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# Multiple solutions with constant sign for a $(p,q)$-elliptic system Dirichlet problem with product nonlinear term

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**Abstract**

In this paper, we consider the existence of multiple solutions of the homogeneous Dirichlet problem for a $(p,q)$-elliptic system with nonlinear product term as follows:

$$\begin{cases} -\Delta_p u = \lambda\alpha(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \lambda\beta(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega. \end{cases}$$

We emphasize that the potential $F(x,u,v)$ might contain a nonlinear product term which includes $F(x,u,v) = |u|^{\theta_1(x)}|v|^{\theta_2(x)}\ln(1+|u|)\ln(1+|v|)$ as a prototype, and does not require $F(x,u,v) \to +\infty$ as $|u|+|v| \to +\infty$. With novel growth conditions on $F(x,u,v)$, we develop a new method to check the Cerami compactness condition. Through arguments of critical point theory, we prove the existence of multiple constant-sign solutions for our elliptic system without requiring the well-known Ambrosetti–Rabinowitz condition. Moreover, we also give a result guaranteeing the existence of infinitely many solutions.

**MSC:** 35J47; 35J50; 35J57

**Keywords:** Elliptic system; Dirichlet problem; Cerami condition; Without Ambrosetti–Rabinowitz condition; Constant-sign solution

## 1 Introduction

We consider the existence of multiple solutions of the Dirichlet problem for the $(p,q)$-elliptic system with nonlinear product term as follows:

$$\begin{cases} -\Delta_p u = \lambda\alpha(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \lambda\beta(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $\alpha(\cdot), \beta(\cdot) > 1$ belong to the space $C(\overline{\Omega})$, $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^1$, and $\lambda > 0$ is a parameter. The main feature of the above problem is the presence of the nonlinear product term.

Our goal is to obtain existence results for problem $(P)$ without requiring the usual Ambrosetti–Rabinowitz condition. To this end, we provide novel growth conditions on the potential $F(x,u,v)$ allowing us to develop a new method to check the Cerami compactness condition, which is crucial to applying critical point theory.

The Ambrosetti–Rabinowitz type conditions are rather restrictive and exclude significant classes of nonlinearities. Numerous papers deal with the elliptic equations without the Ambrosetti–Rabinowitz type conditions, some of them even weakening growth condition such as $f(x,t)/|t|^{p-2}t \to +\infty$ as $|t| \to +\infty$ (see [1–8]). It is worth mentioning that there are some results related to system $(P)$ without the Ambrosetti–Rabinowitz type growth conditions, but requiring conditions such as $F(x,u,v)/(|u|^p + |v|^q) \to +\infty$ as $|u|^p + |v|^q \to +\infty$ (see [9, 10]). In [11] for $N = 1$ and $\lambda = 0$, the authors study problem $(P)$ without the Ambrosetti–Rabinowitz type condition, but requiring the integral coercive condition $\int_0^T F(t,u,v)\,dt \to +\infty$ as $|u| + |v| \to +\infty$.

Recently, in [12] the authors extended the results in [13] establishing an existence result of multiple solutions for a Dirichlet problem with variable exponents involving an elliptic system without Ambrosetti–Rabinowitz condition as follows:

$$\begin{cases} -\Delta_{p(x)}u = \lambda\alpha(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_{q(x)}v = \lambda\beta(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega. \end{cases}$$

We point out that in these results the condition $F(x,u,v)/(|u|^{p(x)} + |v|^{q(x)}) \to +\infty$ as $|u|^{p(x)} + |v|^{q(x)} \to +\infty$ is required.

In the present paper, we extend in the case of $(P)$ the results in [12] obtaining multiple constant-sign solutions. A relevant contribution consists in the fact that the restrictive requirement $F(x,u,v) \to +\infty$ as $|u| + |v| \to +\infty$ is not needed anymore. A typical form of the admissible potential is $F(x,u,v) = |u|^{\theta_1(x)}|v|^{\theta_2(x)}\ln(1+|u|)\ln(1+|v|)$.

Before stating our main results, we list the following conditions:

$(H_{\alpha,\beta})$ $\frac{\alpha(\cdot)}{p} + \frac{\beta(\cdot)}{q} < 1$ on $\overline{\Omega}$.

$(H_0)$ $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $C^1$-function, and

$$\bigl|F_u(x,u,v)u\bigr| + \bigl|F_v(x,u,v)v\bigr| \le C\bigl(1 + |u|^\gamma + |v|^\delta\bigr), \quad \forall (x,u,v) \in \Omega \times \mathbb{R},$$

with $p < \gamma < p^*, q < \delta < q^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p}, & p < N, \\ \infty, & p \ge N, \end{cases}$$

$$q^* = \begin{cases} \frac{Nq}{N-q}, & q < N, \\ \infty, & q \ge N. \end{cases}$$

$(H_1)$ There exist constants $M, C_{1*}, C_{2*} > 0$, and continuous functions $1 < \theta_1(\cdot) < p$, $1 <$

$\theta_2(\cdot) < q$, $\frac{\theta_1(\cdot)}{p} + \frac{\theta_2(\cdot)}{q} \equiv 1$ on $\overline{\Omega}$ such that

$$\begin{aligned} & C_{1*}|u|^{\theta_1(x)}|v|^{\theta_2(x)} \\ & \quad \le C_{2*}\left(\frac{F_u(x,u,v)u}{\ln(e+|u|)} + \frac{F_v(x,u,v)v}{\ln(e+|v|)}\right) \\ & \quad \le \frac{1}{p}F_u(x,u,v)u + \frac{1}{q}F_v(x,u,v)v - F(x,u,v), \quad \forall |u|+|v| \ge M, x \in \Omega, \end{aligned}$$

and

$$F_u(x,u,v)u \ge 0 \quad \text{and} \quad F_v(x,u,v)v \ge 0, \quad \forall |u|+|v| \ge M, \forall x \in \Omega.$$

($H_2$) $F(x,u,v) = o(|u|^p + |v|^q)$ uniformly for $x \in \Omega$ as $u, v \to 0$.
($H_3$) $F$ satisfies $F_u(x,u,v) = 0, F_v(x,u,v) = 0$, $\forall x \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$ with $uv = 0$.
($H_4$) $F(x,-u,-v) = F(x,u,v)$, $\forall x \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$.
($H_{p,q}$) $p = q$.

Our results are stated as follows.

**Theorem 1.1** *If* $\lambda > 0$ *is small enough and assumptions* ($H_{\alpha,\beta}$), ($H_0$), ($H_2$), ($H_3$) *hold, then problem* ($P$) *has at least four nontrivial constant-sign solutions.*

**Theorem 1.2** *If* $\lambda > 0$ *is small enough and assumptions* ($H_{\alpha,\beta}$), ($H_0$)–($H_3$) *hold, then problem* ($P$) *has at least eight nontrivial constant-sign solutions.*

**Theorem 1.3** *If assumptions* ($H_{\alpha,\beta}$), ($H_0$), ($H_1$), ($H_4$), *and* ($H_{p,q}$) *hold, then there are infinitely many pairs of symmetric solutions to problem* ($P$).

*Remark*

(i) Let

$$F(x,u,v) = |u|^{\theta_1(x)}|v|^{\theta_2(x)} \ln\bigl(1+|u|\bigr) \ln\bigl(1+|v|\bigr),$$

with $1 < \theta_1(x) < p, 1 < \theta_2(x) < q, \frac{\theta_1(x)}{p} + \frac{\theta_2(x)}{q} = 1, \forall x \in \overline{\Omega}$. Then $F$ satisfies conditions ($H_0$)–($H_4$), but $F$ does not satisfy the Ambrosetti–Rabinowitz condition, and does not satisfy $F(x,u,v) \to +\infty$ as $|u|+|v| \to +\infty$.

(ii) We do not assume any monotonicity condition on $F(x,\cdot,\cdot)$.

(iii) Our method can be applied to other relevant cases, for instance,

$$F(x,u,v) = |u|^{\theta_1(x)}|v|^{\theta_2(x)}\bigl[\ln\bigl(1+\ln\bigl(1+|u|\bigr)\bigr)\bigr]\bigl[\ln\bigl(1+\ln\bigl(1+|v|\bigr)\bigr)\bigr].$$

The rest of the paper is organized as follows. In Sect. 2 we do some preparation work focusing on certain Sobolev spaces and Nemytskii operators. In Sect. 3 we prove our main results.

## 2 Preliminary results

In order to study problem ($P$), we first recall some basic properties of the space $W_0^{1,p}(\Omega)$ that will be used later (for details, see [14–19]).

Denote

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; u \text{ is measurable}, \int_\Omega \left|u(x)\right|^p dx < \infty \right\}.$$

Endowed with the norm

$$|u|_p = \left( \int_\Omega \left|u(x)\right|^p dx \right)^{\frac{1}{p}},$$

$(L^p(\Omega), |\cdot|_p)$ becomes a Banach space.

The space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \nabla u \in \left(L^p(\Omega)\right)^N \right\},$$

and is endowed with the norm

$$\|u\|_p = |u|_p + |\nabla u|_p.$$

We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

**Proposition 2.1** (see [14, 16, 18])

(i) *$W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are separable reflexive Banach spaces.*

(ii) *If $\eta \in [1, p^*)$, then the embedding of $W^{1,p}(\Omega)$ into $L^\eta(\Omega)$ is compact.*

(iii) *There is a constant $C > 0$ such that*

$$|u|_p \le C|\nabla u|_p, \quad \forall u \in W_0^{1,p}(\Omega).$$

We know from Proposition 2.1 that $|\nabla u|_p$ and $\|u\|_p$ are equivalent norms on $W_0^{1,p}(\Omega)$. From now on we will use $|\nabla u|_p$ to replace $\|u\|_p$ as the norm on $W_0^{1,p}(\Omega)$, and use $|\nabla v|_q$ to replace $\|v\|_q$ as the norm on $W_0^{1,q}(\Omega)$.

**Proposition 2.2** (see [18, 20]) *The first eigenvalue $\lambda_p$ of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ is positive.*

Denote $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. The norm $\|\cdot\|$ on $X$ is defined by

$$\left\|(u, v)\right\| = \max\left\{ \|u\|_p, \|v\|_q \right\}.$$

For any $(u, v)$ and $(\phi, \psi)$ in $X$, let

$$\Phi_1(u) = \int_\Omega \frac{1}{p} |\nabla u|^p \, dx, \qquad \Phi_2(v) = \int_\Omega \frac{1}{q} |\nabla v|^q \, dx,$$

$$\Phi(u, v) = \Phi_1(u) + \Phi_2(v), \qquad \Psi(u, v) = \int_\Omega \left( \lambda |u|^{\alpha(x)} |v|^{\beta(x)} + F(x, u, v) \right) dx.$$

From Proposition 2.1, conditions $(H_{\alpha,\beta})$, $(H_0)$, and the continuity of Nemytskii operator (see [13, Proposition 2.2] as well as [18]), it follows that $\Phi_1, \Phi_2, \Phi, \Psi \in C^1(X, \mathbb{R})$ and

$$\left(\Phi'(u, v), (\phi, \psi)\right) = \left(D_1 \Phi(u, v), \phi\right) + \left(D_2 \Phi(u, v), \psi\right),$$

$$\big(\Psi'(u,v),(\phi,\psi)\big) = \big(D_1\Psi(u,v),\phi\big) + \big(D_2\Psi(u,v),\psi\big),$$

where

$$\big(D_1\Phi(u,v),\phi\big) = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi\,dx = \big(\Phi_1'(u),\phi\big),$$
$$\big(D_2\Phi(u,v),\psi\big) = \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi\,dx = \big(\Phi_2'(v),\psi\big),$$
$$\big(D_1\Psi(u,v),\phi\big) = \int_\Omega \left[\lambda\alpha(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} + \frac{\partial}{\partial u}F(x,u,v)\right]\phi\,dx,$$
$$\big(D_2\Psi(u,v),\psi\big) = \int_\Omega \left[\lambda\beta(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v + \frac{\partial}{\partial v}F(x,u,v)\right]\psi\,dx.$$

The integral functional associated with problem $(P)$ is

$$\varphi(u,v) = \Phi(u,v) - \Psi(u,v).$$

Without loss of generality, we may assume that $F(x,0,0) = 0, \forall x \in \overline{\Omega}$. Then we have

$$F(x,u,v) = \int_0^1 \big[u\partial_2F(x,tu,tv) + v\partial_3F(x,tu,tv)\big]\,dt, \quad \forall x \in \overline{\Omega}, \tag{1}$$

where $\partial_j$ denotes the partial derivative of $F$ with respect to its $j$th variable. From (1) and assumptions $(H_0)$–$(H_1)$, it holds

$$\big|F(x,u,v)\big| \le c\big(|u|^\gamma + |v|^\delta + 1\big), \quad \forall x \in \overline{\Omega}, \tag{2}$$

with a constant $c > 0$.

Through Proposition 2.1, assumptions $(H_{\alpha,\beta})$–$(H_0)$, and the continuity of Nemytskii operator (see [13, Proposition 2.2] as well as [18]), it follows that $\varphi \in C^1(X,\mathbb{R})$ and satisfies

$$\big(\varphi'(u,v),(\phi,\psi)\big) = \big(D_1\varphi(u,v),\phi\big) + \big(D_2\varphi(u,v),\psi\big),$$

with

$$\big(D_1\varphi(u,v),\phi\big) = \big(D_1\Phi(u,v),\phi\big) - \big(D_1\Psi(u,v),\phi\big),$$
$$\big(D_2\varphi(u,v),\psi\big) = \big(D_2\Phi(u,v),\psi\big) - \big(D_2\Psi(u,v),\psi\big).$$

We recall that $(u,v) \in X$ is a critical point of $\varphi$ if

$$\big(\varphi'(u,v),(\phi,\psi)\big) = 0, \quad \forall(\phi,\psi) \in X.$$

The dual space of $X$ will be denoted by $X^*$. Then, for any $H \in X^*$, there exist uniquely $f \in (W_0^{1,p}(\Omega))^*$ and $g \in (W_0^{1,q}(\Omega))^*$ such that $H(u,v) = f(u) + g(v)$ for all $(u,v) \in X$. Denote by $\|\cdot\|_*$, $\|\cdot\|_{*,p}$ and $\|\cdot\|_{*,q}$ the norms of $X^*$, $(W_0^{1,p}(\Omega))^*$and $(W_0^{1,q}(\Omega))^*$, respectively. Since

$X^* = (W_0^{1,p}(\Omega))^* \times (W_0^{1,q}(\Omega))^*$ and

$$\|H\|_* = \|f\|_{*,p} + \|g\|_{*,q},$$

we have

$$\|\varphi'(u,v)\|_* = \|D_1\varphi(u,v)\|_{*,p} + \|D_2\varphi(u,v)\|_{*,q}.$$

It is seen that $\Phi$ is a convex functional and that the following result holds.

**Proposition 2.3** (see [16, 18, 21])

(i) $\Phi' : X \to X^*$ *is a continuous, bounded, and strictly monotone operator*;

(ii) $\Phi'$ *is a mapping of type* $(S_+)$, *i.e., if* $(u_n, v_n) \rightharpoonup (u,v)$ *in* $X$ *and*

$$\overline{\lim_{n\to+\infty}} \big(\Phi'(u_n,v_n) - \Phi'(u,v), (u_n - u, v_n - v)\big) \le 0,$$

*then* $(u_n, v_n) \to (u,v)$ *in* $X$;

(iii) $\Phi' : X \to X^*$ *is a homeomorphism.*

We set forth a useful coercivity property for the potential $F$.

**Lemma 2.4** *Assume* $(H_{\alpha,\beta})$ *and that* $F(x,u,v)$ *verifies*

$$C_1|u|^{\theta_1(x)}|v|^{\theta_2(x)} \min\big\{\ln\big(1+|u|\big), \ln\big(1+|v|\big)\big\} \le F(x,u,v), \quad \forall|u|+|v| \ge M, \forall x \in \Omega, \tag{3}$$

*with a constant* $C_1 > 0$. *Fix* $x_0 \in \Omega$ *and* $\varepsilon > 0$ *such that* $B(x_0,\varepsilon) \subset \Omega$. *Setting*

$$h_0(x) = \begin{cases} 0, & |x - x_0| > \varepsilon, \\ \varepsilon - |x - x_0|, & |x - x_0| \le \varepsilon, \end{cases}$$

*there holds*

$$\varphi\big(t^{\frac{1}{p}} h_0, t^{\frac{1}{q}} h_0\big) \to -\infty \quad \textit{as } t \to +\infty. \tag{4}$$

*Proof* It is known from hypothesis $(H_{\alpha,\beta})$ that

$$\frac{\alpha(x)}{p} + \frac{\beta(x)}{q} < 1 \quad \text{on } \overline{\Omega},$$

which implies

$$\frac{1}{t}\int_\Omega \lambda \big|t^{\frac{1}{p}} h_0\big|^{\alpha(x)} \big|t^{\frac{1}{q}} h_0\big|^{\beta(x)}\,dx \to 0 \quad \text{as } t \to +\infty.$$

By (3) there exists a positive constant $C_2 > 0$, for which one has

$$F(x,u,v) \ge C_1|u|^{\theta_1(x)}|v|^{\theta_2(x)} \min\big\{\ln\big(1+|u|\big), \ln\big(1+|v|\big)\big\} - C_2, \quad \forall u,v \in \mathbb{R}, \forall x \in \Omega.$$

Therefore we may write

$$\begin{aligned}
&\frac{1}{t}F\big(x,t^{\frac{1}{p}}h_0,t^{\frac{1}{q}}h_0\big)\\
&\quad\geq \frac{1}{t}C_1\big|t^{\frac{1}{p}}h_0\big|^{\theta_1(x)}\big|t^{\frac{1}{q}}h_0\big|^{\theta_2(x)}\min\big\{\ln\big(1+t^{\frac{1}{p}}h_0\big),\ln\big(1+t^{\frac{1}{q}}h_0\big)\big\}-C_2\\
&\quad\geq C_1|h_0|^{\theta_1(x)}|h_0|^{\theta_2(x)}\ln\big(1+\big|t^{\frac{1}{p+q}}h_0\big|\big)-C_2\to+\infty\quad \text{as } t\to+\infty, \forall x\in B(x_0,\varepsilon).
\end{aligned}$$

Using the equality

$$\frac{1}{t}\left\{\int_\Omega\big|\nabla t^{\frac{1}{p}}h_0\big|^p\,dx+\int_\Omega\big|\nabla t^{\frac{1}{q}}h_0\big|^q\,dx\right\}=\int_\Omega|\nabla h_0|^p\,dx+\int_\Omega|\nabla h_0|^q\,dx,$$

it is readily seen that $\frac{1}{t}\varphi(t^{\frac{1}{p}}h_0,t^{\frac{1}{q}}h_0)\to-\infty$ as $t\to+\infty$, thus (4) is valid, which completes the proof. □

## 3 Proofs of main results

The solutions to system $(P)$ are understood in the weak sense.

**Definition 3.1** We call $(u,v)\in X$ a weak solution of problem $(P)$ if

$$\begin{aligned}
&\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx=\int_\Omega\big[\lambda\alpha(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}+F_u(x,u,v)\big]\phi\,dx,\quad \forall\phi\in W_0^{1,p}(\Omega),\\
&\int_\Omega|\nabla v|^{q-2}\nabla v\cdot\nabla\psi\,dx=\int_\Omega\big[\lambda\beta(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v+F_v(x,u,v)\big]\psi\,dx,\quad \forall\psi\in W_0^{1,q}(\Omega).
\end{aligned}$$

The energy functional corresponding to problem $(P)$ is

$$\begin{aligned}
\varphi(u,v)&=\Phi(u,v)-\Psi(u,v)\\
&=\int_\Omega\left[\frac{1}{p}|\nabla u|^p+\frac{1}{q}|\nabla v|^q\right]dx-\int_\Omega\big[\lambda|u|^{\alpha(x)}|v|^{\beta(x)}+F(x,u,v)\big]\,dx,\quad \forall(u,v)\in X.
\end{aligned}$$

**Definition 3.2** We say that $\varphi$ satisfies the Cerami condition in $X$, if any sequence $\{(u_n,v_n)\}\subset X$ such that $\{\varphi(u_n,v_n)\}$ is bounded and $\|\varphi'(u_n,v_n)\|_*(1+\|(u_n,v_n)\|)\to 0$ as $n\to\infty$ has a convergent subsequence.

**Lemma 3.3** *If hypotheses* $(H_{\alpha,\beta})$, $(H_0)$, *and* $(H_1)$ *hold, then* $\varphi$ *satisfies the Cerami condition.*

*Proof* Let $\{(u_n,v_n)\}\subset X$ be a Cerami sequence, i.e., one has $\varphi(u_n,v_n)\to c$ and $\|\varphi'(u_n,v_n)\|_*(1+\|(u_n,v_n)\|)\to 0$. If $\{(u_n,v_n)\}$ is bounded, then $\{(u_n,v_n)\}$ contains a weakly convergent subsequence in $X$. We may assume that $(u_n,v_n)\rightharpoonup(u,v)$, so $\Psi'(u_n,v_n)\to\Psi'(u,v)$ in $X^*$. Since $\varphi'(u_n,v_n)=\Phi'(u_n,v_n)-\Psi'(u_n,v_n)\to 0$ in $X^*$, we infer $\Phi'(u_n,v_n)\to\Phi'(u,v)$ in $X^*$. Recalling from Proposition 2.3(iii) that $\Phi'$ is a homeomorphism, we derive $(u_n,v_n)\to(u,v)$, which establishes that $\varphi$ satisfies the Cerami condition.

Next we show the boundedness of the Cerami sequence $\{(u_n,v_n)\}$ arguing by contradiction. Suppose there exist $c\in\mathbb{R}$ and $\{(u_n,v_n)\}\subset X$ satisfying

$$\varphi(u_n,v_n)\to c,\qquad \big\|\varphi'(u_n,v_n)\big\|_*\big(1+\big\|(u_n,v_n)\big\|\big)\to 0,\qquad \big\|(u_n,v_n)\big\|\to+\infty.$$

Since $\|(\frac{1}{p}u_n, \frac{1}{q}v_n)\| \leq C\|(u_n, v_n)\|$, we obtain

$$\left(\varphi'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right) \to 0.$$

For $n$ sufficiently large, it turns out that

$$\begin{aligned}
c+1 &\geq \varphi(u_n, v_n) - \left(\varphi'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right) \\
&= \int_\Omega \left(\frac{1}{p}|\nabla u_n|^p + \frac{1}{q}|\nabla v_n|^q\right) dx - \int_\Omega F(x, u_n, v_n)\, dx \\
&\quad - \left\{\int_\Omega \frac{1}{p}|\nabla u_n|^p\, dx - \int_\Omega \frac{1}{p}F_u(x, u_n, v_n)u_n\, dx\right\} \\
&\quad - \left\{\int_\Omega \frac{1}{q}|\nabla v_n|^q\, dx - \int_\Omega \frac{1}{q}F_v(x, u_n, v_n)v_n\, dx\right\} \\
&\quad + \int_\Omega \lambda\left(\frac{\alpha(x)}{p} + \frac{\beta(x)}{q} - 1\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\, dx \\
&= \int_\Omega \left[\frac{1}{p}F_u(x, u_n, v_n)u_n + \frac{1}{q}F_v(x, u_n, v_n)v_n - F(x, u_n, v_n)\right] dx \\
&\quad + \int_\Omega \lambda\left(\frac{\alpha(x)}{p} + \frac{\beta(x)}{q} - 1\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\, dx.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{p}F_u(x, u_n, v_n)u_n + \frac{1}{q}F_v(x, u_n, v_n)v_n\, dx - F(x, u_n, v_n)\right] dx \\
&\quad \leq C_1 + \int_\Omega \lambda\left(1 - \frac{\alpha(x)}{p} - \frac{\beta(x)}{q}\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\, dx,
\end{aligned} \tag{5}$$

with a constant $C_1 > 0$.

From condition $(H_1)$ and (5) we get

$$\begin{aligned}
&\int_\Omega \left[\frac{|F_u(x, u_n, v_n)u_n|}{\ln(e + |u_n|)} + \frac{|F_v(x, u_n, v_n)v_n|}{\ln(e + |v_n|)}\right] dx \\
&\quad \leq \frac{1}{C_{2*}} \int_\Omega \left[\frac{F_u(x, u_n, v_n)u_n}{p} + \frac{F_v(x, u_n, v_n)v_n}{q} - F(x, u_n)\right] dx + C_2 \\
&\quad \leq \frac{1}{C_{2*}} \int_\Omega \lambda\left(1 - \frac{\alpha(x)}{p} - \frac{\beta(x)}{q}\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\, dx + C_3,
\end{aligned}$$

with constants $C_2, C_3 > 0$.

Since $\|\varphi'(u_n, v_n)\|_*(1 + \|(u_n, v_n)\|) \to 0$, by the preceding inequality and $(H_0)$, we have

$$\begin{aligned}
&\int_\Omega \left[|\nabla u_n|^p + |\nabla v_n|^q\right] dx \\
&\quad = \int_\Omega \left[F_u(x, u_n, v_n)u_n + F_v(x, u_n, v_n)v_n\right] dx \\
&\qquad + \int_\Omega \lambda\left(\alpha(x) + \beta(x)\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\, dx + o(1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega \left|F_u(x,u_n,v_n)u_n\right|^\varepsilon \left[\ln\left(e+|u_n|\right)\right]^{1-\varepsilon} \left|F_u(x,u_n,v_n)\frac{u_n}{\ln(e+|u_n|)}\right|^{1-\varepsilon} dx \\
&\quad + \int_\Omega \left|F_v(x,u_n,v_n)v_n\right|^\varepsilon \left[\ln\left(e+|v_n|\right)\right]^{1-\varepsilon} \left|F_v(x,u_n,v_n)\frac{v_n}{\ln(e+|v_n|)}\right|^{1-\varepsilon} dx + C_4 \\
&\quad + \int_\Omega \lambda\left(1+\alpha(x)+\beta(x)\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\,dx \\
&\leq C_5\left(1+\left\|(u_n,v_n)\right\|\right)^{\sqrt{\varepsilon}} \int_\Omega \left[\frac{(1+|u_n|^\gamma+|v_n|^\delta)^{1+\varepsilon}+C_6}{(1+\|(u_n,v_n)\|)^{\frac{\sqrt{\varepsilon}}{\varepsilon}}}\right]^\varepsilon \left[\frac{|F_u(x,u_n,v_n)u_n|}{\ln(e+|u_n|)}\right]^{1-\varepsilon} dx \\
&\quad + C_5\left(1+\left\|(u_n,v_n)\right\|\right)^{\sqrt{\varepsilon}} \int_\Omega \left[\frac{(1+|u_n|^\gamma+|v_n|^\delta)^{1+\varepsilon}+C_6}{(1+\|(u_n,v_n)\|)^{\frac{1}{\sqrt{\varepsilon}}}}\right]^\varepsilon \left[\frac{|F_v(x,u_n,v_n)v_n|}{\ln(e+|v_n|)}\right]^{1-\varepsilon} dx \\
&\quad + C_7 + \int_\Omega \lambda\left(1+\alpha(x)+\beta(x)\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\,dx \\
&\leq C_8\left(1+\left\|(u_n,v_n)\right\|\right)^{\sqrt{\varepsilon}} \int_\Omega \lambda\left(1+\alpha(x)+\beta(x)\right)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\,dx + C_9,
\end{aligned}
$$

with constants $C_i > 0$ for $4 \leq i \leq 9$ and $\varepsilon \in (0,1)$.

Due to the fact that $\frac{\alpha(\cdot)}{p} + \frac{\beta(\cdot)}{q} < 1$ on $\overline{\Omega}$, there exists a small enough $\varepsilon > 0$ such that $\frac{\alpha(\cdot)}{p} + \frac{\beta(\cdot)}{q} < 1 - 2\sqrt{\varepsilon}$ on $\overline{\Omega}$. Then, by Young's inequality, we get

$$
\begin{aligned}
&\int_\Omega \left[|\nabla u_n|^p + |\nabla v_n|^q\right] dx \\
&\quad \leq C_{10}\left(1+\left\|(u_n,v_n)\right\|\right)^{\sqrt{\varepsilon}} \left(1 + \int_\Omega |u_n|^{p(1-2\sqrt{\varepsilon})}\,dx + \int_\Omega |v_n|^{q(1-2\sqrt{\varepsilon})}\,dx\right) + C_9,
\end{aligned}
$$

with a constant $C_{10} > 0$.

When $\varepsilon > 0$ is sufficiently small, it is straightforward to reach a contradiction. Thus $\{(u_n,v_n)\}$ is bounded, which completes the proof. □

In order to prove Theorem 1.1, consider the truncation $F^{++}(x,u,v) = F(x,S(u),S(v))$, where $S(t) = \max\{0,t\}$. For any $(u,v) \in X$, we say $(u,v)$ belong to the first, the second, the third, and the fourth quadrant of $X$, if $u \geq 0$ and $v \geq 0$, $u \leq 0$ and $v \geq 0$, $u \leq 0$ and $v \leq 0$, $u \geq 0$ and $v \leq 0$, respectively.

*Proof of Theorem* 1.1 On the basis of hypothesis $(H_3)$, it is easy to check that $F^{++} \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$, and

$$
F_u^{++}(x,u,v) = F_u\left(x,S(u),S(v)\right), \qquad F_v^{++}(x,u,v) = F_v\left(x,S(u),S(v)\right).
$$

Let us consider the auxiliary problem

$$
\begin{cases}
-\Delta_p u = \lambda\alpha(x)|S(u)|^{\alpha(x)-2}S(u)|S(v)|^{\beta(x)} + F_u^{++}(x,u,v) & \text{in } \Omega, \\
-\Delta_q v = \lambda\beta(x)|S(u)|^{\alpha(x)}|S(v)|^{\beta(x)-2}S(v) + F_v^{++}(x,u,v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{$P^{++}$}
$$

The corresponding functional is expressed by

$$\varphi^{++}(u,v) = \Phi(u,v) - \Psi^{++}(u,v), \quad \forall (u,v) \in X,$$

where

$$\Psi^{++}(u,v) = \int_\Omega \bigl[\lambda \bigl|S(u)\bigr|^{\alpha(x)} \bigl|S(v)\bigr|^{\beta(x)} + F\bigl(x, S(u), S(v)\bigr)\bigr]\,dx, \quad \forall (u,v) \in X.$$

Let $\sigma > 0$ satisfy $\sigma \le \frac{1}{4}\min\{\lambda_p, \lambda_q\}$, where $\lambda_p$ and $\lambda_q$ are the first eigenvalues of $-\Delta_p$ and $-\Delta_q$, respectively. By assumptions $(H_0)$ and $(H_2)$, we have

$$F(x,u,v) \le \sigma\left(\frac{1}{p}|u|^p + \frac{1}{q}|v|^q\right) + C(\sigma)\bigl(|u|^\gamma + |v|^\delta\bigr), \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2,$$

with a constant $C(\sigma) > 0$ depending on $\sigma$. Notice that $\lambda_p, \lambda_q > 0$ (see Proposition 2.2) and

$$\begin{aligned} &\int_\Omega \frac{1}{p}|\nabla u|^p\,dx - \sigma\int_\Omega \frac{1}{p}|u|^p\,dx \ge \frac{3}{4}\int_\Omega \frac{1}{p}|\nabla u|^p, \\ &\int_\Omega \frac{1}{q}|\nabla v|^q\,dx - \sigma\int_\Omega \frac{1}{q}|v|^q\,dx \ge \frac{3}{4}\int_\Omega \frac{1}{q}|\nabla v|^q. \end{aligned}$$

Denote

$$\epsilon = \min\{\gamma - p, \delta - q\}.$$

When $\|u\|_p$ is small enough, by Proposition 2.1 we have

$$\begin{aligned} C(\sigma)\int_\Omega |u|^\gamma\,dx &= C(\sigma)|u|_\gamma^\gamma \\ &\le C_{11}\|u\|_p^\gamma \\ &\le C_{12}\|u\|_p^\epsilon \int_\Omega \frac{1}{p}|\nabla u|^p\,dx \\ &\le \frac{1}{4}\int_\Omega \frac{1}{p}|\nabla u|^p\,dx, \end{aligned}$$

with constants $C_{11}, C_{12} > 0$. Similarly, if $\|v\|_q$ is small enough, we obtain

$$C(\sigma)\int_\Omega |v|^\delta\,dx \le \frac{1}{4}\int_\Omega \frac{1}{q}|\nabla v|^q\,dx.$$

Then, when $\lambda > 0$ is sufficiently small, for any $(u,v) \in X$ with small enough norm, through Young's inequality, we find the estimate

$$\begin{aligned} \varphi^{++}(u,v) &= \Phi(u,v) - \Psi^{++}(u,v) \\ &\ge \frac{1}{2}\int_\Omega \frac{1}{p}|\nabla u|^p\,dx + \frac{1}{2}\int_\Omega \frac{1}{q}|\nabla v|^q\,dx - \int_\Omega \lambda|u|^{\alpha(x)}|v|^{\beta(x)}\,dx \\ &\ge \frac{1}{4}\int_\Omega \frac{1}{p}|\nabla u|^p\,dx + \frac{1}{4}\int_\Omega \frac{1}{q}|\nabla v|^q\,dx. \end{aligned}$$

We conclude that if $\lambda > 0$ is sufficiently small, there exist $r > 0$ and $\varepsilon > 0$ such that $\varphi^{++}(u,v) \geq \varepsilon$ for every $(u,v) \in X$ and $\|(u,v)\| = r$.

For a possibly smaller $\varepsilon > 0$, let $\Omega_0$ be an open ball of radius $\varepsilon$ contained in $\Omega$. Set

$$\alpha^+_{\overline{\Omega_0}} := \max_{x\in\overline{\Omega_0}} \alpha(x), \qquad \beta^+_{\overline{\Omega_0}} := \max_{x\in\overline{\Omega_0}} \beta(x).$$

By $(H_{\alpha,\beta})$ we may suppose that $\varepsilon > 0$ is small enough such that

$$\frac{\alpha^+_{\overline{\Omega_0}}}{p} + \frac{\beta^+_{\overline{\Omega_0}}}{q} < 1.$$

Fix $u_0, v_0 \in C^2_0(\overline{\Omega}_0)$ which are positive in $\Omega_0$. From hypothesis $(H_2)$ it follows that

$$\begin{aligned}
&\varphi^{++}\bigl(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0\bigr) \\
&\quad = \Phi\bigl(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0\bigr) - \Psi^{++}\bigl(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0\bigr) \\
&\quad \leq \Phi\bigl(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0\bigr) + \int_\Omega \bigl[\bigl|t^{\frac{1}{p}}u_0\bigr|^p + \bigl|t^{\frac{1}{q}}v_0\bigr|^q\bigr]\,dx \\
&\qquad - \lambda \int_\Omega \bigl|t^{\frac{1}{p}}u_0\bigr|^{\alpha(x)} \cdot \bigl|t^{\frac{1}{q}}v_0\bigr|^{\beta(x)}\,dx \\
&\quad \leq t\Phi(u_0,v_0) + t\int_\Omega \bigl[|u_0|^p + |v_0|^q\bigr]\,dx \\
&\qquad - \lambda t^{\frac{\alpha^+_{\overline{\Omega_0}}}{p} + \frac{\beta^+_{\overline{\Omega_0}}}{q}} \int_\Omega |u_0|^{\alpha(x)}|v_0|^{\beta(x)}\,dx < 0 \quad \text{as } t \to 0^+.
\end{aligned}$$

The discussion above enables us to see through local minimization that $\varphi^{++}(u,v)$ has at least one nontrivial critical point $(u_1^*, v_1^*)$ with $\varphi^{++}(u_1^*, v_1^*) < 0$. Furthermore, from assumption $(H_3)$ it is clear that $(u_1^*, v_1^*)$ is situated in the first quadrant of $X$.

Using $S(-u_1^*) \in W_0^{1,p(\cdot)}(\Omega)$ as a test function and invoking assumption $(H_3)$, we have

$$\begin{aligned}
&\int_\Omega \bigl|\nabla u_1^*\bigr|^{p-2}\nabla u_1^* \nabla S\bigl(-u_1^*\bigr)\,dx \\
&\quad = \int_\Omega \bigl[\lambda\alpha(x)\bigl|S\bigl(u_1^*\bigr)\bigr|^{\alpha(x)-2}S\bigl(u_1^*\bigr)\bigl|S\bigl(v_1^*\bigr)\bigr|^{\beta(x)} + F_u\bigl(x, S\bigl(u_1^*\bigr), S\bigl(v_1^*\bigr)\bigr)\bigr]S\bigl(-u_1^*\bigr)\,dx \\
&\quad = \int_\Omega F_u\bigl(x, S\bigl(u_1^*\bigr), S\bigl(v_1^*\bigr)\bigr)S\bigl(-u_1^*\bigr)\,dx = 0,
\end{aligned}$$

thus $u_1^* \geq 0$. Similarly, we can prove $v_1^* \geq 0$. Therefore, $(u_1^*, v_1^*)$ is a nontrivial constant-sign solution of $(P)$ with $\varphi(u_1^*, v_1^*) < 0$. From condition $(H_3)$ it follows that $u_1^*, v_1^*$ are both nontrivial. Along the same lines, we can show that $(P)$ possesses a nontrivial constant-sign solution $(u_i^*, v_i^*)$ in the $i$th quadrant of $X$ such that $\varphi(u_i^*, v_i^*) < 0$, $i = 2,3,4$. Hence system $(P)$ has at least four nontrivial constant-sign solutions. The proof of Theorem 1.1 is complete. □

*Proof of Theorem* 1.2 According to the proof of Theorem 1.1, if $\lambda > 0$ is small enough, there exist $r > 0$ and $\varepsilon > 0$ such that $\varphi^{++}(u,v) \geq \varepsilon$ for every $(u,v) \in X$ with $\|(u,v)\| = r$.

From $(H_1)$ and (1) we infer that

$$F(x,u,v) \geq C_2|u|^{\theta_1(x)}|v|^{\theta_2(x)} \min\{\ln(1+|u|), \ln(1+|v|)\} - C_3, \tag{6}$$

with positive constants $C_2$, $C_3$, for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

Fix $x_* \in \Omega$ and $\varepsilon > 0$ such that $\overline{B(x_*,\varepsilon)} \subset \Omega$. Define $h_* \in C_0(\overline{B(x_*,\varepsilon)})$ by

$$h_*(x) = \begin{cases} 0, & |x-x_*| \geq \varepsilon, \\ \varepsilon - |x-x_*|, & |x-x_*| < \varepsilon. \end{cases}$$

From Lemma 2.4 we know that $\varphi^{++}(t^{\frac{1}{p}}h_*, t^{\frac{1}{q}}h_*) \to -\infty$ as $t \to +\infty$. Since $\varphi^{++}(0,0) = 0$, $\varphi^{++}$ satisfies the geometry conditions of the mountain pass theorem. Similar to the proof of Lemma 3.3, it follows that $\varphi^{++}$ fulfils the Cerami condition. So $\varphi^{++}$ has at least one nontrivial critical point $(u_1,v_1)$ with $\varphi^{++}(u_1,v_1) > 0$, and by assumption $(H_3)$ the components $u_1, v_1$ are both nontrivial. As in the proof of Theorem 1.1, it is easy to see that $(u_1,v_1)$ is in the first quadrant of $X$. Thus, $(u_1,v_1)$ is a nontrivial constant-sign solution of problem $(P)$ in the first quadrant of $X$ with $\varphi(u_1,v_1) > 0$. Recall from Theorem 1.1 that $(P)$ has also a nontrivial constant-sign solution $(u_1^*,v_1^*)$ in the first quadrant of $X$ verifying $\varphi(u_1^*,v_1^*) < 0$.

As before we can see that $(P)$ admits constant-sign solutions $(u_i,v_i)$ and $(u_i^*,v_i^*)$ in the $i$th quadrant in $X$ $(i = 1,2,3,4)$ satisfying $\varphi(u_i,v_i) > 0$ and $\varphi(u_i^*,v_i^*) < 0$, and $u_i, v_i, u_i^*, v_i^*$ are all nontrivial, which completes the proof of Theorem 1.2. □

In order to prove Theorem 1.3, we need to do some preparation. Note that $X$ is a reflexive and separable Banach space (see [22, Sect. 17], [18]), so there are sequences $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}}\{e_j, j = 1,2,\ldots\}, \qquad X^* = \overline{\operatorname{span}}^{W^*}\{e_j^*, j = 1,2,\ldots\},$$

and

$$\langle e_j^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we set $X_j = \operatorname{span}\{e_j\}$, $Y_k = \overline{\bigoplus_{j=1}^{k} X_j}$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$.

**Lemma 3.4** *For $\gamma, \delta \geq 1$ with $\gamma < p^*$ and $\delta < q^*$, denote*

$$\beta_k = \sup\{|u|_\gamma + |v|_\delta \mid \|(u,v)\| = 1, (u,v) \in Z_k\}.$$

*Then there holds* $\lim_{k\to\infty} \beta_k = 0$.

*Proof* Obviously, $0 < \beta_{k+1} \leq \beta_k$, so we have $\beta_k \to \beta \geq 0$ as $k \to \infty$. Let $(u_k,v_k) \in Z_k$ satisfy

$$\|(u_k,v_k)\| = 1, \qquad 0 \leq \beta_k - |u_k|_\gamma - |v_k|_\delta < \frac{1}{k}.$$

Then there exists a subsequence of $\{(u_k, v_k)\}$ (still denoted by $(u_k, v_k)$) such that $(u_k, v_k) \rightharpoonup (u, v)$, and

$$\langle e_j^*, (u,v)\rangle = \lim_{k\to\infty}\langle e_j^*, (u_k, v_k)\rangle = 0, \quad \forall j.$$

This implies $(u,v) = (0,0)$, thus $(u_k, v_k) \rightharpoonup (0,0)$. Since the embedding of $W_0^{1,p}(\Omega)$ into $L^\gamma(\Omega)$ is compact, we get $u_k \to 0$ in $L^\gamma(\Omega)$. Similarly, we have $v_k \to 0$ in $L^\delta(\Omega)$. Hence we derive $\beta_k \to 0$ as $k \to \infty$. The proof is complete. □

In order to prove Theorem 1.3, we need the following lemma whose proof can be found in [23, Theorem 4.7]. If the Cerami condition is replaced by the Palais–Smale condition, it is proven in [24, Theorem 3.6].

**Lemma 3.5** *Suppose $\varphi \in C^1(X, \mathbb{R})$ is even and satisfies the Cerami condition. Let $V^+$ and $V^- \subset X$ be closed subspaces of $X$ with* $\operatorname{codim} V^+ + 1 = \dim V^-$, *and suppose there hold*
- $(1^0)$ $\varphi(0,0) = 0$;
- $(2^0)$ $\exists \tau > 0$ *and* $R > 0$ *such that* $\forall (u,v) \in V^+$: $\|(u,v)\| = R \Rightarrow \varphi(u,v) \geq \tau$;
- $(3^0)$ $\exists \rho > 0$ *such that* $\forall (u,v) \in V^-$: $\|(u,v)\| \geq \rho \Rightarrow \varphi(u,v) \leq 0$.

*Denoting*

$$\Gamma = \bigl\{g \in C^0(X,X) \mid g \textit{ is odd}, g(u,v) = (u,v) \textit{ if } (u,v) \in V^- \textit{ and } \bigl\|(u,v)\bigr\| \geq \rho\bigr\},$$

*it holds*
- (a) $\forall \delta > 0$, $\forall g \in \Gamma$, *one has* $S_\delta^+ \cap g(V^-) \neq \varnothing$, *where*

$$S_\delta^+ = \bigl\{(u,v) \in V^+ \mid \bigl\|(u,v)\bigr\| = \delta\bigr\};$$

- (b) *the number*

$$\varpi := \inf_{g\in\Gamma} \sup_{(u,v)\in V^-} \varphi\bigl(g(u,v)\bigr) \geq \tau$$

*is a critical value of $\varphi$.*

*Proof of Theorem* 1.3 According to assumptions $(H_{\alpha,\beta})$, $(H_0)$, $(H_1)$, and $(H_4)$, the functional $\varphi$ is even and satisfies the Cerami condition (see Lemma 3.3). Setting $V_k^+ = Z_k$, then $V_k^+$ is a closed linear subspace of $X$ and $V_k^+ \oplus Y_{k-1} = X$.

Take mutually distinct points $x_n \in \Omega$ and define $h_n \in C_0(\overline{B(x_n, \varepsilon_n)})$ by

$$h_n(x) = \begin{cases} 0, & |x - x_n| \geq \varepsilon_n, \\ \varepsilon_n - |x - x_n|, & |x - x_n| < \varepsilon_n, \end{cases}$$

for $\varepsilon_n > 0$ with $\overline{B(x_n, \varepsilon_n)} \subset \Omega$.

From $(H_1)$ and (1), we obtain the estimate

$$C_0|u|^{\theta_1(x)}|v|^{\theta_2(x)} \min\bigl\{\ln\bigl(1 + |u|\bigr), \ln\bigl(1 + |v|\bigr)\bigr\} \leq F(x,u,v), \quad \forall |u| + |v| \geq M, \forall x \in \Omega,$$

with a constant $C_0 > 0$. Consequently, the requirement (3) in Lemma 2.4 is fulfilled.

Note that $(H_{p,q})$ yields $p = q$, and from Lemma 2.4 we have $\varphi(th_n, th_n) \to -\infty$ as $t \to +\infty$. Without loss of generality, we may assume that

$$\operatorname{supp} h_i \cap \operatorname{supp} h_j = \varnothing, \quad \forall i \neq j. \tag{7}$$

Setting $V_k^- = \operatorname{span}\{(h_1, h_1), \dots, (h_k, h_k)\}$, we will prove for every pair of spaces $V_k^+$ and $V_k^-$ that the functional $\varphi$ satisfies the conditions of Lemma 3.5 and has a sequence of critical values

$$\varpi_k := \inf_{g\in\Gamma} \sup_{(u,v)\in V_k^-} \varphi\bigl(g(u,v)\bigr) \to +\infty \quad \text{as } k \to +\infty.$$

This results in the fact that there are infinitely many pairs of symmetric solutions to problem $(P)$.

For any $k = 1, 2, \dots$, we check that there exist $\rho_k > R_k > 0$ such that

$(A_1)$ $b_k := \inf\{\varphi(u,v) \mid (u,v) \in V_k^+, \|(u,v)\| = R_k\} \to +\infty$ $(k \to +\infty)$;

$(A_2)$ $a_k := \max\{\varphi(u,v) \mid (u,v) \in V_k^-, \|(u,v)\| = \rho_k\} \leq 0$.

First, we prove assertion $(A_1)$.

By direct computation based on (2) and the expression of $\beta_k$ in Lemma 3.4, we find that

$$\begin{aligned}
\varphi(u,v) &= \int_\Omega \frac{1}{p}|\nabla u|^p\,dx + \int_\Omega \frac{1}{q}|\nabla v|^q\,dx \\
&\quad - \int_\Omega \lambda |u|^{\alpha(x)}|v|^{\beta(x)}\,dx - \int_\Omega F(x,u,v)\,dx \\
&\geq \frac{1}{p}\|u\|_p^p - C_{13}|u|_\gamma^\gamma + \frac{1}{q}\|v\|_q^q - C_{13}|v|_\delta^\delta - C_{14} \\
&\geq \frac{1}{p}\|u\|_p^p - C_{13}\beta_k^\gamma\|u\|_p^\gamma + \frac{1}{q}\|v\|_q^q - C_{13}\beta_k^\delta\|v\|_q^\delta - C_{14} \\
&\geq \frac{1}{pq}\bigl\|(u,v)\bigr\|^{\min\{p,q\}} - C_{15}\beta_k\bigl\|(u,v)\bigr\|^{\max\{\gamma,\delta\}} - C_{16} \\
&= \frac{1}{2pq}(2pqC_{15}\beta_k)^{\min\{p,q\}/(\min\{p,q\}-\max\{\gamma,\delta\})} - C_{16},
\end{aligned}$$

with positive constants $C_{13}, \dots, C_{16}$ for all $(u,v) \in Z_k$ with

$$\bigl\|(u,v)\bigr\| = R_k = (2pqC_{15}\beta_k)^{1/(\min\{p,q\}-\max\{\gamma,\delta\})}.$$

Therefore $\varphi(u,v) \geq \frac{1}{2pq}R_k^{\min\{p,q\}} - C_{16}$, $\forall (u,v) \in Z_k$ with $\|(u,v)\| = R_k$, which yields $b_k \to +\infty$ as $k \to \infty$.

Next we prove assertion $(A_2)$.

Recall that assumption $(H_{p,q})$ ensures $p = q$. From (7) and the definition of $h_n$, it is easy to see that

$$\varphi(h_i + h_j, h_i + h_j) = \varphi(h_i, h_i) + \varphi(h_j, h_j), \quad \forall i \neq j.$$

Since (6) and Lemma 2.4 guarantee

$$\varphi(th_i, th_i) \to -\infty \quad \text{as } t \to +\infty, \forall i = 1, 2, \dots,$$

we have

$$\varphi(th,th) \to -\infty \quad \text{as } t \to +\infty$$

for all $(h,h) \in V_k^-$ with $\|(h,h)\| = 1$. Then one can provide $\rho_k$ from which assertion $(A_2)$ follows.

Now it is sufficient to combine $(A_1)$ and $(A_2)$ for completing the proof of Theorem 1.3. □

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