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Rotating periodic solutions for second order systems with Hartman-type nonlinearity

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Abstract

In this paper, by a constructive proof based on the homotopy continuation method, we prove that the second order system $x'' = g(t, x)$ admits rotating periodic solutions with form $u(t + T) = Qu(t)$ for any orthogonal matrix Q when the nonlinearity g admits the Hartman-type condition.

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1 Introduction and main results

In this paper we study the following second order system:

$$x'' = g(t, x), \tag{1.1}$$

where g is continuous from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and satisfies the following Hartman-type condition:

(H) there exists $\eta > 0$ such that

$$g(t, x) \cdot x \geq 0, \quad \forall t \in [0, T], \forall x \in \mathbb{R} \text{ with } |x| = \eta. \tag{1.2}$$

In 1960, Hartman [1] firstly studied the existence of system (1.1) with Dirichlet boundary value conditions under the assumption (1.2). In 1971, Knobloch [2] obtained the existence of periodic solutions for (1.1) under the Hartman-type condition (H) when g is locally Lipschitzian in x. The result was extended to the case that g is continuous by Rouche and Mawhin [3] in 1973. In [4], Mawhin applied the Schauder fixed point theorem to obtain periodic solutions of p-Laplacian ordinary differential systems under assumption (H). In [5, 6], some results are obtained on periodic solutions of ordinary differential systems involving singular ϕ-Laplace operator and $\phi(t)$-Laplace operator under the Hartman-type condition (H). For some variants and extensions, one can see [7–11] and the references therein.

This paper is devoted to showing that, under the Hartman condition (H), the system (1.1) admits the type of solutions with form $u(t + T) = Qu(t)$ for any $Q \in O(n)$ when the

nonlinearity g amidts the corresponding rotating periodic structure. Here $O(n)$ denotes the orthogonal group on $\mathbb{R}^n$. Usually, we call the solutions by rotating periodic solutions (or Q-rotating periodic solutions related to the matrix Q).

Theorem 1.1 *Assume that $g \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and there exists some $Q \in O(n)$ such that $g(t + T, x) = Qg(t, Q^{-1}x), \forall t \in \mathbb{R}$. If the assumption (H) holds, then (1.1) has at least one Q-rotating periodic solution.*

Example 1.2 Define $g(t, x) = a(t)|x|^{p-2}x, \forall x \in \mathbb{R}^n$, where $a \in C(\mathbb{R}, \mathbb{R})$ with $a(t) > 0$ and $a(t + T) = a(t)$ for all $t \in \mathbb{R}$, $p \geq 2$. Clearly, by Theorem 3.1 in [4] we can obtain the existence of at least one periodic solution, while here by Theorem 1.1 we can get infinitely many periodic solutions and quasi-periodic solutions of (1.1) by taking different Q. It should be noticed that if $Q \equiv I_n$, then the corresponding rotating periodic solution is just a periodic solution of (1.1), while if $Q \not\equiv I_n$, we may obtain some existence results on quasi-periodic solutions of (1.1). In fact, we can take Q by

$$\Lambda_1 \doteq \begin{pmatrix} R(\theta_1) & & \\ & \cdots & \\ & & R(\theta_m) \end{pmatrix}$$

if $n = 2l$ for some $m, l \in \mathbb{Z}^+$, or

$$\Lambda_2 \doteq \begin{pmatrix} R(\theta_1) & & & \\ & \cdots & & \\ & & R(\theta_m) & \\ & & & 1 \end{pmatrix}$$

if $n = 2l + 1$ for some $m, l \in \mathbb{Z}^+$, where $\theta_i \in [0, 2\pi]$, $i = 1, 2, \ldots, m$, and there exist $i \in \{1, 2, \ldots, m\}$ such that θ_i is an irrational multiplier of $\frac{\pi}{T}$. Here $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Our proof is based on the homotopy continuation method together with a prior estimate. This method was introduced by Kellogg, Li and Yorke [12] and afterward many authors applied this method to obtain a constructive proof for the existence of periodic solutions of second order ordinary differential, one can see [13–15] and the references. For the periodic solution case, i.e., $Q \equiv I_n$, as shown in [15], one can transform the problem into one of finding certain solutions of the Cauchy problem by introducing a homotopy parameter $\lambda \in [0, 1]$. Following the C^1 path of solutions of the Cauchy problem starting from a simpler auxiliary equation corresponding to $\lambda = 0$, one can obtain the desired periodic solutions corresponding to $\lambda = 1$. However, when $\text{Ker}(I_n - Q) \not\equiv \mathbb{R}^n$, it seems difficult to apply this idea to our problem directly. Here we modify this idea and gluing two C^1 paths of solutions together to obtain a continuous path of solutions, by which we can track along the trajectory to obtain the rotating periodic solutions (specially, quasi-periodic solutions) of (1.1). For some recent work on rotating periodic solutions of ODEs, one can see [16–19] and the references.

Remark In [15], Lu, Li and Su applied the homotopy continuation method to give a constructive proof of Mawhin's continuation theorem in [20], and thus established a global

method for finding periodic solutions of ordinary differential equations (systems). Our results can be seen as a generalization of the results in [15] from periodic solutions case to the rotating periodic case when g satisfies the Hartman-type nonlinearity.

This paper is organized as follows. In Sect. 2 we give a prior estimate and obtain the proof of Theorem 1.1 by applying the homotopy continuation method. In Sect. 3, some useful facts for the homotopy continuation method are collected. For convenience, we introduce some notations and definitions. $L^p(0, T)$ $(1 < p \leq \infty)$ denotes the usual Lebesgue measurable space with norm $\| \cdot \|_p$. $C^m \doteq C^m(\mathbb{R}, \mathbb{R}^n)$ $(m \in \mathbb{N})$ denotes the space of m-times continuously differential real functions from $\mathbb{R}$ to $\mathbb{R}^n$ with norm

$$\|u\|_{C^m} = \max_{t \in \mathbb{R}} |u(t)| + \max_{t \in \mathbb{R}} |u'(t)| + \cdots + \max_{t \in \mathbb{R}} |u^{(m)}(t)|.$$

Specially, we denote $\| \cdot \| = \|u\|_{C^1}$.

2 Main results

Firstly, we consider the following auxiliary problem:

$$\begin{cases} x'' = \lambda [x - \phi_\eta(x) + g(t, \phi_\eta(x))], \\ x(T) = Qx(0), \end{cases} \tag{2.1}$$

where $\lambda \in (0, 1]$ and $\phi_\eta : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\phi_\eta(x) = \begin{cases} x & \text{if } |x| \leq \eta, \\ \eta \frac{x}{|x|} & \text{if } |x| > \eta, \end{cases}$$

where η is taken in (H).

Lemma 2.1 *Assume that (λ, x) is a solution pair of* (2.1). *Then there exists $\rho > 0$ such that $\|x\| < \rho$. Furthermore, if x is a solution of* (2.1) *with $\lambda = 1$, then x is a Q-rotating periodic solution of* (1.1).

Proof

Assume that (λ, x) is a solution pair of (2.1). Multiplying (2.1) with x, by $Q \in O(n)$ we get

$$\int_0^T |x'(s)|^2 \, ds + \lambda \int_0^T |x(s)|^2 \, ds$$

$$= \lambda \int_0^T \phi_\eta(x) \cdot x(s) \, ds - \lambda \int_0^T g(s, \phi_\eta(x)) \cdot x(s) \, ds$$

$$\leq \lambda \left[\left(\int_0^T |\phi_\eta(x)|^2 \, ds \right)^{\frac{1}{2}} + \left(\int_0^T |g(s, \phi_\eta(x))|^2 \, ds \right)^{\frac{1}{2}} \right] \left(\int_0^T |x(s)|^2 \, ds \right)^{\frac{1}{2}}.$$

By $\lambda \in (0, 1]$, we have

$$\int_0^T |x(s)|^2 \, ds \leq \left[\left(\int_0^T |\phi_\eta(x)|^2 \, ds \right)^{\frac{1}{2}} + \left(\int_0^T |g(s, \phi_\eta(x))|^2 \, ds \right)^{\frac{1}{2}} \right]^2$$

and

$$\int_0^T |x'(s)|^2\, ds \le \left[\left(\int_0^T |\phi_\eta(x)|^2\, ds \right)^{\frac{1}{2}} + \left(\int_0^T |g(s,\phi_\eta(x))|^2\, ds \right)^{\frac{1}{2}} \right] \left(\int_0^T |x(s)|^2\, ds \right)^{\frac{1}{2}}$$

$$\le \left[\left(\int_0^T |\phi_\eta(x)|^2\, ds \right)^{\frac{1}{2}} + \left(\int_0^T |g(s,\phi_\eta(x))|^2\, ds \right)^{\frac{1}{2}} \right]^2.$$

Since ϕ_η is continuous and bounded, it follows that there exists some $C_1 > 0$ independent of λ and x such that

$$\left\| x' \right\|_2 + \left\| x \right\|_2 \le C_1, \tag{2.2}$$

which implies that there exist $\sigma_1, \sigma_2 \in [0, T]$ such that $\max\{|x(\sigma_1)|, |x'(\sigma_2)|\} \le 2\frac{C_1}{\sqrt{T}}$. Denote $h(t,x) = x - \phi_\eta(x) + g(t,\phi_\eta(x))$. Then

$$\begin{aligned}
h(t+T,x) &= x - \phi_\eta(x) + g\big(t+T, \phi_\eta(x)\big) \\
&= Q\big[Q^{-1}x - \phi_\eta\big(Q^{-1}x\big) \big] + Qg\big(t+T, Q^{-1}\phi_\eta(x)\big) \\
&= Q\big[Q^{-1}x - \phi_\eta\big(Q^{-1}x\big) + g\big(t, \phi_\eta\big(Q^{-1}x\big)\big) \big] \\
&= Qh\big(t, Q^{-1}x\big). \tag{2.3}
\end{aligned}$$

It is not difficult to see that $x(t+T) = Qx(t)$ for any $t \in \mathbb{R}$. By $Q \in O(n)$ we can see that

$$\left| x(t+mT) \right| = \left| Q^m x(t) \right| = \left| x(t) \right|, \quad \forall t \in \mathbb{R}, \forall m \in \mathbb{Z}^+. \tag{2.4}$$

By $x(t) = x(\sigma_1) + \int_{\sigma_1}^t x'(s)\, ds$ and (2.2) it follows that

$$\left\| x \right\|_\infty \le C_2 \tag{2.5}$$

for some $C_2 > 0$ independent of λ and x. Integrating (2.1) from σ_2 to t, we obtain

$$x'(t) = x'(\sigma_2) + \lambda \int_{\sigma_2}^t \big[x(s) - \phi_\eta\big(x(s)\big) + g\big(t, \phi_\eta\big(x(s)\big)\big) \big]\, ds.$$

Similar arguments to above imply that

$$\left\| x' \right\|_\infty \le C_3 \tag{2.6}$$

for some $C_3 > 0$ independent of λ and x. Denote $\rho = \max\{C_2, C_3, \eta + 1\} + 1$. Then we get $\|x\| < \rho$.

We claim that $|x(t)| \le \eta$ for all $t \in [0, T]$, i.e., if x is a solution of (2.1) with $\lambda = 1$, then x is a Q-rotating periodic solution of (1.1). Indeed, for each $t \in [0, T]$ such that $|x(t)| > \eta$, by (2.1) with $\lambda = 1$ it follows that

$$\begin{aligned}
\frac{d}{dt}\big[x'(t) \cdot x(t) \big] &= x''(t) \cdot x(t) + \left| x'(t) \right|^2 \\
&= \big[\left| x(t) \right|^2 - \phi_\eta\big(x(t)\big) \cdot x(t) + g\big(t, \phi_\eta\big(x(t)\big)\big) \cdot x(t) \big] + \left| x'(t) \right|^2
\end{aligned}$$

$$\geq \left[|x(t)|^2 - \eta \frac{x(t)}{|x(t)|} \cdot x(t) + g\left(t, \eta \frac{x(t)}{|x(t)|}\right) \cdot \left(\eta \frac{x(t)}{|x(t)|} \frac{|x(t)|}{\eta}\right) \right]$$

$$\geq |x(t)|\left(|x(t)| - \eta\right)$$

$$> 0. \tag{2.7}$$

If $|x(t)| > \eta$ for all $t \in \mathbb{R}$, then (2.7) holds for all $t \in \mathbb{R}$, which is contrary to (2.4). If $|x(t_1)| \leq \eta$ for some $t_1 \in [0, T]$ and $|x(t_2)| > \eta$ for some $t_2 \in [0, T]$, then we can find $\tau, \xi \in \mathbb{R}$ with $\xi > \tau$ such that

$$\left|x(\tau)\right| = \eta, \qquad \left|x(\xi)\right| = \max_{t \in \mathbb{R}}\left|x(t)\right| > \eta, \qquad \left|x(t)\right| > \eta, \quad \forall t \in (\tau, \xi). \tag{2.8}$$

Clearly,

$$\frac{d}{dt}\left[\left|x(t)\right|^2\right]|_{t=\xi} = 2x(\xi) \cdot x'(\xi) = 0.$$

Then, by (2.7) it follows that

$$\frac{d}{dt}\left[\frac{1}{2}\left|x(t)\right|^2\right] = x(t) \cdot x'(t) < x(\xi) \cdot x'(\xi) = 0, \quad \forall t \in (\tau, \xi),$$

which together with (2.8) implies that

$$\eta^2 < \left|x(\xi)\right|^2 < \left|x(\tau)\right|^2 = \eta^2.$$

This is a contradiction. Hence, $|x(t)| \leq \eta$ for all $t \in \mathbb{R}$. Thus x satisfies $x'' = g(t, x)$ and $x(t + T) = Qx(t)$, i.e., x is a Q-rotating periodic solution of (1.1). $\square$

Proof of Theorem 1.1 By $Q \in O(n)$ and the knowledge in linear algebra, there exist an invertible matrix A and an orthogonal matrix B such that $Q = A^{-1}BA$, where B has the form

$$B = \begin{pmatrix} I_{\alpha_1} & \\ & \hat{Q} \end{pmatrix},$$

where $\alpha_1 = \dim \text{Ker}(I_n - B)$, $\hat{Q}$ is a $\beta_1 \times \beta_1$ orthogonal matrix with $\beta_1 = n - \alpha_1$ such that 1 is not an eigenvalue of $\hat{Q}$. Furthermore, we can get

$$\mathbb{R}^n = \text{Ker}(I_n - B) \oplus \text{Im}(I_n - B).$$

Let $\hat{P}_1, \hat{P}_2 : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal projectors such that $\text{Im}\,\hat{P}_1 = \text{Ker}(I_n - B)$ and $\text{Im}\,\hat{P}_2 = \text{Im}(I_n - B)$, respectively. Let $B_{r_1}^{\alpha_1}(0)$ be the open sphere in $\text{Ker}(I_n - B)$ with center 0 and radius $r_1 > 0$, and $B_{r_2}^{\beta_1}(0)$ be the open sphere in $\text{Im}(I_n - B)$ with center 0 and radius $r_2 > 0$.

In view of $g(t + T, x) = Qg(t, Q^{-1}x)$, letting $f(t, x) = Ag(t, A^{-1}x)$, we get

$$f(t + T, x) = Ag\left(t + T, A^{-1}x\right)$$

$$= AQg\left(t, Q^{-1}A^{-1}x\right)$$

$$= BAg(t, A^{-1}B^{-1}x)$$
$$= Bf(t, B^{-1}x), \quad \forall x \in \mathbb{R}^n. \tag{2.9}$$

Set $y = Ax$. If x is a solution of (2.1), then y satisfies

$$\begin{cases} y'' = \lambda[y - \phi_\eta(y) + Ag(t, A^{-1}\phi_\eta(y))] \doteq \lambda\varphi(t, y), \\ y(T) = By(0). \end{cases} \tag{2.10}$$

By (2.9) it is clear that $\varphi(t + T, y) = B\varphi(t, B^{-1}y)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

Case 1: $\mathrm{Ker}(I_n - B) \neq \{0\}$.

Define

$$H(\xi, \mu) = \mu\xi + (1 - \mu)\frac{1}{T}\int_0^T \hat{P}_1\varphi(s, \xi)\,ds, \quad \forall \xi \in \mathrm{Ker}(I_n - B), \forall \mu \in [0, 1].$$

For $\xi \in \mathrm{Ker}(I_n - B)$ with $|\xi| \geq \rho$, where ρ is taken from Lemma 2.1, we have

$$H(\xi, \mu) \cdot \xi = \mu|\xi|^2 + (1 - \mu)\frac{1}{T}\int_0^T \hat{P}_1\varphi(s, \xi)\,ds \cdot \xi$$
$$= \mu|\xi|^2 + (1 - \mu)\frac{1}{T}\int_0^T \left[|\xi|^2 - \eta|\xi| + \hat{P}_1g\left(s, \eta\frac{\xi}{|\xi|}\right)\cdot\eta\frac{\xi}{|\xi|}\frac{|\xi|}{\eta}\right]ds$$
$$> 0.$$

Then, for any $r \geq \rho$,

$$\deg_B\left(\Phi, B_r^{\alpha_1}(0), 0\right) \neq 0, \tag{2.11}$$

where $\Phi(\xi) \doteq \frac{1}{T}\int_0^T \hat{P}_1\varphi(s, \xi)\,ds, \forall \xi \in \mathrm{Ker}(I_n - B)$.

Applying the Weierstrass's theorem, we can approximate φ by a C^2 function sequence $\{\varphi_k\}$ such that

$$\lim_{k \to +\infty} \varphi_k(t, y) = \varphi(t, y), \tag{2.12}$$

uniformly in $t \in [0, T]$ and $y \in \overline{B_\rho(0)}$. Without loss of generality, we may assume that

$$\|\varphi_k - \varphi\|_\infty \leq \frac{1}{2}, \quad \forall k \in \mathbb{Z}^+. \tag{2.13}$$

By the continuation and definition of the Brouwer degree, there exists a constant $\sigma_1 > 0$ sufficiently small such that

$$\deg_B\left(\Phi_k, B_\rho^{\alpha_1}(0), d\right) \neq 0 \tag{2.14}$$

for all $d \in B_{\sigma_1}^{\alpha_1}(0) \setminus \{0\}$, where $\Phi_k(\xi) \doteq \frac{1}{T}\int_0^T \hat{P}_1\varphi_k(s, \xi)\,ds, \forall \xi \in \mathrm{Ker}(I_n - B)$. Then the perturbed boundary value problem

$$\begin{cases} y'' = \lambda\varphi_k(t, y) + (1 - \lambda)\frac{1}{T}\int_0^T \hat{P}\varphi_k(s, y(s))\,ds - (1 - \lambda)d, \\ y(T) = By(0), \end{cases} \tag{2.15}$$

has at least one solution $y_0(t) \equiv \xi_0 \in B_\rho^{\alpha_1}(0)$ for any $d \in B_{\sigma_1}^{\alpha_1}(0) \setminus \{0\}$ when $\lambda = 0$.

For $\lambda \in (0, 1]$, denote $y_k(t, \xi, \lambda)$ as the solution of $y'' = \lambda \varphi_k(t, y)$ with initial value $y_k(0, \xi, \lambda) = \xi$ for $\xi \in \mathbb{R}^n$.

By the Sard's theorem, for almost all $d \in B_{\sigma_1}^{\alpha_1}(0)$, d is a regular value of Φ_k. Denote the set of all such points of d's by Θ. Denote $\Lambda_1 \equiv \Theta \oplus B_\rho^{\beta_1}(0)$. For $\xi \in \mathbb{R}^n$ we can write it as $\xi = \hat{\xi} + \check{\xi}$, where $\hat{\xi} \in \text{Ker}(I_n - B)$ and $\check{\xi} \in \text{Im}(I_n - B)$. Define the homotopy map $H_k : \Lambda_1 \times \mathbb{R}^n \times [0, 1]$ by

$$H_k(d, \xi, \lambda) = \Psi_k(\xi, \lambda) - (1 - \lambda)d,$$

where $\Psi_k(\xi, \lambda) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ is defined by

$$\Psi_k(\xi, \lambda) = \left(\frac{1}{T} \int_0^T \hat{P}_1 \varphi_k \big(s, y_k(s, \xi, \lambda)\big) \, ds, \right.$$

$$\left. \frac{1}{T}(I_n - B)\check{\xi} + \frac{\lambda}{T} \int_0^T \hat{P}_2 \varphi_k \big(s, y_k(s, \xi, \lambda)\big) \, ds \right).$$

By (2.14), for any $d \in \Theta$, it follows that

$$\deg_B\big(H_k(d, \cdot, 0), B_\rho^{\alpha_1}(0) \oplus B_\rho^{\beta_1}(0), d\big)$$

$$= \deg_B\big(\Phi_k, B_\rho^{\alpha_1}(0), d\big) \cdot \deg_B\left(\frac{1}{T}(I_n - B)\check{\xi}, B_\rho^{\beta_1}(0), 0 \right) \neq 0. \tag{2.16}$$

It is easily seen that there exists $\lambda_0 > 0$ small enough such that, for all $d \in \Theta$, 0 is a regular value of $H_k(d, \xi, \lambda)$ for any $\lambda \in [0, \lambda_0)$. Denote the set of all such points of d's by Λ_1^*. For any $d \in \Lambda_1^*$, denote $H_{k,d}(\xi, \lambda) = H_k(d, \xi, \lambda)$ for all $(\xi, \lambda) \in \mathbb{R}^n \times [0, 1)$. Applying the parametrized Sard theorem, we know that for almost all the points $d \in \Lambda_1^*$, 0 is a regular value of $H_{k,d}(\xi, \lambda)|_{\mathbb{R}^n \times (0, \lambda_0)}$. Denote the set of all such points of d's as Λ_2^*. Clearly, for any $d \in \Lambda_2^*$, 0 is a regular value of $H_{k,d}(\xi, \lambda)|_{\mathbb{R}^n \times [0, \lambda_0)}$. Then, by Lemma 3.3, for every $d \in \Lambda_2^*$, there exists a C^1 curve $(\xi(s), \lambda(s))$ $(s \geq 0)$ such that

$$H_{k,d}\big(\xi(s), \lambda(s)\big) = 0, \quad 0 \leq \lambda < \lambda_0,$$

$$\big(\xi(0), \lambda(0)\big) = (\xi_0, 0).$$

By Lemma 3.5 we can see that $(\xi(s), \lambda(s))$ satisfies the following differential equation:

$$\begin{cases} \frac{d\xi_i}{ds}(s) = (-1)^{i+1} \det H'_{k,d,i}, & i = 1, \dots, n, \\ \frac{d\lambda}{ds}(s) = (-1)^{n+2} \det H'_{k,d,n+1}, \\ (\xi(0), \lambda(0)) = (\xi_0, 0). \end{cases} \tag{2.17}$$

Now we show that the curve $(\xi(s), \lambda(s))$ is not a loop. In fact, since $\overline{B_\rho^{\alpha_1}(0)}$ is compact, it is not difficult to see that the number of such ξ_0 is finite, without loss of generality, we denote them as $\xi_{01}, \xi_{02}, \dots, \xi_{0m}$. Let $(\xi(s, \xi_{0i}), \lambda(s, \xi_{0i}))$ be a path of the homotopy $H_{k,d}(\xi(s), \lambda(s))$ with $(\xi(0), \lambda(0)) = (\xi_{0i}, 0)$. When $\lambda = 0$ and $d \in \Lambda_2^*$, denoting $H'_{k,d,\xi}(\xi_{0i}) = \frac{\partial H_{k,d}}{\partial \xi}|_{\xi = \xi_{0i}}$, by the fact that 0 is a regular value of $H_{k,d}(\xi, \lambda)|_{\mathbb{R}^n \times [0, \lambda_0)}$ for any $d \in \Lambda_2^*$ it follows that

$\det H'_{k,d,\xi}(\xi_{0i}) \neq 0$ for $i = 1, 2, \ldots, m$. By the implicit function theorem we can see that the conclusion follows.

Using Lemma 3.4 it follows that each curve $(\xi(s, \xi_{0i}), \lambda(s, \xi_{0i}))$ $(i \in \{1, 2, \ldots, m\})$ is an interval, which maybe open, closed or semi-closed. We claim there exists some $i_0 \in \{1, 2, \ldots, m\}$ such that the curve $(\xi(s, \xi_{0i_0}), \lambda(s, \xi_{0i_0}))$ starts from $\overline{B_\rho^{\alpha_1}(0)} \times \{0\}$ and doesn't end on $\overline{B_\rho(0)} \times \{0\}$. Indeed, if not, then $m = 2l$ for some $l \in \mathbb{Z}^+$. Without loss of generality, we may assume that $(\xi(s, \xi_{0i}), \lambda(s, \xi_{0i}))$ starts from $(\xi_{0i}, 0) \in \overline{B_\rho^{\alpha_1}(0)} \times \{0\}$ and ends at $(\xi_{0(l+i)}, 0) \in \overline{B_\rho(0)} \times \{0\}$ for all $i = 1, 2, \ldots, l$. By Lemma 3.6 we can get that

$$\operatorname{sign} \det H'_{k,d,\xi}(\xi_{0i}) = -\operatorname{sign} \det H'_{k,d,\xi}(\xi_{0(l+i)}), \quad i = 1, 2, \ldots, l.$$

Thus by the continuation and definition of the Brouwer degree it follows that, for $d \in \Theta$,

$$\deg_B\big(H_k(d, \cdot, 0), B_\rho^{\alpha_1}(0) \oplus B_\rho^{\beta_1}(0), d\big)$$
$$= \deg_B\big(H_{k,d}(\cdot, 0), B_\rho^{\alpha_1}(0) \oplus B_\rho^{\beta_1}(0), 0\big) = \sum_{i=1}^{2l} \operatorname{sign} \det H'_{k,d,\xi}(\xi_{0i}) = 0.$$

However, this is contrary to (2.16).

Note that $\xi_0 \in \overline{B_\rho^{\alpha_1}(0)}$, we can see that there exists $\lambda^* \in (0, \lambda_0)$ small enough such that the curve $\xi(s)$ will stay in $\overline{B_\rho^{\alpha_1}(0)}$ when $0 < \lambda(s) \leq \lambda^*$. Denote the position $\xi(s^*)$ by ξ^* such that $\lambda^* = \lambda(s^*)$ for some $s^* > 0$. Clearly, by the homotopy invariance of Brouwer degree and (2.16), we get

$$\deg_B\big(H_{k,d}(\cdot, \lambda^*), B_\rho^{\alpha_1}(0) \oplus B_\rho^{\beta_1}(0), d\big)$$
$$= \deg_B\big(H_{k,d}(\cdot, 0), B_\rho^{\alpha_1}(0) \oplus B_\rho^{\beta_1}(0), 0\big) \neq 0, \quad \forall d \in \Lambda_2^*. \tag{2.18}$$

For any $\lambda \in [\lambda^*, 1)$ and $\xi \in \mathbb{R}^n$, since $(\frac{\partial H_k}{\partial d}, \frac{\partial H_k}{\partial \xi}, \frac{\partial H_k}{\partial \lambda}) = (-(1-\lambda)I_n, \frac{\partial H_k}{\partial \xi}, \frac{\partial H_k}{\partial \lambda})$, we can see that, for all $d \in \Lambda_2^*$, 0 is a regular value of $H_{k,d}(\xi, \lambda)$ for any $\lambda \in [\lambda^*, 1)$. Denote the set of all such points of d's by Γ_1^*. Applying the parametrized Sard theorem, we know that for almost all the points $d \in \Gamma_1^*$, 0 is a regular value of $H_{k,d}(\xi, \lambda)|_{\mathbb{R}^n \times [\lambda^*, 1)}$. Denote the set of all such points of d's as Γ_2^*. Clearly, for any $d \in \Gamma_2^*$, 0 is a regular value of $H_{k,d}(\xi, \lambda)|_{\mathbb{R}^n \times [\lambda^*, 1)}$. Then, by Lemma 3.3, for every $d \in \Gamma_2^*$, there exists a C^1 curve $(\tilde{\xi}(s), \tilde{\lambda}(s))$, such that

$$H_{k,d}\big(\tilde{\xi}(s), \tilde{\lambda}(s)\big) = 0, \quad \lambda^* \leq \tilde{\lambda} < 1,$$
$$\big(\tilde{\xi}(0), \tilde{\lambda}(0)\big) = \big(\xi^*, \lambda^*\big).$$

By Lemma 3.5 we can see that $(\tilde{\xi}(s), \tilde{\lambda}(s))$ satisfies the following differential equation:

$$\begin{cases} \frac{d\tilde{\xi}_i}{ds}(s) = (-1)^{i+1} \det H'_{k,d,i}, & i = 1, \ldots, n, \\ \frac{d\tilde{\lambda}}{ds}(s) = (-1)^{n+2} \det H'_{k,d,n+1}, \\ (\tilde{\xi}(0), \tilde{\lambda}(0)) = (\xi^*, \lambda^*). \end{cases} \tag{2.19}$$

Since φ_k is of C^2 and $(\xi^*, \lambda^*) \in B_\rho(0) \times [\lambda^*, 1)$, it follows that Cauchy problem (2.19) has the maximal solution $(\tilde{\xi}(s), \tilde{\lambda}(s))_{s \in [0, \tau)}$ in $\overline{B_\rho(0)} \times [\lambda^*, 1)$. In what follows, we shall proceed

to prove that the path tends to $\overline{B_\rho(0)} \times \{1\}$. By similar arguments as above, together with the implicit function theorem and (2.18), we get:

(a) The path $(\tilde{\xi}(s), \tilde{\lambda}(s))_{s \in [0, \tau)}$ is not a loop.

(b) The path $(\tilde{\xi}(s), \tilde{\lambda}(s))_{s \in [0, \tau)}$ will not return to some point (ξ_1, λ^*).

Furthermore, by the above claim it follows that the path $(\tilde{\xi}(s), \tilde{\lambda}(s))_{s \in [0, \tau)}$ will not touch the boundary $\partial B_\rho(0) \times [\lambda^*, 1)$. By the idea of the homotopy continuation method (see [15, 21]), we can obtain that there exists $j^* \in \{1, 2, \ldots, m\}$ such that $\lambda_{\max, j^*} = \sup\{\lambda(s, \xi_{0j^*}) : s \in [0, \tau)\} = 1$. Hence $H_k(d, \xi_{0, j^*}, 1) = 0$ and $y_k(t, \xi_{0, j^*}, 1)$ is a solution of (2.1) with $\lambda = 1$.

Assume that $\{y_k\}$ is a solution sequence of (2.10) with $\lambda = 1$ corresponding to $\{\varphi_k\}$. Firstly, it is easily seen that $\{y_k\} \subset \Omega$ and thus is uniformly bounded. Secondly, to show $\{y_k\}$ is equicontinuous on $[0, T]$, we can obtain by (2.12)–(2.13) and the smooth of φ_k that

$$\left|\varphi_k(t, y_k) - \varphi(t, y_k)\right| \le \frac{1}{2},$$

$$\left|\varphi(t, y_k)\right| \le M,$$

which implies $|\varphi_k(t, y_k)| \le M + \frac{1}{2}, \forall k \in \mathbb{Z}^+$. By (2.10) with $\lambda = 1$ corresponding to $\{\varphi_k\}$ we get $|y_k''| \le M + \frac{1}{2}$. Then $\{y_k\}$ and $\{y_k'\}$ are uniformly bounded and equicontinuous on $[0, T]$. By the Arzela–Ascoli theorem it follows that there is a uniformly convergent subsequence of $\{y_k\}$ on $[0, T]$, which is still denoted by $\{y_k\}$, such that

$$\lim_{k \to +\infty} y_k(t) = y^*(t), \qquad \lim_{k \to +\infty} y_k'(t) = (y^*)'(t).$$

Hence $\lim_{k \to +\infty} \varphi_k(t, y_k(t)) = \varphi(t, y^*(t))$ and we can easily verify that $x^* = Ay^*$ is a solution of (2.1) with $\lambda = 1$.

Case 2: $\operatorname{Ker}(I_n - B) = \{0\}$.

In this case $\alpha_1 = 0$. By approximating φ by a C^2 function sequence $\{\varphi_k\}$, we can consider the homotopy map

$$\tilde{H}_k(d, \xi, \lambda) = \tilde{\Psi}_k(\xi, \lambda) - (1 - \lambda)d,$$

where $\tilde{\Psi}_k(\xi, \lambda) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ is defined by

$$\tilde{\Psi}_k(\xi, \lambda) = \frac{1}{T}(I_n - B)\xi + \frac{\lambda}{T} \int_0^T \varphi_k\big(s, y_k(s, \xi, \lambda)\big) \, ds.$$

By similar arguments as above we can obtain the existence of Q-rotating periodic solutions of (1.1). This completes the proof. □

3 Appendix

In this section, we collect some basic results on the homotopy continuation method.

Lemma 3.1 ([22] Sard's theorem) *If $\Phi : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is a C^k map on an open set U with $k > \max\{0, m - n\}$, then the set of singular values of Φ has n-dimensional Lebesgue measure zero. Consequently, the set of regular values of Φ is dense in $\mathbb{R}^n$.*

Lemma 3.2 ([23] Parametrized Sard's theorem) *Let $V \subset \mathbb{R}^n, U \subset \mathbb{R}^m$ be open sets and $\Phi : V \times U \to \mathbb{R}^k$ a C^r map, where $r > \max\{0, m - k\}$. If $0 \in \mathbb{R}^k$ is a regular value of Φ, then for almost all $a \in V$, 0 is a regular value of $\Phi_a = \Phi(a, \cdot)$.*

Lemma 3.3 ([22]) *Let $\Phi : R^{n+1} \to R^n$ be a C^1 map and 0 a regular value of Φ. Then $\Phi^{-1}(0)$ is a C^1-manifold of dimension 1.*

Lemma 3.4 ([22]) *A C^1-manifold of dimension 1 is homeomorphic to a loop or an interval (open, closed, or semiclosed).*

Lemma 3.5 ([21]) *Let the homotopy $\Phi : R^{n+1} \to R^n$ be a C^1 map and 0 be regular value of Φ. Then each solution $x(s)$ of the Cauchy problem*

$$\frac{dx_i}{ds} = (-1)^{i+1} \det \Phi_i{}', \qquad x_i(0) = x_{i_0} \quad (i = 1, \dots, n + 1)$$

is a C^1 path of $\Phi^{-1}(0)$, where s is a parameter and

$$\Phi_i' = (\Phi_{x_1}, \dots, \Phi_{x_{i-1}}, \Phi_{x_{i+1}}, \dots, \Phi_{x_{n+1}}).$$

Lemma 3.6 ([21]) *Let $V \subset R^n$ be an open bounded set, $H : V \times [0,1] \to R^n$ a C^1 homotopy, 0 a regular value of H, and the Jacobi matrices $H_p'(p, 0)$ $(p \in H_0^{-1}(0))$, $H_p'(p, 1)$ $(p \in H_0^{-1}(0))$ be nonsingular, where*

$$H_0 = H(\cdot, 0), \qquad H_1 = H(\cdot, 1).$$

If a path solution $(p(s), \lambda(s))$ connects point $(p_0, 0)$ with $(p_1, 0)$, then

$$\operatorname{sgn} \det H_\lambda'(P_0, 0) = - \operatorname{sgn} \det H_\lambda'(P_1, 0).$$

If a path solution $(p(s), \lambda(s))$ of H connects point $(p_0, 0)$ with $(p_1, 1)$, then

$$\operatorname{sgn} \det H_\lambda'(P_0, 0) = \operatorname{sgn} \det H_\lambda'(P_1, 0).$$

4 Conclusions

In this paper, by a constructive proof based on the homotopy continuation method, we prove that the second order system $x'' = g(t, x)$ admits rotating periodic solutions with form $u(t + T) = Qu(t)$ for any orthogonal matrix Q when the nonlinearity g admits the Hartman-type condition. In [3], Rouche and Mawhin obtained the existence of periodic solutions of second order system under the Hartman-type condition. In [15], Lu, Li and Su applied the homotopy continuation method to give a constructive proof of Mawhin's continuation theorem in [20], which can be used to obtain the periodic solutions by track along a C^1 trajectory. However, when $\operatorname{Ker}(I_n - Q) \not\equiv \mathbb{R}^n$, it seems difficult to apply this idea to our problem directly. Here we modify this idea and gluing two C^1 paths of solutions together to obtain a continuous path of solutions, by which we can obtain the rotating periodic solutions (specially, quasi-periodic periodic solutions) of (1.1). Our results can be seen as a generalization of the results in [15] from periodic solutions case to the rotating periodic case when g satisfies the Hartman-type nonlinearity.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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