beta_i < 1, i = 1, 2, \ldots, m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} < 1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. Here our nonlinearity f may be singular at $u = 0$.

Lemma 3.1 *Given* $y \in C(0,1)$ *the problem*

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{aligned} \tag{3.2}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where $G(t,s) = G_1(t,s) + G_2(t,s)$,

$$G_1(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}-\beta_1 t^{\alpha-1}(\eta_1-s)^{\alpha-1}-(t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \le \eta_1, \\ \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 < \eta_1 \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-1}-\beta_1 t^{\alpha-1}(\eta_1-s)^{\alpha-1}}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le \eta_1 < 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1, \eta_1 \le s. \end{cases} \tag{3.3}$$

$$G_2(t,s) = H(s)t^{\alpha-1}, \qquad H(s) = M_0 \sum_{i=1}^{m-2}\left[\frac{K_i}{1-K_1}q_i(s) + \frac{K_1K_i}{1-K_1}p_i(s)\right], \tag{3.4}$$

where $K_i = \beta_i\eta_i^{\alpha-1}, M_0 = \frac{1}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}$,

$$q_i(s) = (1-s)^{\alpha-1} - \left(1-\frac{s}{\eta_i}\right)^{\alpha-1} I_{[0\le s\le\eta_i]},$$

$$p_i(s) = \left(1-\frac{s}{\eta_i}\right)^{\alpha-1} I_{[0\le s\le\eta_i]} - \left(1-\frac{s}{\eta_1}\right)^{\alpha-1} I_{[0\le s\le\eta_1]},$$

$$I_{[0\le s\le\eta_i]} = \begin{cases} 1, & s \in [0,\eta_i], \\ 0, & s \notin [0,\eta_i]. \end{cases}$$

Proof By Lemma 2.1, the solution of (3.2) can be written as

$$u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds.$$

From $u(0) = 0$, we know that $C_2 = 0$.

On the other hand, together with $u(1) = \sum_{i=1}^{m-2}\beta_i u(\eta_i)$, we have

$$C_1 = \frac{1}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds\right].$$

Therefore, the unique solution of fractional boundary value problem (3.2) is

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds\right] \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right] \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \\ &\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right] \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})} \end{aligned}$$

$$\begin{aligned}&\times\left[\int_0^1(1-s)^{\alpha-1}y(s)\,ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds\right]\\&=u_1(t)+u_2(t),\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}u_1(t)={}&\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\int_0^1(1-s)^{\alpha-1}y(s)\,ds-\beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right]\\&-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds,\end{aligned}\tag{3.6}$$

$$\begin{aligned}u_2(t)={}&-\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\int_0^1(1-s)^{\alpha-1}y(s)\,ds-\beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right]\\&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\int_0^1(1-s)^{\alpha-1}y(s)\,ds\right.\\&\left.-\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds\right].\end{aligned}\tag{3.7}$$

When $t\le\eta_1$, we have

$$\begin{aligned}u_1(t)={}&-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\left(\int_0^t+\int_t^{\eta_1}+\int_{\eta_1}^1\right)(1-s)^{\alpha-1}y(s)\,ds\right]\\&-\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left(\int_0^t+\int_t^{\eta_1}\right)\beta_1(\eta_1-s)^{\alpha-1}y(s)\,ds\\={}&\int_0^t\frac{[t(1-s)]^{\alpha-1}-\beta_1(\eta_1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}y(s)\,ds\\&+\int_t^{\eta_1}\frac{[t(1-s)]^{\alpha-1}-\beta_1(\eta_1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}y(s)\,ds+\int_{\eta_1}^1\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}y(s)\,ds\\={}&\int_0^1G_1(t,s)y(s)\,ds.\end{aligned}$$

When $t\ge\eta_1$, we have

$$\begin{aligned}u_1(t)={}&-\left(\int_0^{\eta_1}+\int_{\eta_1}^t\right)\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\\&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\left(\int_0^{\eta_1}+\int_{\eta_1}^t+\int_t^1\right)(1-s)^{\alpha-1}y(s)\,ds\right]\\&-\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\int_0^{\eta_1}\beta_1(\eta_1-s)^{\alpha-1}y(s)\,ds\\={}&\int_0^{\eta_1}\frac{[t(1-s)]^{\alpha-1}-\beta_1(\eta_1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}y(s)\,ds\\&+\int_{\eta_1}^t\frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}y(s)\,ds+\int_t^1\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}y(s)\,ds\\={}&\int_0^1G_1(t,s)y(s)\,ds,\end{aligned}$$

where $G_1(t,s)$ is defined by (3.3).

$$
\begin{aligned}
u_2(t) = {} & -\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right] \\
& + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right. \\
& \left. - \sum_{i=2}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds\right] \\
= {} & \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds\right. \\
& \left. - \frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds\right] \\
& - \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\sum_{i=2}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds \\
= {} & \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds\right. \\
& - \frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\beta_1\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}y(s)\,ds \\
& \left. - \frac{1-\beta_1\eta_1^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\sum_{i=2}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\,ds\right] \\
= {} & \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds\right. \\
& - \frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\beta_1\eta_1^{\alpha-1}\int_0^{\eta_1}\left(1-\frac{s}{\eta_1}\right)^{\alpha-1}y(s)\,ds \\
& - \frac{1}{1-\beta_1\eta_1^{\alpha-1}}\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}\int_0^{\eta_i}\left(1-\frac{s}{\eta_i}\right)^{\alpha-1}y(s)\,ds \\
& \left. + \frac{\beta_1\eta_1^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}\int_0^{\eta_i}\left(1-\frac{s}{\eta_i}\right)^{\alpha-1}y(s)\,ds\right] \\
= {} & \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}\left[\frac{\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\left(\int_0^1 (1-s)^{\alpha-1}y(s)\,ds\right.\right. \\
& \left. - \int_0^{\eta_i}\left(1-\frac{s}{\eta_i}\right)^{\alpha-1}y(s)\,ds\right) \\
& \left. + \frac{\beta_1\eta_1^{\alpha-1}\sum_{i=2}^{m-2}\beta_i\eta_i^{\alpha-1}}{1-\beta_1\eta_1^{\alpha-1}}\left(\int_0^{\eta_i}\left(1-\frac{s}{\eta_i}\right)^{\alpha-1}y(s)\,ds - \int_0^{\eta_1}\left(1-\frac{s}{\eta_1}\right)^{\alpha-1}y(s)\,ds\right)\right] \\
= {} & \int_0^1 G_2(t,s)y(s)\,ds,
\end{aligned}
$$

where $G_2(t,s)$ is defined by (3.4). The proof is complete. □

Lemma 3.2 ([18]) *The Green's function $G_1(t,s)$ defined by Lemma* 3.1 *has the following properties*:

(1)

$$G(t,s) \le G(s,s) \le \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \quad \text{for } t,s \in [0,1], \beta_1\eta_1^{\alpha-2} \le 1; \tag{3.8}$$

(2)

$$\frac{Mt^{\alpha-1}s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \le G(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \quad \text{for } t,s \in [0,1]; \tag{3.9}$$

(3)

$$\begin{aligned} &\frac{Mt^{\alpha-1}s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \le G(t,s) \le \frac{s(1-s)^{\alpha-1}t^{\alpha-2}(1+\beta_1\eta_1^{\alpha-2})}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \\ &\quad \text{for } t,s \in (0,1]; \end{aligned} \tag{3.10}$$

(4)

$$\begin{aligned} &\frac{Mts(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \le t^{2-\alpha}G(t,s) \le \frac{s(1-s)^{\alpha-1}(1+\beta_1\eta_1^{\alpha-2})}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} \\ &\quad \text{for } t,s \in [0,1], \end{aligned} \tag{3.11}$$

where $0 < M = \min\{1-\beta_1\eta_1^{\alpha-1}, \beta_1\eta_1^{\alpha-2}(1-\eta_1)(\alpha-1), \beta_1\eta_1^{\alpha-1}\} < 1$.

Theorem 3.1 *The Green's function $G(t,s)$ defined by Lemma* 3.1 *has the following properties*:

(1)

$$H(s) \ge 0, \tag{3.12}$$

(2)

$$\begin{aligned} &Mt\left[\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} + H(s)\right] \\ &\quad \le t^{2-\alpha}G(t,s) \\ &\quad \le \left(1+\beta_1\eta_1^{\alpha-2}\right)\left[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right] \quad \text{for } t,s \in [0,1], \end{aligned} \tag{3.13}$$

(3)

$$\begin{aligned} &Mt^{\alpha-1}\left[\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} + H(s)\right] \\ &\quad \le G(t,s) \le t^{\alpha-1}\left[H(s) + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right] \quad \text{for } t,s \in [0,1], \end{aligned} \tag{3.14}$$

where $0 < M = \min\{1-\beta_1\eta_1^{\alpha-1}, \beta_1\eta_1^{\alpha-2}(1-\eta_1)(\alpha-1), \beta_1\eta_1^{\alpha-1}\} < 1$.

Proof From (3.4), we know that $K_i = \beta_i \eta_i^{\alpha-1} > 0, M_0 = \frac{1}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})} > 0$, and $p_i(s) \geq 0$, $q_i(s) \geq 0$ since $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$. Thus, $H(s) \geq 0$. From Lemma 3.2 and the definition of $G_2(t,s)$, it is easy to check that (2)–(3) hold. Thus the proof of Theorem 3.1 is complete. □

For convenience, in this article, we let $\omega = \frac{M}{1+\beta_1\eta_1^{\alpha-2}}, G^*(t,s) = t^{2-\alpha}G(t,s)$.

Lemma 3.3 *Suppose* $e \in C[0,1], e(t) > 0$, *for* $t \in (0,1)$, $1 < \alpha \leq 2$, *and* $\gamma(t)$ *is the unique solution of*

$$\mathbf{D}_{0+}^{\alpha}u(t) + e(t) = 0, \quad 0 < t < 1,$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2}\beta_i u(\eta_i),$$

where $1 < \alpha < 2, 0 < \beta_i < 1, i = 1,2,\dots,m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1} < 1$, *then there exists a constant* C_0 *such that*

$$0 \leq \phi(t) := t^{2-\alpha}\gamma(t) \leq \omega C_0 t, \quad 0 \leq t \leq 1,$$

here $C_0 = \frac{1+\beta_1\eta_1^{\alpha-2}}{M}\int_0^1 [H(s) + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}]e(s)\,ds$.

Proof In fact, from Lemma 3.1, we have

$$\gamma(t) = \int_0^1 G(t,s)e(s)\,ds.$$

Thus, from Theorem 3.1, we have

$$\phi(t) = t^{2-\alpha}\int_0^1 G(t,s)e(s)\,ds \leq \int_0^1 t^{2-\alpha}t^{\alpha-1}\left[H(s) + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right]e(s)\,ds = \omega C_0 t.$$ □

The above lemma together with Lemma 2.2 and the Leray–Schauder alternative principle establishes our main result.

Theorem 3.2 *Suppose the following conditions are satisfied*:

(H_1) $f : (0,1] \times (0,\infty) \to \mathbf{R}$ *is continuous and there exists a function* $e(t) \in C[0,1], e(t) > 0$ *for* $t \in (0,1)$, *with* $f(t,u) + e(t) \geq 0$ *for* $(t,u) \in (0,1] \times (0,\infty)$;

(H_2) $f^*(t,u) = f(t,u) + e(t)$, *and* $f^*(t,t^{\alpha-2}u) \leq q(t)[g(u) + h(u)]$ *on* $(0,1] \times (0,\infty)$ *with* $g > 0$ *continuous and nonincreasing on* $(0,\infty), h \geq 0$ *continuous on* $[0,\infty)$ *and* $\frac{h}{g}$ *nondecreasing on* $(0,\infty), q \in L^1[0,1], q > 0$ *on* $(0,1)$;

(H_3) $a_0 = \int_0^1 q(s)g(s)\,ds < +\infty$;

(H_4) $\exists K_0$ *with* $g(ab) \leq K_0 g(a)g(b), \forall a > 0, b > 0$;

(H_5) $\exists r > C_0$ *with*

$$\frac{r}{g(\omega(r - C_0))\{1 + \frac{h(r)}{g(r)}\}} > K_0 b_0,$$

where

$$b_0 = \int_0^1 \left(1 + \beta_1 \eta_1^{\alpha-2}\right)\left[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right] q(s)g(s)\,ds; \tag{3.15}$$

(H_6) *there exists* $0 < a < \frac{1}{2}$ (*choose and fix it*) *and a continuous, nonincreasing function* $g_1 : (0,\infty) \to (0,\infty)$, *and a continuous function* $h_1 : [0,\infty) \to (0,\infty)$ *with* $\frac{h_1}{g_1}$ *nondecreasing on* $(0,\infty)$ *and with* $f^*(t, t^{\alpha-2}u) \geq q_1(t)[g_1(u) + h_1(u)]$ *for* $(t,u) \in [a,1] \times (0,\infty)$, $q_1(t) \in C([0,1],[0,\infty))$;

(H_7) $\exists R > r$ *with*

$$\frac{Rg_1(\varepsilon a\omega R)}{g_1(R)g_1(\varepsilon\omega aR) + g_1(R)h_1(\varepsilon\omega aR)} \leq \int_a^1 G^*(\sigma,s)q_1(s)\,ds \tag{3.16}$$

here $\varepsilon > 0$ *is any constant* (*choose and fix it*) *so that* $1 - \frac{C_0}{R} \geq \varepsilon$ (*note* ε *exists since* $R > r > C_0$) *and* $0 \leq \sigma \leq 1$ *is such that*

$$\int_a^1 q_1(s)G^*(\sigma,s)\,ds = \sup_{t\in[0,1]} \int_a^1 q_1(s)G^*(t,s)\,ds;$$

(H_8) *for each* $L > 0$, *there exists a function* $\varphi_L \in C[0,1]$, $\varphi_L > 0$ *for* $t \in (0,1)$ *such that* $f^*(t, t^{\alpha-2}u) \geq \varphi_L(t)$ *for* $(t,u) \in (0,1) \times (0,L]$, *and* $\varphi_r(t) > e(t), t \in (0,1)$, *where r is as in* H_5.

Then (3.1) *has at least two solution* u_1, u_2 *with* $u_1(t) > 0, u_2(t) > 0$ *for* $t \in (0,1)$.

Proof To show (3.1) has two nonnegative solutions we will look at the boundary value problem

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} u(t) + f^*\bigl(t, u(t) - \gamma(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{aligned} \tag{3.17}$$

where $1 < \alpha < 2, 0 < \beta_i < 1, i = 1,2,\ldots,m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} < 1$, and γ is as in Lemma 3.1.

We will show, using Lemma 2.2 and the Leray–Schauder alternative principle, that there exist two solutions, u_1, u_2 to (3.17) with $u_1(t) > \gamma(t), u_2(t) > \gamma(t)$ for $t \in (0,1)$. If this is true then $\bar{u}_i(t) = u_i(t) - \gamma(t), 0 \leq t \leq 1$ are nonnegative solutions (positive on (0,1)) of (3.1), $i = 1,2$, since

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} \bar{u}_i(t) \\ &\quad = \mathbf{D}_{0+}^{\alpha} u_i(t) - \mathbf{D}_{0+}^{\alpha} \gamma(t) = -f^*\bigl(t, u_i(t) - \gamma(t)\bigr) + e(t) \\ &\quad = -\bigl[f\bigl(t, u_i(t) - \gamma(t)\bigr) + e(t)\bigr] + e(t) \\ &\quad = -f\bigl(t, \bar{u}_i(t)\bigr), \quad 0 < t < 1. \end{aligned}$$

As a result, we will concentrate our study on (3.17). Suppose that u is a solution of (3.17). Then

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)f^*\big(s,u(s)-\gamma(s)\big)\,ds \\ &= \int_0^1 t^{\alpha-2}G^*(t,s)f^*\big(s,u(s)-\gamma(s)\big)\,ds, \quad 0 \le t \le 1. \end{aligned} \tag{3.18}$$

Let $y(t) := t^{2-\alpha}u(t)$, then from (3.18) we have

$$\begin{aligned} y(t) &= \int_0^1 G^*(t,s)f^*\big(s,s^{\alpha-2}y(s)-\gamma(s)\big)\,ds \\ &= \int_0^1 G^*(t,s)f^*\big(s,s^{\alpha-2}\big(y(s)-\phi(s)\big)\big)\,ds, \end{aligned} \tag{3.19}$$

where γ and ϕ are as in Lemma 3.2.

Let $E = C[0,1]$ be endowed with maximum norm, $\|u\| = \max_{0\le t\le 1}|u(t)|$, and define the cone $K \subset E$ by

$$K = \big\{u \in E | u(t) \ge \omega t\|u\|\big\},$$

and let

$$\Omega_1 = \big\{u \in E; \|u\| < r\big\}, \qquad \Omega_2 = \big\{u \in E; \|u\| < R\big\}.$$

Next let $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to E$ be defined by

$$(Ay)(t) = \int_0^1 G^*(t,s)f^*\big(s,s^{\alpha-2}\big(y(s)-\phi(s)\big)\big)\,ds, \quad 0 \le t \le 1. \tag{3.20}$$

First we show A is well defined. To see this notice if $y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ then $r \le \|y\| \le R$ and $y(t) \ge \omega t\|y\| \ge \omega tr, 0 \le t \le 1$. Also notice for $t \in (0,1)$ that Lemma 3.1 implies

$$y(t) - \phi(t) \ge \omega tr - \omega tC_0 = \omega t(r - C_0), \quad t \in [0,1],$$

and for $t \in (0,1)$, from (H_2) we have

$$\begin{aligned} f^*\big(t,t^{\alpha-2}\big(y(t)-\phi(t)\big)\big) &= f\big(t,t^{\alpha-2}\big(y(t)-\phi(t)\big)\big) + e(t) \\ &\le q(t)\big[g\big(y(t)-\phi(t)\big) + h\big(y(t)-\phi(t)\big)\big] \\ &= q(t)g\big(y(t)-\phi(t)\big)\left\{1 + \frac{h(y(t)-\phi(t))}{g(y(t)-\phi(t))}\right\} \\ &\le q(t)g\big(\omega t(r-C_0)\big)\left\{1 + \frac{h(R)}{g(R)}\right\} \\ &\le K_0q(t)g(t)g\big(\omega(r-C_0)\big)\left\{1 + \frac{h(R)}{g(R)}\right\}. \end{aligned}$$

These inequalities with (H_3) guarantee that $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to E$ is well defined.

If $y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, then we have

$$\begin{cases} \|Ay\| \leq \int_0^1 (1 + \beta_1 \eta_1^{\alpha-2}[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}] f^*(s, s^{\alpha-2}(y(s) - \phi(s)))\,ds, \\ (Ay)(t) \geq \int_0^1 Mt[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}] f^*(s, s^{\alpha-2}(y(s) - \phi(s)))\,ds \\ \qquad \geq \omega t\|Ay\|, \quad t \in [0,1], \end{cases}$$

i.e., $Ay \in K$ so $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$.

Similarly reasoning as in the proof of Theorem 3.1 [18] shows that $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is continuous and completely continuous.

We now show

$$\|Ay\| \leq \|y\| \quad \text{for } K \cap \partial\Omega_1. \tag{3.21}$$

To see this, let $y \in K \cap \partial\Omega_1$. Then $\|y\| = r$ and $y(t) \geq \omega tr$ for $t \in [0,1]$. Now for $t \in (0,1)$ (as above)

$$y(t) - \phi(t) \geq \omega t(r - C_0) > 0,$$

then we have

$$\begin{aligned} (Ay)(t) &= \int_0^1 G^*(t,s) f^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds \\ &\leq \int_0^1 \bigl(1 + \beta_1\eta_1^{\alpha-2}\bigr)\left[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right] \\ &\quad \times q(s)\bigl[g\bigl(y(s) - \phi(s)\bigr) + h\bigl(y(s) - \phi(s)\bigr)\bigr]\,ds \\ &\leq \int_0^1 \bigl(1 + \beta_1\eta_1^{\alpha-2}\bigr)\left[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right] q(s) g\bigl(\omega s(r - C_0)\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\} ds \\ &\leq \int_0^1 \bigl(1 + \beta_1\eta_1^{\alpha-2}\bigr)\left[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right] \\ &\quad \times q(s) K_0 g\bigl(\omega(r - C_0)\bigr) g(s)\left\{1 + \frac{h(r)}{g(r)}\right\} ds. \end{aligned}$$

This together with (H_5) yields

$$\|Ay\| < r = \|y\|,$$

so (3.21) is satisfied.

Next we show

$$\|Ay\| \geq \|y\| \quad \text{for } K \cap \partial\Omega_2. \tag{3.22}$$

To see this let $y \in K \cap \partial\Omega_2$, then $\|y\| = R$ and $y(t) \geq \omega tR$ for $t \in [0,1]$. Also for $t \in [0,1]$ we have

$$y(t) - \phi(t) \geq \omega t(R - C_0) \geq \omega tR\left(1 - \frac{C_0}{R}\right) \geq \varepsilon\omega tR.$$

Now with σ as in the statement of Theorem 3.1, we have

$$\begin{aligned}(Ay)(\sigma) &= \int_0^1 G^*(\sigma,s)f^*\big(s,s^{\alpha-2}\big(y(s)-\phi(s)\big)\,ds \\ &\geq \int_a^1 G^*(\sigma,s)q_1(s)\big[g_1\big(y(s)-\phi(s)\big)+h_1\big(y(s)-\phi(s)\big]\,ds \\ &= \int_a^1 G^*(\sigma,s)q_1(s)g_1\big(y(s)-\phi(s)\big)\left\{1+\frac{h_1(y(s)-\phi(s))}{g_1(y(s)-\phi(s))}\right\}ds \\ &\geq g_1(R)\int_a^1 G^*(\sigma,s)q_1(s)\left\{1+\frac{h_1(\varepsilon\omega aR)}{g_1(\varepsilon\omega aR)}\right\}ds.\end{aligned}$$

This together with (3.16) yields

$$(Ay)(\sigma)\geq R=\|y\|.$$

Thus $\|Ay\|\geq\|y\|$, (3.22) holds.

Now Lemma 2.2 implies A has a fixed point $y_1\in K\cap(\bar{\Omega}_2\setminus\Omega_1)$, i.e. $r\leq\|y_1\|\leq R$ and $y_1(t)\geq\omega tr$ for $t\in[0,1]$. Thus $y_1(t)$ is a solution of (3.19) with $y_1(t)>\phi(t)$ for $t\in(0,1)$ and $t^{\alpha-2}y_1(t)$ is a solution of (3.17) for $t\in[0,1]$. Therefore, $t^{\alpha-2}y_1(t)-\gamma(t)$ is a positive solution of (3.1).

The existence of another solution is proved using the Leray–Schauder alternative principle, together with a truncation technique.

Since (H_5) holds, we can choose $n_0\in\{1,2,\ldots\}$ such that $\frac{1}{n_0}<r-C_0$ and

$$K_0 g\big(\omega(r-C_0)\big)\left\{1+\frac{h(r)}{g(r)}\right\}b_0+\frac{1}{n_0}<r. \tag{3.23}$$

Let $N_0=\{n_0,n_0+1,\ldots\}$. Consider the family of equations

$$(T_n y)(t)=\int_0^1 G^*(t,s)f_n^*\big(s,s^{\alpha-2}\big(y(s)-\phi(s)\big)\big)\,ds+\frac{1}{n}, \tag{3.24}$$

where $n\in N_0$ and

$$f_n^*\big(s,s^{\alpha-2}u\big)=\begin{cases} f^*(s,s^{\alpha-2}u), & u\geq\frac{1}{n},\\ f^*(s,s^{\alpha-2}\frac{1}{n}), & u\leq\frac{1}{n}.\end{cases}$$

Similarly reasoning as in the proof of Theorem 3.1 shows that T_n is well defined and maps E into K. Moreover, T_n is continuous and completely continuous.

We consider

$$y=\lambda T_n y+(1-\lambda)\frac{1}{n}$$

i.e.

$$y(t)=\lambda\int_0^1 G^*(t,s)f_n^*\big(s,s^{\alpha-2}\big(y(s)-\phi(s)\big)\big)\,ds+\frac{1}{n}, \tag{3.25}$$

where $\lambda \in [0,1]$. We claim that any fixed point y of (3.25) for any $\lambda \in [0,1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that y is a fixed point of (3.25) for some $\lambda \in [0,1]$ such that $\|y\| = r$. Note that

$$\begin{aligned} y(t) - \frac{1}{n} &= \lambda \int_0^1 G^*(t,s) f_n^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds \\ &\le \lambda \int_0^1 \bigl(1 + \beta_1 \eta_1^{\alpha-2}\bigr)\biggl[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta_1\eta_1^{\alpha-1})}\biggr] f_n^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds, \end{aligned}$$

then we have

$$\biggl\| y - \frac{1}{n} \biggr\| \le \lambda \int_0^1 \bigl(1 + \beta_1 \eta_1^{\alpha-2}\bigr)\biggl[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta_1\eta_1^{\alpha-1})}\biggr] f_n^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds.$$

On the other hand, we have

$$\begin{aligned} y(t) - \frac{1}{n} &= \lambda \int_0^1 G^*(t,s) f_n^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds \\ &\ge \lambda M t \int_0^1 \biggl[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta_1\eta_1^{\alpha-1})}\biggr] f_n^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds \\ &\ge \omega t \biggl\| y - \frac{1}{n} \biggr\|. \end{aligned}$$

By the choice of n_0, $\frac{1}{n} \le \frac{1}{n_0} < r - C_0$. Hence, for all $t \in [0,1]$, we have

$$y(t) \ge \omega t \biggl\| y - \frac{1}{n} \biggr\| + \frac{1}{n} \ge \omega t \biggl(\|y\| - \frac{1}{n} \biggr) + \frac{1}{n} \ge \omega t r + (1 - \omega t)\frac{1}{n}.$$

Therefore,

$$y(t) - \phi(t) \ge \omega t r + (1 - \omega t)\frac{1}{n} - \omega t C_0 \ge \omega t \biggl(r - C_0 - \frac{1}{n} \biggr) + \frac{1}{n} > \frac{1}{n}$$

and

$$y(t) - \phi(t) \ge \omega t r + (1 - \omega t)\frac{1}{n} - \omega t C_0 \ge \omega t (r - C_0) + [1 - \omega t]\frac{1}{n} > \omega t (r - C_0).$$

Thus we have from condition (H_2), for all $t \in [0,1]$,

$$\begin{aligned} y(t) &= \lambda \int_0^1 G^*(t,s) f_n^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds + \frac{1}{n} \\ &= \lambda \int_0^1 G^*(t,s) f^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\,ds + \frac{1}{n} \\ &\le \int_0^1 \bigl(1 + \beta_1 \eta_1^{\alpha-2}\bigr)\biggl[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta_1\eta_1^{\alpha-1})}\biggr] \\ &\quad \times q(s)\bigl[g\bigl(y(s) - \phi(s)\bigr) + h\bigl(y(s) - \phi(s)\bigr)\bigr]\,ds + \frac{1}{n} \\ &\le \int_0^1 \bigl(1 + \beta_1 \eta_1^{\alpha-2}\bigr)\biggl[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta_1\eta_1^{\alpha-1})}\biggr] q(s) g\bigl(\omega s (r - C_0)\bigr)\biggl\{1 + \frac{h(r)}{g(r)}\biggr\}\,ds + \frac{1}{n} \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \left(1+\beta_1\eta_1^{\alpha-2}\right)\left[H(s)+\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right]K_0 q(s) g\big(\omega(r-C_0)\big) \\
&\quad \times g(s)\left\{1+\frac{h(r)}{g(r)}\right\}ds+\frac{1}{n} \\
&= K_0 g\big(\omega(r-C_0)\big)\left\{1+\frac{h(r)}{g(r)}\right\}b_0+\frac{1}{n}.
\end{aligned}$$

Therefore,

$$r=\|y\|\leq K_0 g\big(\omega(r-C_0)\big)\left\{1+\frac{h(r)}{g(r)}\right\}b_0+\frac{1}{n}.$$

This is a contradiction to the choice of n_0 and the claim is proved.

From this claim, the Leray–Schauder alternative principle guarantees that

$$y(t)=\int_0^1 G^*(t,s)f_n^*\big(s,s^{\alpha-2}\big(y(s)-\phi(s)\big)\big)\,ds+\frac{1}{n}$$

has a fixed point, denoted by y_n, in $B_r=\{y\in E,\|y\|<r\}$.

Next we claim that $y_n(t)-\phi(t)$ has a uniform positive lower bound, i.e., there exists a constant $\delta>0$, independent of $n\in N_0$, such that

$$\min_{t\in[0,1]}\big\{y_n(t)-\phi(t)\big\}\geq\delta t, \tag{3.26}$$

for all $n\in N_0$. Since (H_8) holds, there exists a continuous function $\varphi_r(t)>0$ such that $f^*(t,t^{\alpha-2}u)>\varphi_r(t)>e(t)$ for all $(t,u)\in(0,1]\times(0,r]$.

Since $y_n(t)-\phi(t)<r$, we have

$$\begin{aligned}
y_n(t)-\phi(t)&=\int_0^1 G^*(t,s)f_n^*\big(s,s^{\alpha-2}\big(y_n(s)-\phi(s)\big)\big)\,ds+\frac{1}{n}-\int_0^1 G^*(t,s)e(s)\,ds \\
&\geq\int_0^1 G^*(t,s)\big(\varphi_r(s)-e(s)\big)\,ds \\
&\geq\int_0^1 Mt\left[H(s)+\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right]\big(\varphi_r(s)-e(s)\big)\,ds:=\delta t.
\end{aligned}$$

Next note

$$\{y_n\}\text{ is equicontinuous on }[0,1]\text{ for all }n\in N_0. \tag{3.27}$$

Similarly reasoning as in the proof of Theorem 3.2 [18], we can easily verify it.

The facts $\|y_n\|<r$ and (3.27) show that $\{y_n\}_{n\in N_0}$ is a bounded and equicontinuous family on $[0,1]$. Now the Arzela–Ascoli Theorem guarantees that $\{y_n\}_{n\in N_0}$ has a subsequence $\{y_{n_k}\}_{n_k\in N_0}$, converging uniformly on $[0,1]$ to a function $y\in E$. From the facts $\|y_n\|<r$ and (3.26), y satisfies $\delta t\leq y(t)-\phi(t)<r$ for all $t\in[0,1]$. Moreover, y_{n_k} satisfies the integral equation

$$y_{n_k}=\int_0^1 G^*(t,s)f_n^*\big(s,s^{\alpha-2}\big(y_{n_k}(s)-\phi(s)\big)\big)\,ds+\frac{1}{n_k}.$$

Letting $k \to \infty$, we arrive at

$$y = \int_0^1 G^*(t,s) f^*\bigl(s, s^{\alpha-2}\bigl(y(s) - \phi(s)\bigr)\bigr)\, ds.$$

Therefore, y is a positive solution of (3.19) and satisfies $0 < \|y\| < r$. Therefore, $t^{\alpha-2}y(t) - \gamma(t)$ is a positive solution of (3.1).

Thus (3.1) has at least two positive solutions $u_1 = t^{\alpha-2}y_1(t) - \gamma(t), u_2(t) = t^{\alpha-2}y(t) - \gamma(t)$. □

It is easy to see that if $e(t) \equiv 0$, we can have a corollary for the singular positone boundary value problems. Here we omit it.

Example 3.1 Consider the boundary value problem

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} u(t) + \mu\bigl(u^{-a}(t) + u^{b}(t) - At^{(2-\alpha)a}\bigr) = 0, \\ &u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{aligned} \tag{3.28}$$

where $0 < a < \alpha - 1 < 1 < b < \frac{1-a}{2-\alpha}, 1 < \alpha < 2, 0 < \beta_i < 1, i = 1, 2, \dots, m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} < 1, 0 < A < 1$, $\mu \in (0, \mu_0)$ is such that

$$\frac{\mu_0^2}{(1-\mu_0 c_0)^a} < \frac{\omega^a}{2d_0}, \qquad \mu_0 < \frac{1}{c_0}; \tag{3.29}$$

here

$$\begin{aligned} c_0 = {}& \frac{A(1+\beta_1\eta_1^{\alpha-2})\Gamma(1+a(2-\alpha))}{M\Gamma(\alpha+1+a(2-\alpha))} \\ &\times \biggl[\frac{1-\beta_1\eta_1^{\alpha+(2-\alpha)a}}{1-\beta_1\eta_1^{\alpha-1}} + \frac{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha+(2-\alpha)a}}{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1}} + \frac{1}{1-\beta_1\eta_1^{\alpha-1}}\biggr], \\ d_0 = {}& \frac{(1+\beta_1\eta_1^{\alpha-2})\Gamma(1-a-b(2-\alpha))}{\Gamma(\alpha+1-a-b(2-\alpha))} \\ &\times \biggl[\frac{1-\beta_1\eta_1^{\alpha-a-b(2-\alpha)}}{1-\beta_1\eta_1^{\alpha-1}} + \frac{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-a-b(2-\alpha)}}{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1}} + \frac{1-a-b(2-\alpha)}{(1-\beta_1\eta_1^{\alpha-1})(2-a-b(2-\alpha))}\biggr]. \end{aligned}$$

Then (3.28) has at least two positive solutions, u_1, u_2 with $u_1(t) > 0, u_2(t) > 0$ for $t \in (0,1)$.

Proof We will apply Theorem 3.2. To this end we take

$$f(t,u) = \mu\bigl(u^{-a} + u^{b} - At^{(2-\alpha)a}\bigr),$$

then

$$f\bigl(t, t^{\alpha-2}y\bigr) = \mu\bigl(t^{a(2-\alpha)}y^{-a} + t^{b(\alpha-2)}y^{b} - At^{(2-\alpha)a}\bigr).$$

Let $q_1(t) = t^{-a(\alpha-2)}, q(t) = t^{b(\alpha-2)}$ and $g(y) = g_1(y) = \mu y^{-a}, h(y) = h_1(y) = \mu y^{b}$, $e(t) = \mu A t^{(2-\alpha)a}$, $K_0 = 1$, $\varphi_L(t) = \mu t^{(2-\alpha)a} L^{-a}$ then (H_1)–(H_4) and (H_6) are satisfied since $0 < a < \alpha - 1 < 1 < b < \frac{1-a}{2-\alpha}, 1 < \alpha < 2$.

Also we have

$$\begin{aligned}
b_0 &= \int_0^1 \left(1+\beta_1\eta_1^{\alpha-2}\right)\left[H(s)+\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right]q(s)g(s)\,ds \\
&= \mu\frac{(1+\beta_1\eta_1^{\alpha-2})\Gamma(1-a-b(2-\alpha))}{\Gamma(\alpha+1-a-b(2-\alpha))} \\
&\quad\times\left[\frac{1-\beta_1\eta_1^{\alpha-a-b(2-\alpha)}}{1-\beta_1\eta_1^{\alpha-1}}+\frac{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-a-b(2-\alpha)}}{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1}}+\frac{1-a-b(2-\alpha)}{(1-\beta_1\eta_1^{\alpha-1})(2-a-b(2-\alpha))}\right] \\
&:= \mu d_0
\end{aligned}$$

and

$$\begin{aligned}
C_0 &= \frac{1+\beta_1\eta_1^{\alpha-2}}{M}\int_0^1\left[H(s)+\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}\right]e(s)\,ds \\
&= \frac{\mu A(1+\beta_1\eta_1^{\alpha-2})\Gamma(1+a(2-\alpha))}{M\Gamma(\alpha+1+a(2-\alpha))} \\
&\quad\times\left[\frac{1-\beta_1\eta_1^{\alpha+(2-\alpha)a}}{1-\beta_1\eta_1^{\alpha-1}}+\frac{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha+(2-\alpha)a}}{1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1}}+\frac{1}{1-\beta_1\eta_1^{\alpha-1}}\right] \\
&:= \mu c_0.
\end{aligned}$$

Now the existence conditions (H_5) and (H_8) become

$$\mu^2 < \frac{r(r-C_0)^a\omega^a}{d_0(1+r^{a+b})}, \qquad r > C_0,$$

and

$$\mu t^{(2-\alpha)a}r^{-a} > \mu A t^{(2-\alpha)a}.$$

Thus, (H_5) and (H_8) hold with $r=1$ since (3.29) and $A<1$.

Finally, (H_7) becomes

$$\mu \geq \frac{R^{1+a}}{\int_a^1 q_1(s)G^*(\sigma,s)\,ds[1+(\omega\varepsilon a)^{a+b}R^{a+b}]}.$$

Since $b>1$, the right-hand side goes to 0 as $R\to\infty$.

Thus, Eq. (3.28) has at least two positive solutions, u_1, u_2 with $u_1(t)>0, u_2(t)>0$ for $t\in(0,1)$. □

4 Conclusion

Prompted by the application of multi-point boundary value problems to applied mathematics and physics, these problems have provoked a great deal of attention by many authors. In this paper, we considered the properties of Green's function for the nonlinear fractional differential equation boundary value problem

$$\begin{aligned}
&\mathbf{D}_{0+}^{\alpha}u(t) + f\bigl(t,u(t)\bigr) = 0, \quad 0 < t < 1, \\
&u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i),
\end{aligned}$$

where $1 < \alpha < 2, 0 < \beta_i < 1, i = 1, 2, \ldots, m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} < 1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. Here our nonlinearity f may be singular at $u = 0$. Unlike the classical expression, we gave a new expression of the Green's function and obtained some properties. As an application of Green's function, we gave some multiple positive solutions for singular positone and semipositone boundary value problems by means of the Leray–Schauder nonlinear alternative, a fixed point theorem on cones. The results show that:

(1) $G(t,s) = G_1(t,s) + G_2(t,s)$,

$$G_1(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - \beta_1 t^{\alpha-1}(\eta_1 - s)^{\alpha-1} - (t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \eta_1, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-\beta_1\eta_1^{\alpha-1})}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 < \eta_1 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1} - \beta_1 t^{\alpha-1}(\eta_1 - s)^{\alpha-1}}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq \eta_1 < 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-\beta_1\eta_1^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \eta_1 \leq s, \end{cases}$$

$$G_2(t,s) = H(s)t^{\alpha-1}, \qquad H(s) = M_0 \sum_{i=1}^{m-2} \left[\frac{K_i}{1-K_1} q_i(s) + \frac{K_1 K_i}{1-K_1} p_i(s) \right],$$

where $K_i = \beta_i \eta_i^{\alpha-1}, M_0 = \frac{1}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1})}$,

$$\begin{aligned}
&q_i(s) = (1-s)^{\alpha-1} - \left(1 - \frac{s}{\eta_i}\right)^{\alpha-1} I_{[0\leq s\leq \eta_i]}, \\
&p_i(s) = \left(1 - \frac{s}{\eta_i}\right)^{\alpha-1} I_{[0\leq s\leq \eta_i]} - \left(1 - \frac{s}{\eta_1}\right)^{\alpha-1} I_{[0\leq s\leq \eta_1]}.
\end{aligned}$$

(2) The Green's function $G(t,s)$ has the following properties:
   (i) $H(s) \geq 0$,
   (ii) $Mt[\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} + H(s)] \leq t^{2-\alpha}G(t,s) \leq (1 + \beta_1\eta_1^{\alpha-2})[H(s) + \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}]$ for $t, s \in [0,1]$,
   (iii) $Mt^{\alpha-1}[\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})} + H(s)] \leq G(t,s) \leq t^{\alpha-1}[H(s) + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta_1\eta_1^{\alpha-1})}]$ for $t, s \in [0,1]$, where $0 < M = \min\{1 - \beta_1\eta_1^{\alpha-1}, \beta_1\eta_1^{\alpha-2}(1-\eta_1)(\alpha-1), \beta_1\eta_1^{\alpha-1}\} < 1$.

(3) Suppose the conditions (H_1)–(H_8) hold. Then (3.1) has at least two solutions, u_1, u_2 with $u_1(t) > 0, u_2(t) > 0$ for $t \in (0,1)$.

(4) The boundary value problem

$$\begin{aligned}
&\mathbf{D}_{0+}^{\alpha}u(t) + \mu\bigl(u^{-a}(t) + u^{b}(t) - At^{(2-\alpha)a}\bigr) = 0, \\
&u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i),
\end{aligned}$$

where $0 < a < \alpha - 1 < 1 < b < \frac{1-a}{2-\alpha}, 1 < \alpha < 2, 0 < \beta_i < 1, i = 1, 2, \ldots, m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} < 1, 0 < A < 1$, has at least two positive solutions, u_1, u_2 with $u_1(t) > 0, u_2(t) > 0$ for $t \in (0,1)$, $\mu \in (0, \mu_0)$, μ_0 is some constant.

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Authors' contributions
XX developed the study concept and design, analyzed and drafted the manuscript. HZ corrected some spelling mistakes and inaccuracies of the expressions. All authors read and approved the final manuscript.

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