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Blow-up of arbitrarily positive initial energy solutions for a viscoelastic wave system with nonlinear damping and source terms

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Abstract

This work is concerned with the Dirichlet initial boundary problem for a semilinear viscoelastic wave system with nonlinear weak damping and source terms. For nonincreasing positive functions g and h, we show the finite time blow-up of some solutions whose initial data have arbitrarily high initial energy.

MSC: 35L53; 35B44

Keywords: Viscoelastic wave system; Nonlinear damping; Blow-up; Arbitrarily positive initial energy

1 Introduction and main result

We consider a semilinear viscoelastic wave system with nonlinear damping and source terms,

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + |u_t|^{m-1}u_t = f_1(u,v), \quad x \in \Omega, t > 0, \tag{1}$$

$$v_{tt} - \Delta v + \int_0^t h(t-\tau)\Delta v(\tau)\,d\tau + |v_t|^{r-1}v_t = f_2(u,v), \quad x \in \Omega, t > 0, \tag{2}$$

subject to null Dirichlet boundary and initial conditions

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{3}$$

$$\begin{aligned} &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \\ &v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, \end{aligned} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $m > 1$, $r > 1$, and the relaxation functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ are positive nonincreasing. Problems of this type arise in viscoelasticity and systems governing the longitudinal motion of a viscoelastic configuration obeying the nonlinear Boltzmann model.

During the past decades, there has been much work dealing with the well-posedness and qualitative properties of solutions for damped viscoelastic wave equation. In this paper, we would like to investigate the blow-up phenomena with high initial energy for a semilinear damped viscoelastic wave system. To motivate our work, let us recall some results regarding viscoelastic wave models. For the single viscoelastic wave equation, we refer the reader to [1, 2] (the case $g = 0$) and [3–7] (the case $g \neq 0$), where blow-up solutions with initial negative energy, positive energy and arbitrarily positive energy are [1–7], respectively. Moreover, for general energy decay estimates on global solutions of a nonlinear abstract viscoelastic equation with variable density and the oscillation criteria and numerical solution of damped wave models, we refer the reader to [8–10].

Concerning wave systems without viscoelastic term ($g = 0$), Agre and Rammaha [11] investigated the following coupled semilinear wave system with nonlinear damping terms:

$$\begin{aligned} &u_{tt} - \Delta u + |u_t|^{m-1}u_t = (p+1)\big[a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}u|v|^{\frac{p+1}{2}}\big], \\ &v_{tt} - \Delta v + |v_t|^{r-1}v_t = (p+1)\big[a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}v|u|^{\frac{p+1}{2}}\big], \end{aligned}$$

in $\Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^N$ $(N = 1,2,3)$, $m \geq 1$, $r \geq 1$, $a > 1$, $b > 0$, $p \geq 3$. Using the Galerkin method and the method in [2] different from the concavity method we already know, that is, differential inequality techniques, they determined local and global existence of weak solutions and showed that any weak solution with negative initial energy blows up in finite time. Thereafter, Said-Houari [12] considered the blow-up result for a larger class of initial data with positive initial energy combining potential well method and differential inequality techniques ([2]). Pişkin [13] studied a coupled semilinear Klein–Gordon system with nonlinear damping terms,

$$\begin{aligned} &u_{tt} - \Delta u + m_1^2 u + |u_t|^{m-1}u_t = (p+1)\big[a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}u|v|^{\frac{p+1}{2}}\big], \\ &v_{tt} - \Delta v + m_2^2 v + |v_t|^{r-1}v_t = (p+1)\big[a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}v|u|^{\frac{p+1}{2}}\big], \end{aligned}$$

in $\Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^N$ $(N = 1,2,3)$, $m \geq 1$, $r \geq 1$, $m_1, m_2 > 0$, $a, b > 0$, $p > 1$. The decay estimates of the solution are established by using Nakao's inequality. Meanwhile, similar to [2], he also proved the blow-up of the solution in finite time with negative initial energy, using the technique of appropriate modification for energy functional.

In the presence of the viscoelastic term ($g \neq 0$), Han and Wang [14] discussed semilinear coupled viscoelastic wave system with nonlinear damping terms,

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + |u_t|^{m-1}u_t \\ &\quad = (p+1)\big[a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}u|v|^{\frac{p+1}{2}}\big], \\ &v_{tt} - \Delta v + \int_0^t h(t-\tau)\Delta v(\tau)\,d\tau + |v_t|^{r-1}v_t \\ &\quad = (p+1)\big[a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}v|u|^{\frac{p+1}{2}}\big], \end{aligned}$$

in $\Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^N$ ($N = 1,2,3$), $m \geq 1$, $r \geq 1$, $a > 1$, $b > 0$, $p \geq 3$. They established several results concerning the global existence, uniqueness and finite time blow-up of weak solutions with negative initial energy by utilizing the Galerkin and the concavity method. Recently, Messaoudi and Said-Hauari [15] dealt with our problem (1)–(4) and improved the result in [12] to a larger class of initial data for which the initial energy can take positive values. Besides, for the work on quasilinear wave equations, we refer the reader to [16–18] and the references therein.

In view of the work mentioned above, one can find that research on the blow-up phenomena of the solutions with high initial energy for a semilinear damped viscoelastic wave system (1)–(4) has not been started yet. Since the viscoelastic terms, nonlinear damping and source terms are included in the system, the classical method employed in single equation cannot be directly used to prove the blow-up result. The main difficulty of the present paper is to find the technique to deal with nonlinear damping and source terms. In order to overcome the difficulty, combining an argument of contradiction, property of convex function ([7]) and important inequalities in [15] (cf. Lemma 2.1), we consider problem (1)–(4) and prove a blow-up result of certain solutions at a high energy level.

Firstly, let us present some notations and assumptions used throughout this article.

Taking

$$f_1(u,v) = \bigl[a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2}\bigr],$$
$$f_2(u,v) = \bigl[a|u+v|^{2(p+1)}(u+v) + b|v|^p v|u|^{p+2}\bigr], \quad a,b > 0,$$

one can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(p+2)F(u,v), \quad \forall (u,v) \in \mathbb{R}^2,$$

where

$$F(u,v) = \frac{1}{2(p+2)}\bigl[a|u+v|^{2(p+2)} + 2b|uv|^{(p+2)}\bigr].$$

For the relaxation functions $g(s)$, $h(s)$ and real number p, we give the following assumptions:

(H_1) $g \in C^1([0,\infty])$, $h \in C^1([0,\infty])$ *are nonnegative functions satisfying*

$$g'(s) \leq 0, \quad 1 - \int_0^\infty g(s)\,ds = l > 0,$$
$$h'(s) \leq 0, \quad 1 - \int_0^\infty h(s)\,ds = k > 0.$$

(H_2)

$$-1 < p < \infty, \quad N = 1,2,$$
$$-1 < p \leq \frac{3-N}{N-2}, \quad N \geq 3.$$

Remark 1 Condition (H_1) is necessary to guarantee the hyperbolicity and well-posedness of the system (1)–(4).

Note that we easily obtain the following local existence and uniqueness of weak solution for problem (1)–(4) by using the Faedo–Galerkin approximation methods and the Banach contraction mapping principle, which is similar to [2] with slight modification. The process of this proof is standard, so we omit it here.

Proposition *Under the assumptions* (H_1) *and* (H_2), *let the initial data* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *are given, then the problem* (1)–(4) *has a unique local solution*

$$\begin{aligned}
&(u, v) \in C\bigl([0,T]; H_0^1(\Omega)\bigr) \times C\bigl([0,T]; H_0^1(\Omega)\bigr),\\
&(u_t, v_t) \in C\bigl([0,T]; L^2(\Omega)\bigr) \cap L^{m+1}\bigl(\Omega \times (0,T)\bigr) \times C\bigl([0,T]; L^2(\Omega)\bigr) \cap L^{m+1}\bigl(\Omega \times (0,T)\bigr),
\end{aligned}$$

for the maximum existence time $T > 0$, *where* $T \in (0, \infty]$.

The energy related to problem (1)–(4) is

$$\begin{aligned}
E(t) = {}& \frac{1}{2}\bigl(\|u_t\|_2^2 + \|v_t\|_2^2\bigr) + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 \\
&+ \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\
&+ \frac{1}{2}\bigl[(g \circ \nabla u) + (h \circ \nabla v)\bigr] - \int_\Omega F(u, v)\,dx,
\end{aligned} \tag{5}$$

where

$$(g \circ \nabla v)(t) = \int_0^t g(t - \tau)\bigl\|v(t) - v(\tau)\bigr\|_2^2\,d\tau.$$

Now we are in a position to state our main result.

Theorem 1 *Under the assumptions* (H_1) *and* (H_2), *assume that* $m > 1$, $r > 1$, $2(p+2) > \max\{m+1, r+1\}$, *and*

$$\max\left\{\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right\} < \frac{p+1}{p+1+\frac{1}{4(p+2)}}. \tag{6}$$

Let (u, v) *be a solution of Eqs*. (1)–(4), *satisfying*

$$\int_\Omega u(0)u_t(0)\,dx + \int_\Omega v(0)v_t(0)\,dx > ME(0) > 0, \tag{7}$$

then (u, v) *blows up in finite time, where*

$$M = \frac{\sigma}{\sigma+1}\left(\frac{1-\xi}{2c_0\varepsilon_0(p+2)}\right)^{\frac{1}{\sigma^*}},$$

$\varepsilon_0 \in (0,1)$ *is a root of the equation* $\frac{\sigma}{\sigma+1}(\frac{1-\xi}{2c_0\varepsilon_0(p+2)})^{\frac{1}{\sigma^*}} = \frac{2(p+2)(1-\varepsilon_0)}{\alpha(\varepsilon_0)}$, *and*

$$\sigma = \max\{m,r\}, \qquad \sigma^* = \min\{m,r\},$$
$$\xi = \frac{2(p+2)-(\sigma+1)}{2(p+2)-2}, \qquad \xi^* = \frac{2(p+2)-(\sigma^*+1)}{2(p+2)-2},$$
$$\alpha(\varepsilon) = 2\sqrt{\bigl((p+2)(1-\varepsilon)+1\bigr)\biggl(k(\varepsilon)\lambda_1 - \frac{c_0\varepsilon(p+2)\xi^*}{1-\xi}\biggr)},$$
$$k(\varepsilon) = \bigl((p+2)(1-\varepsilon)-1\bigr)\min\{k,l\} - \frac{1-\min\{k,l\}}{4(p+2)(1-\varepsilon)},$$

λ_1 *being the first eigenvalue of* $-\Delta$.

The outline of the paper is as follows. In Sect. 2, we introduce three lemmas related to the study of problem (1)–(4). Section 3 devoted to the proof of our main result.

2 Preliminary results

In the section, we give some lemmas which are useful for the proof of our blow-up result.

Lemma 1 *Assume* (H_1) *and* (H_2) *hold. Let* (u,v) *be a solution of* (1)–(4), *then* $E(t)$ *is non-increasing, that is,* $E'(t) \le 0$.

Proof By the multiplier method, multiplying (1), (2) by u_t, v_t, respectively, and then using (5), we get

$$E'(t) = -\bigl(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1}\bigr) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}h(t)\|\nabla v\|_2^2 + \frac{1}{2}\bigl[\bigl(g' \circ \nabla u\bigr)(t) + \bigl(h' \circ \nabla v\bigr)(t)\bigr],$$

for $t \ge 0$, $E'(t) \le 0$. Moreover, the following energy inequality holds:

$$E'(t) \le -\bigl(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1}\bigr), \quad \forall t \ge 0.$$

□

Lemma 2 ([15], Lemma 2.1) *There exist two positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0}{2(p+2)}\bigl(|u|^{2(p+2)} + |v|^{2(p+2)}\bigr) \le F(u,v) \le \frac{c_1}{2(p+2)}\bigl(|u|^{2(p+2)} + |v|^{2(p+2)}\bigr). \tag{8}$$

Next, we present the following crucial lemma which repeats the same one of Han and Wang [14], Theorem 2.4, with slight modification, so we will omit its proof.

Lemma 3 ([14]) *Under the assumptions* (H_1) *and* (H_2), *assume that* $m > 1$, $r > 1$, $2(p+2) > \max\{m+1, r+1\}$ *and satisfying* (6). *If* $\exists t_0 \ge 0$ *such that* $E(t_0) < 0$, *then the solution of the problem* (1)–(4) *blows up in finite time*.

3 Proof of Theorem 1

In the section, using an argument of contradiction and the property of a convex function, we prove our main result.

Proof of Theorem 1 Assume (u,v) is a global solution of problem (1)–(4). Multiplying (1), (2) by u, v, respectively, and integrating over Ω, we derive that

$$\begin{aligned}
&(u_{tt},u)+\|\nabla u\|_2^2-\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\nabla u(\tau)\,dx\,d\tau+\int_\Omega |u_t|^{m-1}u_t u\,dx \\
&\quad=\int_\Omega uf_1(u,v)\,dx, \\
&(v_{tt},v)+\|\nabla v\|_2^2-\int_0^t h(t-\tau)\int_\Omega \nabla v(t)\nabla v(\tau)\,dx\,d\tau+\int_\Omega |v_t|^{r-1}v_t v\,dx=\int_\Omega vf_2(u,v)\,dx.
\end{aligned}$$

Thus the following equalities are obtained:

$$\begin{aligned}
\frac{d}{dt}(u,u_t)&=\|u_t\|_2^2-\|\nabla u\|_2^2+\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\nabla u(\tau)\,dx\,d\tau+\int_\Omega uf_1(u,v)\,dx \\
&\quad-\int_\Omega |u_t|^{m-1}u_t u\,dx, \qquad (9)\\
\frac{d}{dt}(v,v_t)&=\|v_t\|_2^2-\|\nabla v\|_2^2+\int_0^t h(t-\tau)\int_\Omega \nabla v(t)\nabla v(\tau)\,dx\,d\tau+\int_\Omega vf_2(u,v)\,dx \\
&\quad-\int_\Omega |v_t|^{r-1}v_t v\,dx. \qquad (10)
\end{aligned}$$

Using the Cauchy inequality, we estimate the third terms on the right side of (9) and (10), for $\forall\varepsilon\in(0,1)$,

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\nabla u(\tau)\,dx\,d\tau \\
&\quad=\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\big[\nabla u(\tau)-\nabla u(t)\big]\,dx\,d\tau+\int_0^t g(\tau)\,d\tau\|\nabla u\|_2^2 \\
&\quad\geq-\frac{2(p+2)(1-\varepsilon)}{2}(g\circ\nabla u)(t)-\frac{1}{4(p+2)(1-\varepsilon)}\int_0^t g(\tau)\,d\tau\|\nabla u\|_2^2 \\
&\qquad+\int_0^t g(\tau)\,d\tau\|\nabla u\|_2^2, \qquad (11)\\
&\int_0^t h(t-\tau)\int_\Omega \nabla v(t)\nabla v(\tau)\,dx\,d\tau \\
&\quad=\int_0^t h(t-\tau)\int_\Omega \nabla v(t)\big[\nabla v(\tau)-\nabla v(t)\big]\,dx\,d\tau+\int_0^t h(\tau)\,d\tau\|\nabla v\|_2^2 \\
&\quad\geq-\frac{2(p+2)(1-\varepsilon)}{2}(h\circ\nabla v)(t)-\frac{1}{4(p+2)(1-\varepsilon)}\int_0^t h(\tau)\,d\tau\|\nabla v\|_2^2 \\
&\qquad+\int_0^t h(\tau)\,d\tau\|\nabla v\|_2^2. \qquad (12)
\end{aligned}$$

Combining (11) and (12), we derive that

$$\begin{aligned}
\frac{d}{dt}(u,u_t)+\frac{d}{dt}(v,v_t)&\geq-\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2-\left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\
&\quad+\|u_t\|_2^2+\|v_t\|_2^2-\frac{2(p+2)(1-\varepsilon)}{2}\big((g\circ\nabla u)(t)+(h\circ\nabla v)(t)\big)
\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega |v_t|^{r-1} v_t v \, dx - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t g(\tau)\, d\tau \|\nabla u\|_2^2 \\
&- \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t h(\tau)\, d\tau \|\nabla v\|_2^2 - \int_\Omega |u_t|^{m-1} u_t u \, dx \\
&+ 2(p+2) \int_\Omega F(u,v)\, dx. \qquad (13)
\end{aligned}
$$

For the right side of (13) to add $2(p+2)(1-\varepsilon)E(t)$, one can get

$$
\begin{aligned}
&\frac{d}{dt}(u,u_t) + \frac{d}{dt}(v,v_t) \\
&\quad \geq \bigl((p+2)(1-\varepsilon)-1\bigr)\biggl(1 - \int_0^t g(s)\, ds\biggr) \|\nabla u\|_2^2 \\
&\qquad + \bigl((p+2)(1-\varepsilon)+1\bigr)\bigl(\|u_t\|_2^2 + \|v_t\|_2^2\bigr) - \int_\Omega |u_t|^{m-1} u_t u \, dx \\
&\qquad + \bigl((p+2)(1-\varepsilon)-1\bigr)\biggl(1 - \int_0^t h(s)\, ds\biggr) \|\nabla v\|_2^2 - \int_\Omega |v_t|^{r-1} v_t v \, dx \\
&\qquad + 2(p+2)\varepsilon \int_\Omega F(u,v)\, dx - 2(p+2)(1-\varepsilon)E(t) \\
&\qquad - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t g(\tau)\, d\tau \|\nabla u\|_2^2 \\
&\qquad - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t h(\tau)\, d\tau \|\nabla v\|_2^2. \qquad (14)
\end{aligned}
$$

For the third and fifth terms on the right side of (14), Hölder's and Young's inequalities give us

$$
\biggl| \int_\Omega |u_t|^{m-1} u_t u \, dx \biggr| \leq \|u\|_{m+1} \|u_t\|_{m+1}^m \leq \varepsilon_1^{m+1} \frac{\|u\|_{m+1}^{m+1}}{m+1} + \varepsilon_1^{-\frac{m+1}{m}} \frac{m}{m+1} \|u_t\|_{m+1}^{m+1}.
$$

By the convexity of the function $\frac{u^y}{y}$ in y, for $u \geq 0$ and $y > 0$, we have

$$
\frac{\|u\|_{m+1}^{m+1}}{m+1} \leq \theta \frac{\|u\|_2^2}{2} + (1-\theta) \frac{\|u\|_{2(p+2)}^{2(p+2)}}{2(p+2)},
$$

where $\theta = \frac{2(p+2)-(m+1)}{2(p+2)-2}$, then one can get

$$
\biggl| \int_\Omega |u_t|^{m-1} u_t u \, dx \biggr| \leq \varepsilon_1^{m+1} \biggl(\theta \frac{\|u\|_2^2}{2} + (1-\theta) \frac{\|u\|_{2(p+2)}^{2(p+2)}}{2(p+2)} \biggr) + \varepsilon_1^{-\frac{m+1}{m}} \frac{m}{m+1} \|u_t\|_{m+1}^{m+1}. \qquad (15)
$$

Similarly,

$$
\biggl| \int_\Omega |v_t|^{r-1} v_t v \, dx \biggr| \leq \varepsilon_1^{r+1} \biggl(\eta \frac{\|v\|_2^2}{2} + (1-\eta) \frac{\|v\|_{2(p+2)}^{2(p+2)}}{2(p+2)} \biggr) + \varepsilon_1^{-\frac{r+1}{r}} \frac{r}{r+1} \|v_t\|_{r+1}^{r+1}, \qquad (16)
$$

where $\eta = \frac{2(p+2)-(r+1)}{2(p+2)-2}$.

Take

$$\sigma = \max\{m,r\}, \qquad \sigma^* = \min\{m,r\},$$
$$\xi = \frac{2(p+2)-(\sigma+1)}{2(p+2)-2},$$
$$\xi^* = \frac{2(p+2)-(\sigma^*+1)}{2(p+2)-2}.$$

By (14)–(16) and Lemma 1, we have

$$\begin{aligned}
&\frac{d}{dt}\left((u,u_t)+(v,v_t)-\varepsilon_1^{-\frac{\sigma^*+1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right)\\
&\quad\geq \frac{d}{dt}\big((u,u_t)+(v,v_t)\big)+\varepsilon_1^{-\frac{\sigma^*+1}{\sigma^*}}\frac{\sigma}{\sigma+1}\big(\|u_t\|_{m+1}^{m+1}+\|v_t\|_{r+1}^{r+1}\big)\\
&\quad\geq \frac{d}{dt}\big((u,u_t)+(v,v_t)\big)+\varepsilon_1^{-\frac{m+1}{m}}\frac{m}{m+1}\|u_t\|_{m+1}^{m+1}+\varepsilon_1^{-\frac{r+1}{r}}\frac{r}{r+1}\|v_t\|_{r+1}^{r+1}\\
&\quad\geq \big((p+2)(1-\varepsilon)+1\big)\big(\|u_t\|_2^2+\|v_t\|_2^2\big)-2(p+2)(1-\varepsilon)E(t)\\
&\qquad+2(p+2)\varepsilon\int_\Omega F(u,v)\,dx\\
&\qquad-\varepsilon_1^{m+1}\left(\theta\frac{\|u\|_2^2}{2}+(1-\theta)\frac{\|u\|_{2(p+2)}^{2(p+2)}}{2(p+2)}\right)-\varepsilon_1^{r+1}\left(\eta\frac{\|v\|_2^2}{2}+(1-\eta)\frac{\|v\|_{2(p+2)}^{2(p+2)}}{2(p+2)}\right)\\
&\qquad+\left(\big((p+2)(1-\varepsilon)-1\big)\left(1-\int_0^t g(s)\,ds\right)-\frac{1}{4(p+2)(1-\varepsilon)}\int_0^t g(\tau)\,d\tau\right)\|\nabla u\|_2^2\\
&\qquad+\left(\big((p+2)(1-\varepsilon)-1\big)\left(1-\int_0^t h(s)\,ds\right)-\frac{1}{4(p+2)(1-\varepsilon)}\int_0^t h(\tau)\,d\tau\right)\|\nabla v\|_2^2.
\end{aligned}$$

For the formula above, using Lemma 2 and the Poincaré inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\left((u,u_t)+(v,v_t)-\varepsilon_1^{-\frac{\sigma^*+1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right)\\
&\quad\geq \big((p+2)(1-\varepsilon)+1\big)\big(\|u_t\|_2^2+\|v_t\|_2^2\big)-2(p+2)(1-\varepsilon)E(t)\\
&\qquad+c_0\varepsilon\big(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\big)\\
&\qquad+\left(\big((p+2)(1-\varepsilon)-1\big)l-\frac{1-l}{4(p+2)(1-\varepsilon)}\right)\|\nabla u\|_2^2\\
&\qquad-\varepsilon_1^{m+1}\left(\theta\frac{\|u\|_2^2}{2}+(1-\theta)\frac{\|u\|_{2(p+2)}^{2(p+2)}}{2(p+2)}\right)\\
&\qquad+\left(\big((p+2)(1-\varepsilon)-1\big)k-\frac{1-k}{4(p+2)(1-\varepsilon)}\right)\|\nabla v\|_2^2\\
&\qquad-\varepsilon_1^{r+1}\left(\eta\frac{\|v\|_2^2}{2}+(1-\eta)\frac{\|v\|_{2(p+2)}^{2(p+2)}}{2(p+2)}\right)\\
&\quad\geq \big((p+2)(1-\varepsilon)+1\big)\big(\|u_t\|_2^2+\|v_t\|_2^2\big)+\left(k_1(\varepsilon)\lambda_1-\frac{\varepsilon_1^{m+1}\theta}{2}\right)\|u\|_2^2
\end{aligned}$$

$$+\left(k_2(\varepsilon)\lambda_1-\frac{\varepsilon_1^{r+1}\eta}{2}\right)\|v\|_2^2-2(p+2)(1-\varepsilon)E(t)$$
$$+\left(c_0\varepsilon-\frac{\varepsilon_1^{m+1}(1-\theta)}{2(p+2)}\right)\|u\|_{2(p+2)}^{2(p+2)}+\left(c_0\varepsilon-\frac{\varepsilon_1^{r+1}(1-\eta)}{2(p+2)}\right)\|v\|_{2(p+2)}^{2(p+2)}, \tag{17}$$

where $k_1(\varepsilon)=((p+2)(1-\varepsilon)-1)l-\frac{1-l}{4(p+2)(1-\varepsilon)}$, $k_2(\varepsilon)=((p+2)(1-\varepsilon)-1)k-\frac{1-k}{4(p+2)(1-\varepsilon)}$, λ_1 being the first eigenvalue of $-\Delta$.

Take $\varepsilon_1=(\frac{2c_0\varepsilon(p+2)}{1-\xi})^{\frac{1}{\sigma^*+1}}$, we have

$$\begin{aligned}
&\frac{d}{dt}\left((u,u_t)+(v,v_t)-\varepsilon_1^{-\frac{\sigma^*+1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right)\\
&\quad\geq\big((p+2)(1-\varepsilon)+1\big)\big(\|u_t\|_2^2+\|v_t\|_2^2\big)-2(p+2)(1-\varepsilon)E(t)\\
&\qquad+\left(k_1(\varepsilon)\lambda_1-\frac{\varepsilon_1^{m+1}\theta}{2}\right)\|u\|_2^2+\left(k_2(\varepsilon)\lambda_1-\frac{\varepsilon_1^{r+1}\eta}{2}\right)\|v\|_2^2,\\
&\frac{d}{dt}\left((u,u_t)+(v,v_t)-\left(\frac{2c_0\varepsilon(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right)\\
&\quad\geq\big((p+2)(1-\varepsilon)+1\big)\big(\|u_t\|_2^2+\|v_t\|_2^2\big)-2(p+2)(1-\varepsilon)E(t)\\
&\qquad+\left(k_1(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\theta}{1-\xi}\right)\|u\|_2^2+\left(k_2(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\eta}{1-\xi}\right)\|v\|_2^2.
\end{aligned}\tag{18}$$

Since

$$\begin{aligned}
&\max\left\{\int_0^\infty g(s)\,ds,\int_0^\infty h(s)\,ds\right\}<\frac{p+1}{p+1+\frac{1}{4(p+2)}},\\
&\delta_1=\big(2(p+2)-2\big)l-\frac{1-l}{2(p+2)}>0,\\
&\delta_2=\big(2(p+2)-2\big)k-\frac{1-k}{2(p+2)}>0.
\end{aligned}$$

Then we can take ε small enough such that

$$\begin{aligned}
&k_1(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\theta}{1-\xi}>0,\\
&k_2(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\eta}{1-\xi}>0.
\end{aligned}$$

The Cauchy inequality gives us

$$\begin{aligned}
&\big((p+2)(1-\varepsilon)+1\big)\|u_t\|_2^2+\left(k_1(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\theta}{1-\xi}\right)\|u\|_2^2\\
&\quad\geq2\sqrt{\big((p+2)(1-\varepsilon)+1\big)\left(k_1(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\theta}{1-\xi}\right)}(u,u_t),
\end{aligned}\tag{19}$$

$$\begin{aligned}
&\big((p+2)(1-\varepsilon)+1\big)\|v_t\|_2^2+\left(k_2(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\eta}{1-\xi}\right)\|v\|_2^2\\
&\quad\geq2\sqrt{\big((p+2)(1-\varepsilon)+1\big)\left(k_2(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\eta}{1-\xi}\right)}(v,v_t).
\end{aligned}\tag{20}$$

Combining (19) and (20), we get

$$\begin{aligned}
&\frac{d}{dt}\left((u,u_t)+(v,v_t)-\left(\frac{2c_0\varepsilon(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right)\\
&\quad\geq \alpha_1(\varepsilon)(u,u_t)+\alpha_2(\varepsilon)(v,v_t)-2(p+2)(1-\varepsilon)E(t)\\
&\quad\geq \alpha(\varepsilon)\left((u,u_t)+(v,v_t)-\frac{2(p+2)(1-\varepsilon)}{\alpha(\varepsilon)}E(t)\right),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\alpha_1(\varepsilon)&=2\sqrt{\big((p+2)(1-\varepsilon)+1\big)\left(k_1(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\theta}{1-\xi}\right)},\\
\alpha_2(\varepsilon)&=2\sqrt{\big((p+2)(1-\varepsilon)+1\big)\left(k_2(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\eta}{1-\xi}\right)},\\
\alpha(\varepsilon)&=2\sqrt{\big((p+2)(1-\varepsilon)+1\big)\left(k(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\xi^*}{1-\xi}\right)},\\
k(\varepsilon)&=\big((p+2)(1-\varepsilon)-1\big)\min\{k,l\}-\frac{1-\min\{k,l\}}{4(p+2)(1-\varepsilon)}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\alpha(\varepsilon)\to\sqrt{2(p+3)\min\{\delta_1,\delta_2\}\lambda_1},\qquad \left(\frac{2c_0\varepsilon(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}\to+\infty,\\
&\frac{2(p+2)(1-\varepsilon)}{\alpha(\varepsilon)}\to\frac{2(p+2)}{\sqrt{2(p+3)\min\{\delta_1,\delta_2\}\lambda_1}},\quad \text{as } \varepsilon\to 0^+.
\end{aligned} \tag{22}$$

On the other hand, by the definition of $k(\varepsilon)$, we have

$$\begin{aligned}
&k(\varepsilon)\to-\infty,\qquad k(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\xi^*}{1-\xi}\to-\infty,\quad \text{as } \varepsilon\to 1^-,\\
&k(\varepsilon)\lambda_1-\frac{c_0\varepsilon(p+2)\xi^*}{1-\xi}\to\frac{\min\{\delta_1,\delta_2\}\lambda_1}{2},\quad \text{as } \varepsilon\to 0^+.
\end{aligned}$$

Hence, there exists $\varepsilon_*\in(0,1)$ such that

$$\alpha(\varepsilon_*)=0\quad\text{and}\quad\alpha(\varepsilon)>0,\quad\forall\varepsilon\in(0,\varepsilon_*).$$

This implies that

$$\begin{aligned}
&\alpha(\varepsilon)\to 0,\qquad \frac{2(p+2)(1-\varepsilon)}{\alpha(\varepsilon)}\to+\infty,\\
&\left(\frac{2c_0\varepsilon(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}\to\left(\frac{2c_0\varepsilon_*(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1},\quad \text{as } \varepsilon\to\varepsilon_*^-.
\end{aligned} \tag{23}$$

Using (22), (23) and the continuity in ε of $\frac{2(p+2)(1-\varepsilon)}{\alpha(\varepsilon)}$ and $(\frac{2c_0\varepsilon(p+2)}{1-\xi})^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}$, there exists $\varepsilon_0 \in (0,\varepsilon_*) \subset (0,1)$ such that

$$\left(\frac{2c_0\varepsilon_0(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1} = \frac{2(p+2)(1-\varepsilon_0)}{\alpha(\varepsilon_0)}.$$

Then (21) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\left((u,u_t)+(v,v_t)-\left(\frac{2c_0\varepsilon_0(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right)\\ &\quad\geq \alpha(\varepsilon_0)\left((u,u_t)+(v,v_t)-\left(\frac{2c_0\varepsilon_0(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)\right).\end{aligned} \tag{24}$$

Now, setting $H(t) = (u,u_t)+(v,v_t)-(\frac{2c_0\varepsilon_0(p+2)}{1-\xi})^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(t)$. Then we exploit (7), this tells us that

$$H(0) = \int_\Omega u(0)u_t(0)\,dx + \int_\Omega v(0)v_t(0)\,dx - \left(\frac{2c_0\varepsilon_0(p+2)}{1-\xi}\right)^{-\frac{1}{\sigma^*}}\frac{\sigma}{\sigma+1}E(0) > 0$$

and

$$\frac{d}{dt}H(t) \geq \alpha(\varepsilon_0)H(t). \tag{25}$$

A simple integration of (25) over $(0,t)$ then yields

$$H(t) \geq e^{\alpha(\varepsilon_0)t}H(0), \quad \forall t \geq 0.$$

Since (u,v) is global, by Lemma 2 and Lemma 3, for $t \geq 0$, we have $0 \leq E(t) \leq E(0)$. Hence, we obtain

$$(u,u_t)+(v,v_t) \geq e^{\alpha(\varepsilon_0)t}H(0).$$

So, we get the estimate

$$\begin{aligned}\|u_t\|_2^2+\|v_t\|_2^2 &= \|u(0)\|_2^2+\|v(0)\|_2^2+2\int_0^t\big[(u,u_t)+(v,v_t)\big]\,d\tau\\ &\geq \|u(0)\|_2^2+\|v(0)\|_2^2+2\int_0^t e^{\alpha(\varepsilon_0)t}H(0)\,d\tau\\ &\geq \|u(0)\|_2^2+\|v(0)\|_2^2+\frac{2}{\alpha(\varepsilon_0)}\big(e^{\alpha(\varepsilon_0)t}-1\big)H(0).\end{aligned} \tag{26}$$

On the other hand, by Lemma 1, Lemma 3 and the Hölder inequality, we derive

$$\begin{aligned}\|u_t\|_2+\|v_t\|_2 &\leq \|u(0)\|_2+\|v(0)\|_2+\int_0^t\|u_t(\tau)\|_2\,d\tau+\int_0^t\|v_t(\tau)\|_2\,d\tau\\ &\leq \|u(0)\|_2+\|v(0)\|_2+C_0\left(\int_0^t\|u_t(\tau)\|_{m+1}\,d\tau+\int_0^t\|v_t(\tau)\|_{r+1}\,d\tau\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \|u(0)\|_2 + \|v(0)\|_2 + C_0 t^{\frac{m}{m+1}} \int_0^t \|u_t(\tau)\|_{m+1}^{m+1} \, d\tau^{\frac{1}{m+1}} \\
&\quad + C_0 t^{\frac{r}{r+1}} \int_0^t \|v_t(\tau)\|_{r+1}^{r+1} \, d\tau^{\frac{1}{r+1}} \\
&\leq \|u(0)\|_2 + \|v(0)\|_2 + C t^{\frac{\sigma}{\sigma+1}} \int_0^t \left(\|u_t(\tau)\|_{m+1}^{m+1} + \|v_t(\tau)\|_{r+1}^{r+1}\right) d\tau^{\frac{1}{\sigma^*+1}} \\
&\leq \|u(0)\|_2 + \|v(0)\|_2 + C t^{\frac{\sigma}{\sigma+1}} \left(E(0) - E(t)\right)^{\frac{1}{\sigma^*+1}} \\
&\leq \|u(0)\|_2 + \|v(0)\|_2 + C t^{\frac{\sigma}{\sigma+1}} E(0)^{\frac{1}{\sigma^*+1}}.
\end{aligned}
$$

This contradicts (26) and we get the finite time blow-up result. □

4 Conclusion

We prove the finite time blow-up of some solutions for a semilinear viscoelastic wave system with nonlinear weak damping and source terms whose initial data have arbitrarily high initial energy. We point out that the methods for a single equation in [6, 7] are not necessarily applicable to our system. We also notice that the result in Theorem 1 extends the results for the system in [14, 15].

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The authors declare that they have no competing interests.

Authors' contributions
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