Wang et al. *Boundary Value Problems* (2018) 2018:28
https://doi.org/10.1186/s13661-018-0948-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On a uniqueness theorem of Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter

Yu Ping Wang[1], Ko Ya Lien[2] and Chung Tsun Shieh[2*]

*Correspondence:
ctshieh@mail.tku.edu.tw
[2]Department of Mathematics, Tamkang University, New Taipei City, Taiwan
Full list of author information is available at the end of the article

Abstract

Inverse nodal problems for Sturm–Liouville equations associated with boundary conditions polynomially dependent on the spectral parameter are studied. The authors show that a twin-dense subset $W_B([a,b])$ can uniquely determine the operator up to a constant translation of eigenparameter and potential, where $[a,b]$ is an arbitrary interval which contains the middle point of the domain of the operator and B is a subset of $\mathbb{N}$ which satisfies some condition (see Theorem 4.2).

MSC: 34A55; 34B24; 47E05

Keywords: Inverse spectral problem; Inverse nodal problem; Spectral parameter; Potential; Weyl m-function

1 Introduction

The inverse problems of the differential operator $L := L(q, U_0, U_1)$:

$$\begin{cases} lu := -u'' + q(x)u = \lambda u, \quad x \in (0,1), & (1.1)\\ U_0(u) := R_{01}(\lambda)u'(0,\lambda) + R_{00}(\lambda)u(0,\lambda) = 0, & (1.2)\\ U_1(u) := R_{11}(\lambda)u'(1,\lambda) + R_{10}(\lambda)u(1,\lambda) = 0, & (1.3)\end{cases}$$

are considered, where λ is called the spectral parameter, q is a real-valued L^2-function on $(0,1)$ and

$$R_{\xi k}(\lambda) = \sum_{l=0}^{r_{\xi k}} a_{\xi kl}\lambda^{r_{\xi k}-l}, \quad r_{\xi 1} = r_{\xi 0} = r_\xi \geq 0, a_{\xi 10} = 1, \xi, k = 0, 1,$$

are arbitrary polynomials of degree r_ξ with real coefficients such that $R_{\xi 1}(\lambda)$ and $R_{\xi 0}(\lambda)$ have no common zeros for $\xi = 0, 1$. The inverse spectral problem for the Sturm–Liouville equation with boundary conditions dependent on the spectral parameter was studied in [1–7] respectively. In particular, Freiling and Yurko [4] studied three inverse spectral problems for L and showed that this operator L can be recovered either from the Weyl function, or from discrete spectral data, or from two spectra. Recently, the inverse spectral problem for L was studied with mixed spectral data in [7–9]. For the case $R_{00}(\lambda) = 1$, $R_{01}(\lambda) = -h$ in (1.2) and $R_{10}(\lambda) = 1$, $R_{11}(\lambda) = H$ in (1.3), the operator $U(q, U_0, U_1)$ turns to a classical Sturm–Liouville problem $L(q, h, H)$. Inverse spectral problems and inverse nodal problems of $L(q, h, H)$ have been well studied, the readers can refer to [2, 10–21] and the references therein.

The aim of this article is to investigate the inverse spectral and nodal problems for the BVP L. We show that the result on the Weyl m-function for L also holds by an alternative approach, which is a generalization of the result for the classical Sturm–Liouville operator in [16]. Moreover, the authors show that the operator $L(q, U_0, U_1)$ can be uniquely determined up to constant translation by the twin-dense subset in the interior interval.

2 Preliminaries

Let $u_-(x, \lambda)$ and $u_+(x, \lambda)$ be solutions of equation (1.1) with initial conditions

$$u_-(0,\lambda) = R_{01}(\lambda), \qquad u'_-(0,\lambda) = -R_{00}(\lambda),$$
$$u_+(1,\lambda) = R_{11}(\lambda), \qquad u'_+(1,\lambda) = -R_{10}(\lambda).$$

Denote $\lambda = \rho^2$, $\tau = |\operatorname{Im}\rho|$, for sufficiently large $|\lambda|$, we have

$$u_-(x,\lambda) = \lambda^{r_0}\left(\cos\rho x + O\left(\frac{e^{\tau x}}{\rho}\right)\right), \tag{2.1}$$

$$u'_-(x,\lambda) = \lambda^{r_0}\left(-\rho\sin\rho x + O\left(e^{\tau x}\right)\right), \tag{2.2}$$

$$u_+(x,\lambda) = \lambda^{r_1}\left(\cos\rho(1-x) + O\left(\frac{e^{\tau(1-x)}}{\rho}\right)\right), \tag{2.3}$$

$$u'_+(x,\lambda) = \lambda^{r_1}\left(\rho\sin\rho(1-x) + O\left(e^{\tau(1-x)}\right)\right). \tag{2.4}$$

Denote

$$\Delta(\lambda) := [u_+, u_-](x,\lambda),$$

where $[y,z](x) := y(x)z'(x) - y'(x)z(x)$ is the Wronskian of y and z. Then

$$\Delta(\lambda) = -R_{01}(\lambda)u'_+(0,\lambda) - R_{00}(\lambda)u_+(0,\lambda) = -U_0(u_+) = U_1(u_-), \tag{2.5}$$

which is called the characteristic function of L (see [4]). By virtue of (2.1), (2.2), and (2.5), we have

$$\Delta(\lambda) = \lambda^{r_0+r_1}\left(-\rho\sin\rho + \omega\cos\rho + o\left(e^{\tau}\right)\right). \tag{2.6}$$

Define the Weyl m-function $m_\pm(x,\lambda)$ by

$$m_-(x,\lambda) = -\frac{u'_-(x,\lambda)}{u_-(x,\lambda)}, \qquad m_+(x,\lambda) = \frac{u'_+(x,\lambda)}{u_+(x,\lambda)},$$

then

$$m_-(x,\lambda) = i\rho + o(1) \quad \bigl(\text{resp. } m_+(x,\lambda) = i\rho + o(1)\bigr), \tag{2.7}$$

$$\frac{1}{m_-(x,\lambda)} = -\frac{i}{\rho} + o\left(\frac{1}{\rho^2}\right) \quad \left(\text{resp. } \frac{1}{m_+(x,\lambda)} = -\frac{i}{\rho} + o\left(\frac{1}{\rho^2}\right)\right) \tag{2.8}$$

uniformly in $x \in [\delta, 1]$ (resp. $x \in [0, 1-\delta]$) for $|\lambda| \to \infty$ in any sector $\varepsilon < \arg(\lambda) < \pi - \varepsilon$ for $\varepsilon > 0$, where $\delta \in (0,1)$.

Denote the spectrum $\sigma(L) := \{\lambda_n\}_{n=0}^{\infty}$ of L, $\sigma(L)$ consisting of the zeros (counting with multiplicities) of the entire function $\Delta(\lambda)$. For n sufficiently large, λ_n are real and simple and satisfy the asymptotic formulae (see [4])

$$\rho_n := \sqrt{\lambda_n} = (n - r_0 - r_1)\pi + \frac{\omega}{n\pi} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l^2, \tag{2.9}$$

where

$$\omega = \frac{1}{2}\int_0^1 q(t)\,dt - a_{000} + a_{100}. \tag{2.10}$$

3 Inverse spectral problems

For convenience, let $\widetilde{L} = L(\widetilde{q}, \widetilde{U}_0, \widetilde{U}_1)$, where $L(\widetilde{q}, \widetilde{U}_0, \widetilde{U}_1)$ is the operator of the same form as L. If a certain symbol γ denotes an object related to L, then the corresponding symbol $\widetilde{\gamma}$ denotes the analogous object related to $\widetilde{L}$ and $\widehat{\gamma} = \gamma - \widetilde{\gamma}$.

The following two theorems on the Weyl m-function of the BVP L are derived from [4], and they are generalizations of the analogical results for the classical Sturm–Liouville in [16].

Theorem 3.1 *Let $m_-(a_0,\lambda)$ be the Weyl m-function of the BVP L. Then $m_-(a_0,\lambda)$ can uniquely determine functions $R_{0k}(\lambda)$ for $k = 0$ and 1 as well as q (a.e.) on the interval $[0, a_0]$, $0 < a_0 \le 1$.*

Proof Denote by L_D the boundary value problem (1.1), (1.2) together with $\Delta_D(\lambda) := u_-(a_0,\lambda) = 0$ and $\{\mu_{a_0,n}\}_{n=1}^{\infty}$, the zeros (counting with multiplicities) of the entire function $\Delta_D(\lambda)$ (see [10]). Then $\mu_{a_0,n}$ is real and simple for sufficiently large n and

$$u_-(a_0,\lambda) = \lambda^{r_0}\left(\cos(a_0\rho) + \omega_1\frac{\sin(a_0\rho)}{\rho} + o\left(\frac{\mathrm{e}^{a_0\tau}}{\rho}\right)\right) \quad \text{for } |\rho| \gg 1, \tag{3.1}$$

where $\omega_1 = \frac{1}{2}\int_0^{a_0} q(x)\,dx - a_{000}$. Thus we have

$$\sqrt{\mu_{a_0,n}} = \frac{\pi}{a_0}\left(n - r_0 + \frac{1}{2} + \frac{\omega_1}{n\pi} + o\left(\frac{1}{n}\right)\right) \quad \text{for } n \gg 1.$$

By virtue of Hadamard's factorization theorem,

$$u_-(a_0,\lambda) = C_{a_0,0}\lambda^{m_0}\prod_{\mu_{a_0,n}\neq 0}\left(1 - \frac{\lambda}{\mu_{a_0,n}}\right), \tag{3.2}$$

where $C_{a_0,0}$ is a constant and $m_0 \geq 0$. Let $G_{\delta_0} := \{\lambda : |\rho - \frac{\pi}{a_0}(k - r_0 + \frac{1}{2})| > \delta_0, k \in \mathbb{Z}\}$, where δ_0 is sufficiently small, then there exists a constant C_{a_0,δ_0} (see [10, 11]) such that

$$\left|\Delta_D(\lambda)\right| \geq C_{a_0,\delta_0}|\rho|^{2r_0}\mathrm{e}^{a_0\tau}, \quad \forall\lambda \in G_{\delta_0} \text{ and } |\lambda| \gg 1. \tag{3.3}$$

Similarly, denote by L_N the boundary value problem (1.1), (1.2) together with $\Delta_N(\lambda) := u'_-(a_0,\lambda) = 0$ and $\lambda_{a_0,n}$, the zeros (counting with multiplicities) of the entire function $\Delta_N(\lambda)$. Then $\{\mu_{a_0,n}\}_{n=1}^{\infty}$ are real and simple for sufficiently large n and

$$u'_-(a_0,\lambda) = \lambda^{r_0}\left(-\rho\sin(a_0\rho) + c_0\cos(a_0\rho) + o\left(\mathrm{e}^{a_0\tau}\right)\right), \tag{3.4}$$

where $c_0 = 2r_0 + \omega_1$. Therefore we have

$$\sqrt{\lambda_{a_0,n}} = \frac{\pi}{a_0}\left(n - r_0 + \frac{\omega_1}{n\pi} + o\left(\frac{1}{n}\right)\right).$$

Let $G_{\delta_1} := \{\lambda : |\rho - \frac{(k-r_0)\pi}{a_0}| > \delta_1, k \in \mathbb{Z}\}$, where δ_1 is sufficiently small, then there exists a constant C_{a_0,δ_1} such that, for sufficiently large $|\lambda|$,

$$\left|\Delta_N(\lambda)\right| \geq C_{a_0,\delta_1}|\rho|^{2r_0+1}\mathrm{e}^{a_0\tau}, \quad \forall\lambda \in G_{\delta_1}. \tag{3.5}$$

Thus we have

$$u'_-(a_0,\lambda) = C_{a_0,1}\lambda^{m_1}\prod_{\lambda_{a_0,n}\neq 0}\left(1 - \frac{\lambda}{\lambda_{a_0,n}}\right), \tag{3.6}$$

where $m_1 \geq 0$, $C_{a_0,1}$ is a constant. Under the assumption $m_-(a_0,\lambda) = \widetilde{m}_-(a_0,\lambda)$, we obtain

$$\frac{u'_-(a_0,\lambda)}{u_-(a_0,\lambda)} = \frac{\widetilde{u}'_-(a_0,\lambda)}{\widetilde{u}_-(a_0,\lambda)}. \tag{3.7}$$

Since $u_-(a_0,\lambda)$ and $u'_-(a_0,\lambda)$ (resp. $\widetilde{u}_-(a_0,\lambda)$ and $\widetilde{u}'_-(a_0,\lambda)$) have no common zeros, $\frac{\widetilde{u}_-(a_0,\lambda)}{u_-(a_0,\lambda)}$ and $\frac{\widetilde{u}'_-(a_0,\lambda)}{u'_-(a_0,\lambda)}$ are two entire functions in λ and

$$\frac{\widetilde{u}_-(a_0,\lambda)}{u_-(a_0,\lambda)} = \frac{\widetilde{u}'_-(a_0,\lambda)}{u'_-(a_0,\lambda)}. \tag{3.8}$$

By virtue of (3.8) together with (2.1), (2.10), (2.2), and (3.4), this yields

$$\left|\frac{\widetilde{u}_-(a_0,\lambda)}{u_-(a_0,\lambda)}\right| \leq O(1), \quad \forall\lambda \in G_{\delta_0},$$
$$\left|\frac{\widetilde{u}'_-(a_0,\lambda)}{u'_-(a_0,\lambda)}\right| \leq O(1), \quad \forall\lambda \in G_{\delta_1}.$$

Using the maximum modulus principle and Liouville's theorem, we have

$$\begin{cases} \frac{\widetilde{u}_-(a_0,\lambda)}{u_-(a_0,\lambda)} \equiv c, & \forall\lambda \in \mathbb{C}, \\ \frac{\widetilde{u}'_-(a_0,\lambda)}{u'_-(a_0,\lambda)} \equiv c, & \forall\lambda \in \mathbb{C}. \end{cases} \tag{3.9}$$

Letting $\rho = iy \to \infty$, then either the first formula in (3.9) together with (2.3) or the second formula in (3.9) together with (2.4) implies

$$c = \pm 1. \tag{3.10}$$

By virtue of (2.1), (2.2), and (3.10), we have

$$c = 1.$$

Therefore,

$$u_-(a_0,\lambda) = \widetilde{u}_-(a_0,\lambda), \quad \text{and} \quad u'_-(a_0,\lambda) = \widetilde{u}'_-(a_0,\lambda) \quad \forall\lambda \in \mathbb{C}.$$

This implies

$$u_-(x,\lambda) = \widetilde{u}_-(x,\lambda), \quad \forall x \in [0,a_0]. \tag{3.11}$$

Thus (3.11) shows

$$q(x) \overset{a.e.}{=} \widetilde{q}(x) \quad \text{on } [0,a_0] \quad \text{and} \quad R_{0k}(\lambda) = \widetilde{R}_{0k}(\lambda), \quad k = 0,1.$$

Therefore the proof of Theorem 3.1 is completed. □

Analogously, we prove the following theorem on the Weyl m-function $m_+(a_0,\lambda)$.

Theorem 3.2 *Let $m_+(b_0,\lambda)$ be the Weyl m-function of the BVP L. Then $m_+(b_0,\lambda)$ can uniquely determine functions $R_{1k}(\lambda)$ for $k = 0,1$ as well as q (a.e.) on the interval $[b_0,1]$, $0 \le b_0 < 1$.*

4 Inverse nodal problems

By virtue of Lemma 3.1 in [22], we see that, for $n \gg 1$, the eigenfunction $u_-(x,\lambda_n)$ has exactly $n - r_0 - r_1$ zeros $0 < x_n^1 < x_n^2 < \cdots < x_n^j < \cdots < x_n^{n-r_0-r_1} < 1$ inside the interval $(0,1)$ and satisfy the following asymptotic formula:

$$x_n^j = \frac{j-\frac{1}{2}}{n-r_0-r_1} - \frac{(j-\frac{1}{2})\omega}{n(n-r_0-r_1)^2\pi^2} + \frac{\frac{1}{2}\int_0^{x_n^j} q(t)\,dt - a_{000}}{(n-r_0-r_1)^2\pi^2} + O\left(\frac{1}{n^3}\right) \tag{4.1}$$

for $0 < j < n - r_0 - r_1$, where w is as that in (2.10). Denote $x_n^0 = 0$, $x_n^{n-r_0-r_1+1} = 1$. Note that $\sigma(L)$ might contain non-real eigenvalues, hence we write

$$\sigma(L) = \sigma_R(L) \cup \sigma_C(L),$$

where $\sigma_R(L)$ consists of real eigenvalues of L. Denote by X the collection of all zeros of all eigenfunctions of L. Let $B = \{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence in $\mathbb{N}$, where $\lambda_{n_k} \in \sigma_R(L)$. For $0 \le a < b \le 1$, we call the subset $W_B([a,b])$ of $X \cap [a,b]$ an interior twin-dense nodal subset on the interval $[a,b]$ if the following conditions hold:

(1) For all $n_k \in B$, there exists some j_k such that $x_{n_k}^{j_k} \in W_B([a,b])$.

(2) The nodal subset $W_B([a,b])$ is twin on the interval $[a,b]$, i.e., if $x_{n_k}^{j_k} \in W_B([a,b])$, then $x_{n_k}^{j_k+1} \in W_B([a,b])$ or $x_{n_k}^{j_k-1} \in W_B([a,b])$.

(3) The nodal subset $W_B([a,b])$ is dense on the set $[a,b]$, i.e., $\overline{W}_B([a,b]) = [a,b]$, where $\overline{W}_B([a,b])$ denotes the closure of $\overline{W}_B([a,b])$.

The following Lemma 4.1 is necessary for us to prove our main results.

Lemma 4.1 (Theorem 3.2 [8]) *If* $W_B([a,b]) = \widetilde{W}_{\widetilde{B}}([a,b])$, *then*

$$\begin{cases} r_0 + r_1 = \widetilde{r}_0 + \widetilde{r}_1, & (4.2)\\ q(x) - \tilde{q}(x) \overset{a.e.}{=} 2\widehat{\omega} \quad \text{on } [a,b], & (4.3)\\ \lambda_{n_k} - \widetilde{\lambda}_{\widetilde{n}_k} = 2\widehat{\omega}, \quad \forall n_k \in B, & (4.4)\\ n_k = \widetilde{n}_k \quad \text{except for a finite number of } k. & (4.5) \end{cases}$$

Let $S_B = \{\lambda_{n_k} : n_k \in B\}$. For any sequence $S = \{x_n\}_{n=0}^{\infty}$ of positive real numbers, we define

$$N_S(t) = \#\{n \in \mathbb{N} \cup \{0\} : x_n < t\}.$$

The following theorem is our main result which concerns the unique determination of the operator from a twin-dense nodal subset and a partial spectrum.

Theorem 4.2 $0 < a < \frac{1}{2} < b < 1$. *Suppose* $W_B([a,b]) = \widetilde{W}_{\widetilde{B}}([a,b])$ *and*

$$N_{S_B}(t) \geq 2a_1 N_{\sigma_R(L)}(t) + 2(1-a_1)\left(r_0 + r_1 + \frac{1}{2}\right) + 2a_1 k_0 - 1, \quad t \gg 1 \tag{4.6}$$

for $a_1 = a$ *and* $1-b$, *where* k_0 *is the number of elements in* $\sigma_c(L)$. *Then*

$$\widetilde{q}(x) \overset{a.e.}{=} q(x) - c \quad \text{on } [0,1] \quad \text{and} \quad \widetilde{R}_{\xi k}(\lambda) = R_{\xi k}(\lambda + c) \quad \text{for } \xi, k = 0,1,$$

and some constant c.

Proof From Lemma 4.1, we have $r_0 + r_1 = \widetilde{r}_0 + \widetilde{r}_1$, and $r_0 + r_1$ can be reconstructed from (4.1). By virtue of (3.2), one can reconstruct ω by

$$\lim_{k\to\infty} n_k\left(\sqrt{\lambda_{n_k}} - (n_k - r_0 - r_1)\pi\right) = \lim_{k\to\infty}(\omega + \kappa_{n_k}) = \omega, \tag{4.7}$$

$$\lim_{k\to\infty} n_k\left(\sqrt{\widetilde{\lambda}_{n_k}} - (n_k - r_0 - r_1)\pi\right) = \lim_{k\to\infty}(\omega + \kappa_{n_k}) = \widetilde{\omega}, \tag{4.8}$$

and

$$\begin{cases} q(x) - \widetilde{q}(x) \overset{a.e.}{=} 2\widehat{w} \quad \text{on } [a,b], & (4.9)\\ \lambda_{n_k} - \widetilde{\lambda}_{\widetilde{n}_k} = 2\widehat{w}, \quad \forall n_k \in B. & (4.10) \end{cases}$$

Denote

$$F(x, u_-, \widetilde{u}_-, \lambda) = u_-(x,\lambda)\widetilde{u}'_-(x, \lambda - 2\widehat{w}) - u'_-(x,\lambda)\widetilde{u}_-(x, \lambda - 2\widehat{w}).$$

Then

$$\begin{aligned} F(a,u_-,\widetilde{u}_-,\lambda) &= u_-(a,\lambda)\widetilde{u}'_-(a,\lambda) - u'_-(a,\lambda)\widetilde{u}_-(a,\lambda) \\ &= u'_-(a,\lambda)\widetilde{u}'_-(a,\lambda-2\widehat{w})\big(m_-^{-1}(a,\lambda) - \widetilde{m}_-^{-1}(a,\lambda-2\widehat{w})\big). \end{aligned} \tag{4.11}$$

From (2.2), (2.8), and (4.11), we obtain

$$\big|F(a,u_-,\widetilde{u}_-,ix)\big| = o\big(|y|^{r_0+\tilde{r}_0}\mathrm{e}^{|2\operatorname{Im}\sqrt{ix}|a}\big) \quad \text{for } |x| \gg 1, x \in \mathbb{R}. \tag{4.12}$$

Moreover, we can choose $\{x_{n_k}^{j_{n_k}}\} \in W_B([a,b])$ and apply Green's formula to obtain

$$\begin{aligned} &\big(\widetilde{u}(x,\widetilde{\lambda}_{n_k})u'(x,\lambda_{n_k}) - \widetilde{u}'(x,\widetilde{\lambda}_{n_k})u(x,\lambda_{n_k})\big)\Big|_a^{x_{n_k}^{j_{n_k}}} \\ &\quad = -\int_a^{x_{n_k}^{j_{n_k}}} \big[\lambda_{n_k} - \widetilde{\lambda}_{n_k} - \big(q(x)-\widetilde{q}(x)\big)\big]u(x,\lambda_{n_k})\widetilde{u}(x,\widetilde{\lambda}_{n_k})\,dx = 0, \end{aligned}$$

i.e.,

$$F(a,u_-,\widetilde{u}_-,\lambda_{n_k}) = 0 \quad \text{for all } n_k \in B. \tag{4.13}$$

Define the functions

$$G_B(\lambda) = \prod_{n_k\in B}\left(1-\frac{\lambda}{\lambda_{n_k}}\right), \tag{4.14}$$

$$\Delta_R(\lambda) = \prod_{\lambda_n\in\sigma_R(L)}\left(1-\frac{\lambda}{\lambda_n}\right), \tag{4.15}$$

and

$$\Delta_C(\lambda) = \prod_{\lambda_n\in\sigma_C(L)}\left(1-\frac{\lambda}{\lambda_n}\right). \tag{4.16}$$

Then we know $\Delta_C(\lambda)$ is a polynomial of degree k_0 and

$$\Delta(\lambda) = K\Delta_C(\lambda)\Delta_R(\lambda)$$

for some constant K.

Next, we shall use the technique in Appendix B of [15] to get an estimate of $|G_{S_B}(ix)|$. Without loss of generality, we may assume $\lambda > 1$ for $\lambda \in \sigma_R(L)$ (it can be done by a shift of the parameter λ in L). Then $N_{S_B(t)} = N_{\sigma_R(L)}(t) = 0$ for $t \le 1$, and

$$\begin{aligned} \ln\big|G(ix)\big| &= \sum_{k=1}^{\infty}\frac{1}{2}\ln\left(1+\frac{x^2}{\lambda_{n_k}^2}\right) = \frac{1}{2}\int_1^\infty \ln\left(1+\frac{x^2}{t^2}\right)dN_{S_B}(t) \\ &= \frac{1}{2}\left(\ln\left(1+\frac{x^2}{t^2}\right)N_{S_B}(t)\Big|_0^\infty - \int_1^\infty N_{S_B}(t)\frac{d}{dt}\left(\ln\left(1+\frac{x^2}{t^2}\right)\right)dt\right) \\ &= -\frac{1}{2}\int_1^\infty N_{S_B}(t)\frac{d}{dt}\left(\ln\left(1+\frac{x^2}{t^2}\right)\right)dt = \int_1^\infty \left(\frac{x^2}{t^3+tx^2}\right)N_{S_B}(t)\,dt. \end{aligned} \tag{4.17}$$

Hence

$$\ln \frac{|G(ix)|}{|\Delta_R(ix)|^{2a}} = \int_1^\infty \left(\frac{x^2}{t^3 + tx^2}\right)\left(N_{S_B}(t) - 2aN_{\sigma_R(L)}\right) dt. \tag{4.18}$$

By (4.6), we know that there exist a t_0 and a positive number K so that

$$\begin{cases} N_{S_B}(t) - 2a_1 N_{\sigma_R(L)}(t) \ge -K & \text{for } 1 \le t \le t_0, \\ N_{S_B}(t) - 2a_1 N_{\sigma_R(L)}(t) \ge 2(1-a_1)(r_0 + r_1 + \frac{1}{2}) + 2a_1 k_0 - 1 & \text{for } t > t_0, \end{cases}$$

for $a_1 = a$ or $1 - b$. This leads to

$$\begin{aligned} \ln \frac{|G(ix)|}{|\Delta_R(ix)|^{2a_1}} \ge{} & \int_1^{t_0} \left[(K+L)/2\right] \frac{d}{dt}\left(\ln\left(1 + \frac{x^2}{t^2}\right)\right) dt \\ & + L\int_{t_0}^\infty (-1/2)\frac{d}{dt}\left(\ln\left(1 + \frac{x^2}{t^2}\right)\right) dt, \end{aligned}$$

where $L = [2(1-a_1)(r_0 + r_1 + \frac{1}{2}) + 2a_1 k_0 - 1]$. Hence

$$\begin{aligned} \left|G(ix)\right| &= O\left(\left|\Delta_R(ix)\right|^{2a_1}\left|\Delta_C(ix)\right|^{2a_1}\left|\Delta_C(ix)\right|^{-2a_1}|x|^{2(1-a_1))(r_0+r_1+\frac{1}{2})+2a_1k_0-1}\right) \\ &= O\left(|x|^{(2a_1)(r_0+r1+1/2)}e^{2a_1|\operatorname{Im}\sqrt{ix}|}|x|^{-2a_1k_0}|x|^{2(1-a_1)(r_0+r_1+\frac{1}{2})+2a_1k_0-1}\right) \\ &= O\left(|x|^{2(r_0+r_1)}e^{2a_1|\operatorname{Im}(\sqrt{ix})|}\right), \end{aligned}$$

i.e.,

$$\left|G_{S_B}(ix)\right| \ge c|x|^{2(r_0+r_1)}e^{2a_1\operatorname{Im}\sqrt{i}|x|^{\frac{1}{2}}} \quad \text{for } |x| \gg 1 \tag{4.19}$$

and $a_1 = a$ and $1 - b$, where c is a constant. Therefore

$$K_1(\lambda) := \frac{F(a, u, \tilde{u}, \lambda)}{G(\lambda)} \tag{4.20}$$

is an entire function and

$$K_1(ix) \to 0 \quad \text{as } |x| \to \infty. \tag{4.21}$$

In addition, we easily prove that the following formula

$$\inf_{|\lambda|=R_k} \left|G_{S_B}(\lambda)\right| \ge c_0 \exp^{-C_0 R_k^{\frac{1+\varepsilon_0}{2}}}$$

holds for sufficiently large R_k, some $0 < \varepsilon_0 < 1$, c_0 and C_0 are two positive constants. Thus, we have

$$\sup_{|\lambda|=R_k} \left|K_1(\lambda)\right| \le c_0 \exp^{C_0 R_k^{\frac{1+\varepsilon_0}{2}}} \tag{4.22}$$

for sufficiently large $R_k \to \infty$ as $k \to +\infty$. By virtue of (4.21) and (4.22) (see [11], Proposition B.6), we obtain

$$K_1(\lambda) = 0, \quad \forall \lambda \in \mathbb{C}. \tag{4.23}$$

Therefore, we get

$$F(a, u_-, \widetilde{u}_-, \lambda) = 0, \quad \forall \lambda \in \mathbb{C}.$$

This implies

$$m_-(a, \lambda) = \widetilde{m}_-(a, \lambda - 2\widehat{w}). \tag{4.24}$$

By Theorem 3.1 together with (4.24), we get

$$\widetilde{q}(x) \overset{a.e.}{=} q(x) - 2\widehat{w} \quad \text{on } [0, a], \quad \text{and} \quad \widetilde{R}_{0k}(\lambda) = R_{0k}(\lambda + 2\widehat{w}), \quad k = 0, 1. \tag{4.25}$$

Similarly, we can define

$$H(x, u_+, \tilde{u}_+, \lambda) = u_+(x, \lambda)\widetilde{u}'_+(x, \lambda - 2\widehat{w}) - u'_+(x, \lambda)\widetilde{u}_+(x, \lambda - 2\widehat{w}).$$

Then one can repeat the same arguments as above on $H(b, u_+, \tilde{u}_+, \lambda)$ to show

$$\frac{H(b, u_+, \tilde{u}_+, \lambda)}{G_B(\lambda)} = 0.$$

This leads to $H(b, u_+, \tilde{u}_+, \lambda) = 0$ and

$$m_+(b, \lambda) = \widetilde{m}_+(b, \lambda - 2\widehat{w}). \tag{4.26}$$

Hence

$$\widetilde{q}(x) \overset{a.e.}{=} q(x) - 2\widehat{w} \quad \text{on } [b, 1], \quad \text{and} \quad \widetilde{R}_{1k}(\lambda) = R_{1k}(\lambda + 2\widehat{w}), \quad k = 0, 1. \tag{4.27}$$

(4.9), (4.25), and (4.27) imply

$$\widetilde{q}(x) \overset{a.e.}{=} q(x) - 2\widehat{w} \quad \text{on } [0, 1] \quad \text{and} \quad \widetilde{R}_{\xi k}(\lambda) = R_{\xi k}(\lambda + 2\widehat{w}), \quad \xi, k = 0, 1.$$

This completes the proof of Theorem 4.2. □

Corollary 4.3 *Under the assumptions of Theorem* 4.2, *if* $\lambda_{n_k} = \widetilde{\lambda}_{n_k}$ *for* $n_k \gg 1$, *then* $q(x) = \widetilde{q}(x)$ *and* $R_{ij}(\lambda) = \widetilde{R}_{i,j}(\lambda)$ *for* $i, j = 0, 1$.

Remark The readers might be interested in the inverse nodal problem for a more general equation

$$\begin{cases} u'' + A(\lambda)u = 0, \quad x \in (0, 1), \\ R_{01}(\lambda)u'(0) + R_{00}u(0) = 0, \\ R_{11}(\lambda)u'(1) + R_{10}u(1) = 0, \end{cases} \tag{4.28}$$

where $\frac{d^2}{dx^2} + A(\lambda)$ is an operator on $H^2(a, b)$. Some of such problems arise from PDE (please refer to [23, 24] for details). Same arguments for Theorem 4.2 seem to work for (4.28) if $A(\lambda)$ is an appropriate operator.

5 Conclusion

In this paper, the authors show that a twin-dense subset $W_B([a, b])$, $0 < a < 1/2 < b < 1$, can uniquely determine (up to a constant translation on both boundary conditions and potential) the Sturm–Liouville operator associated with boundary conditions polynomially dependent on the spectral parameter. The theorem leads to the same conclusion for classical Sturm–Liouville equation when the coefficient polynomials $R_{ij}(\lambda)$ are all of degree 0 (refer to [25]), but the translation effect on boundary conditions only appears when one of $R_{ij}(\lambda)$ is a non-trivial polynomial.

Acknowledgements
The authors acknowledge the reviewers for their helpful comments.

Funding
The third author was partially supported by the Ministry of Science and Technology of Taiwan grant 106-2115-M-032-004-.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. They have all read and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, Nanjing Forestry University, Nanjing, People's Republic of China. [2]Department of Mathematics, Tamkang University, New Taipei City, Taiwan.

Publisher's Note

Received: 30 November 2017 Accepted: 22 February 2018 Published online: 05 March 2018

References

1. Binding, P.A., Browne, P.J., Seddighi, K.: Sturm–Liouville problems with eigenparameter dependent boundary conditions. Proc. R. Soc. Edinb. **37**, 57–72 (1993)
2. Browne, P.J., Sleeman, B.D.: Inverse nodal problem for Sturm–Liouville equation with eigenparameter dependent boundary conditions. Inverse Probl. **12**, 377–381 (1996)
3. Chernozhukova, A.Y., Freiling, G.: A uniqueness theorem for inverse spectral problems depending nonlinearly on the spectral parameter. Inverse Probl. Sci. Eng. **17**, 777–785 (2009)
4. Freiling, G., Yurko, V.A.: Inverse problems for Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter. Inverse Probl. **26**(6), 055003 (2010)
5. Ozkan, A.S., Keskin, B.: Spectral problems for Sturm–Liouville operator with boundary and jump conditions linearly dependent on the eigenparameter. Inverse Probl. Sci. Eng. **20**(6), 799–808 (2012)
6. Wang, Y.P.: Inverse problems for discontinuous Sturm–Liouville operators with mixed spectral data. Inverse Probl. Sci. Eng. **23**, 1180–1198 (2015)
7. Wang, Y.P.: Inverse problems for a class of Sturm–Liouville operators with the mixed spectral data. Oper. Matrices **11**, 89–99 (2017)
8. Wang, Y.P., Lien, K.Y., Shieh, C.T.: Inverse problems for the boundary value problem with the interior nodal subsets. Appl. Anal. **96**(7), 1229–1239 (2017)
9. Wang, Y.P., Yurko, V.A.: On the inverse nodal problems for discontinuous Sturm–Liouville operators. J. Differ. Equ. **260**, 4086–4109 (2016)
10. Buterin, S.A., Shieh, C.T.: Inverse nodal problem for differential pencils. Appl. Math. Lett. **22**, 1240–1247 (2009)
11. Cheng, Y.H., Law, C.K., Tsay, J.: Remarks on a new inverse nodal problem. J. Math. Anal. Appl. **248**, 145–155 (2000)
12. Chen, X.F., Cheng, Y.H., Law, C.K.: Reconstructing potentials from zeros of one eigenfunction. Trans. Am. Math. Soc. **363**, 4831–4851 (2011)
13. Everitt, W.N.: On a property of the m-coefficient of a second-order linear differential equation. J. Lond. Math. Soc. **4**, 443–457 (1972)

14. Freiling, G., Yurko, V.A.: Inverse Sturm–Liouville Problems and Their Applications. Nova Science Publishers, Huntington (2001)
15. Gesztesy, F., Simon, B.: Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum. Trans. Am. Math. Soc. **352**, 2765–2787 (2000)
16. Marchenko, V.A.: Some questions in the theory of one-dimensional linear differential operators of the second order, I. Tr. Mosk. Mat. Obŝ. **1**, 327–420 (1952) (Russian); English transl. in Amer. Math. Soc. Transl. **2**, 101(1973), 1–104
17. McLaughlin, J.R.: Inverse spectral theory using nodal points as data—a uniqueness result. J. Differ. Equ. **73**, 354–362 (1988)
18. Shieh, C.T., Yurko, V.A.: Inverse nodal and inverse spectral problems for discontinuous boundary value problems. J. Math. Anal. Appl. **347**, 266–272 (2008)
19. Wei, G.S., Xu, H.K.: Inverse spectral problem with partial information given on the potential and norming constants. Trans. Am. Math. Soc. **364**, 3265–3288 (2012)
20. Yang, C.F., Yang, X.P.: Inverse nodal problem for the Sturm–Liouville equation with polynomially dependent on the eigenparameter. Inverse Probl. Sci. Eng. **19**, 951–961 (2011)
21. Yurko, V.A.: Inverse nodal problems for Sturm–Liouville operators on star-type graphs. J. Inverse Ill-Posed Probl. **16**, 715–722 (2008)
22. Yang, X.F.: A new inverse nodal problem. J. Differ. Equ. **169**, 633–653 (2001)
23. Shahmurov, R.: Solution of the Dirichlet and Neumann problems for a modified Helmholtz equation in Besov spaces on an annulus. J. Differ. Equ. **249**, 526–550 (2010)
24. Shahmurov, R.: On strong solutions of a Robin problem modelling heat conduction in materials with corroded boundary. Nonlinear Anal., Real World Appl. **13**, 441–451 (2012)
25. Guo, Y., Wei, G.: Inverse problems: dense nodal subset on an interior subinterval. J. Differ. Equ. **255**, 2002–2017 (2013)