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Global existence of weak solution and regularity criteria for the 2D Bénard system with partial dissipation

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Abstract

In this paper, we first write the velocity equation of the Bénard system in its two components, and consider the global weak solution of the resulting 2D Bénard system with partial dissipation, i.e. (1) $\mu_1 = 0, \mu_2 > 0, \mu_3 = 0, \mu_4 = 0, \kappa_1 > 0, \kappa_2 = 0$; (2) $\mu_1 = 0, \mu_2 = 0, \mu_3 > 0, \mu_4 = 0, \kappa_1 = 0, \kappa_2 > 0$; (3) $\mu_1 > 0, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0, \kappa_1 > 0, \kappa_2 = 0$; (4) $\mu_1 > 0, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0, \kappa_1 = 0, \kappa_2 > 0$; (5) $\mu_1 = 0, \mu_2 > 0, \mu_3 = 0, \mu_4 = 0, \kappa_1 = 0, \kappa_2 > 0$; (6) $\mu_1 = 0, \mu_2 = 0, \mu_3 > 0, \mu_4 = 0, \kappa_1 > 0, \kappa_2 = 0$; (7) $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \mu_4 > 0, \kappa_1 > 0, \kappa_2 = 0$; (8) $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \mu_4 > 0, \kappa_1 = 0, \kappa_2 > 0$, where μ_i ($i = 1,2,3,4$) and κ_j ($j = 1,2$) are the coefficients of dissipation and thermal diffusivity. Furthermore, we establish some regularity criteria for the corresponding system. This work follows the techniques in the paper of Cao and Wu (Adv. Math. 226:1803–1822, 2011).

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Keywords: Bénard system; Weak solution; Regularity criteria; Partial dissipation; Thermal diffusivity

1 Introduction

The Bénard system described the Rayleigh–Bénard convective motion in a heated 2D inviscid incompressible fluid under thermal effects (see e.g. [2–7]). One of the most fundamental problems in fluid dynamics concerning the Bénard system is whether their classical solutions are global regularity for all time or they develop singularities. The Bénard system has been a center of attention to numerous analytical, experimental, and number investigations. The motion of the incompressible Bénard system in $\mathbb{R}^2$ is governed by

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi = \mu\Delta u + \theta e_2, \\ \partial_t\theta + (u \cdot \nabla)\theta = \kappa\Delta\theta + u \cdot e_2, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where the unknown functions u, θ and π denote the 2D velocity field, temperature and pressure, respectively. The constants μ and κ are the coefficients of dissipation and thermal diffusivity. The forcing term θe_2 in the momentum equation describes the acting of the buoyancy force on fluid motion and $u \cdot e_2$ models the Rayleigh–Bénard convection in a heated inviscid fluid, and $e_2 = (0,1)^T$. Physically, $(1.1)_1$ reflects the conservation of momentum, $(1.1)_2$ describes the thermal convection, and in $(1.1)_3$, $\nabla \cdot u = 0$ shows the conservation of mass.

In the fluid dynamics area, the Bénard fluid problem is a very classical problem, which has an important significance in convective motion in a heated fluid, such as the case of planar stationary flows. The incompressible Bénard system have attracted the attention of many physicists and mathematicians due to its important physical background, rich phenomena, mathematical complexity and challenges. Neustupa and Siginer [8] proved the existence of a strong-weak solution (u,π,θ) of the steady Bénard problem in a 2D quadrangular cavity, heated/cooled on two opposite sides and thermally insulated on the other sides. The nonlinear Lyapunov stability of the conduction-diffusion solution of the rotating Bénard problem was studied in [9]. Anh and Son [10] studied the 2D Bénard problem in an arbitrary domain (bounded or unbounded) which satisfying the Poincaré inequality with nonhomogeneous boundary conditions and nonautonomous external force and heat source, and the existence of a weak solution to the problem was proved by using the Galerkin method, and showed that the existence of a unique minimal finite-dimensional pullback D_σ-attractor for the process associated to the problem. Wu and Xue [11] considered the Cauchy problem of the 2D inviscid Bénard system with fractional diffusivity, and showed that the system had a unique global solution (u,θ) such that $u \in C^{0,1}(\mathbf{R}_+, L^2(\mathbb{R}^2))$, $\theta \in C(\mathbb{R}_+, L^2 \cap B^{1-\beta}_{\infty,1}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{\frac{\beta}{2}})$. 2D incompressible Bénard system with critical and supercritical dissipation $(0 \le \alpha \le 1)$ in the velocity was studied in [12]. Cheng and Du [13] considered the Cauchy problem of the 2D magnetic Bénard problem with mixed partial viscosity. More precisely, the global well-posedness of the 2D magnetic Bénard problem without thermal diffusivity and with vertical or horizontal magnetic diffusion was obtained. Moreover, the global regularity and some conditional regularity of strong solutions were obtained for the 2D magnetic Bénard problem with mixed partial viscosity. Zhou–Nakamura [14] studied a 2D magnetic Bénard problem with zero thermal conductivity, and showed a global well-posedness result by a well-known property of Hardy space and BMO. As it is demonstrated in reference [15, 16], we showed the global regularity for the two-and-half-dimensional magnetic Bénard system with zero thermal diffusivity by a well-known property of Hardy space and BMO; resorting to the method of the local-in-time analysis, the global regularity for the two-and-half-dimensional magnetic Bénard system with zero thermal diffusivity and horizontal magnetic diffusion as well as vertical magnetic diffusion are also obtained. Moreover, we proved that, as the initial data satisfy $\|u_0\|^2_{H^1_{\mathbb{R}^3}} + \|b_0\|^2_{H^1_{\mathbb{R}^3}} + \|\theta_0\|^2_{H^1_{\mathbb{R}^3}} \le \varepsilon$, where ε is a suitably small positive number, then the 3D magnetic Bénard system with mixed partial dissipation, magnetic diffusion and thermal diffusivity admits global smooth solutions. Zhang and Tang [17] studied the global regularity for a special family of axisymmetric solutions to the 3D magnetic Bénard problem.

In particular, if $\theta = \mathit{Const.}$, then system (1.1) reduces to the classical Navier–Stokes system which describes the motion of incompressible viscous fluid flows and has been extensively studied by many authors; see [18–24] and the references therein. In addition to this, the reader is referred to [25–29] to find more results about the related fluid flow equations.

Recently, Regmi [30] established global weak solution for the 2D MHD system with partial dissipation and vertical diffusion. Cheng and Li [31] established the global weak solutions for the 2D Boussinesq system with mixed partial dissipation and thermal diffusivity. Chen concerned with the 2D system of the incompressible micropolar fluid flows with mixed partial viscosity and angular viscosity, and the global existence and uniqueness of smooth solution was showed in [32]. Yu considered the global regularity to the initial-boundary value problem of the 2D incompressible MHD system with mixed partial dissipation and magnetic diffusion in [33]. Fan et al. considered the global regularity for the 2D liquid crystal model with mixed partial viscosity and global Cauchy problem of 2D generalized MHD equations, respectively, in [34, 35].

Inspired by this work, we consider the following Bénard system in this paper:

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = \mu_1\partial_{xx}u_1 + \mu_2\partial_{yy}u_1, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = \mu_3\partial_{xx}u_2 + \mu_4\partial_{yy}u_2 + \theta, \\ \partial_t\theta + (u\cdot\nabla)\theta = \kappa_1\partial_{xx}\theta + \kappa_2\partial_{yy}\theta + u_2, \\ \partial_x u_1 + \partial_x u_2 = 0, \end{cases} \tag{1.2}$$

where $u = (u_1, u_2)$. System (1.2) is capable of modeling the motion of anisotropic fluids for which the diffusion properties in different directions are different. Additionally, (1.2) allows us to explore the smooth effects of various partial dissipations. Furthermore, we consider the 2D Bénard system (1.2) with partial dissipation in the following eight cases:

(i) $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$;
(ii) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 > 0$, $\mu_4 = 0$, $\kappa_1 = 0$, $\kappa_2 > 0$;
(iii) $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$;
(iv) $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 = 0$, $\kappa_2 > 0$;
(v) $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 = 0$, $\kappa_2 > 0$;
(vi) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 > 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$;
(vii) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 > 0$, $\kappa_1 > 0$, $\kappa_2 = 0$;
(viii) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 > 0$, $\kappa_1 = 0$, $\kappa_2 > 0$.

In this paper, we equip system (1.2) with the following initial data:

$$\begin{aligned} &u_1(x,y,0) = u_1^0(x,y), \qquad u_2(x,y,0) = u_2^0(x,y), \\ &\theta(x,y,0) = \theta^0(x,y), \quad (x,y)\in\mathbb{R}^2. \end{aligned} \tag{1.3}$$

The plan of this paper is as follows. Firstly, we give two very useful lemmas and establish the global weak solution for the 2D Bénard system with vertical dissipation in the first component of velocity field and horizontal thermal diffusivity in Sect. 2. We shall give the global regularity criteria for weak solution of the 2D Bénard system with vertical dissipation in the first component of velocity field and horizontal thermal diffusivity in Sect. 3. In Sect. 4, we will give the global existence and regularity criteria of weak solution to Bénard system with other cases for partial viscosity and thermal diffusivity.

Notations. We introduce some notations which are used in this paper. For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^2)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. The usual Sobolev space of order n is defined by $H^n = \{f \in L^2(\mathbb{R}^2) | \nabla^n f \in L^2\}$ with the norm $\|f\|_{H^n} = (\|f\|_{L^2}^2 + \|\nabla^n f\|_{L^2}^2)^{\frac{1}{2}}$.

2 Global weak solution for the Bénard system with vertical dissipation and horizontal thermal diffusivity

In this section, we will establish the global weak solution for the 2D Bénard system with vertical dissipation in the first component of velocity field and horizontal thermal diffusivity.

2.1 Preliminaries

In this subsection, we first provide the lemma that bounds a triple-product in terms of the Lebesgue norms of the functions and their directional derivatives; see for example [1, 36, 37]. The following anisotropic Sobolev inequality will play very important roles in proving our main results.

Lemma 2.1 *Assume that* $f, g, h, \partial_x g, \partial_y h \in L^2(\mathbb{R}^2)$. *Then there exists an absolute constant* C *such that*

$$\iint_{\mathbb{R}^2} |fgh| \, dx \, dy \, dz \le C\|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_x g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_y h\|_{L^2}^{\frac{1}{2}}. \tag{2.1}$$

The following simple fact on the boundedness of Riesz transforms will also be used.

Lemma 2.2 (see [1]) *Let* f *be a divergence-free vector field such that* $\nabla f \in L^\gamma$ *for* $\gamma \in (1, \infty)$. *Then there exists a pure constant* $C > 0$ (*independent of* γ) *such that*

$$\|\nabla f\|_{L^\gamma} \le \frac{C\gamma^2}{\gamma - 1} \|\nabla \times f\|_{L^\gamma}. \tag{2.2}$$

For simplicity, throughout this paper, we use the same letter C to denote various generic positive constants whose exact values are unimportant and may vary from line to line.

2.2 Global weak solution

Theorem 2.3 *Let* $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$. *Suppose that* $u_1^0, u_2^0, \theta^0 \in H^1(\mathbb{R}^2)$ *and* $\partial_x u_1^0 + \partial_y u_2^0 = 0$. *Then the problem* (1.2)–(1.3) *admits a global weak solution* (u_1, u_2, θ), *which obeys*

$$u_1, u_2, \theta \in L^\infty\big([0, T); H^1\big(\mathbb{R}^2\big)\big), \qquad \partial_y u_1, \partial_x \theta \in L^2\big([0, T); H^1\big(\mathbb{R}^2\big)\big)$$

for any $T > 0$.

Theorem 2.3 follows from the following two lemmas immediately.

Lemma 2.4 *Consider* (1.2) *with* $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$. *Let* $u_1^0, u_2^0, \theta^0 \in L^2(\mathbb{R}^2)$, *then, for any* $T > 0$ *and* $0 < t < T$, *we have*

$$\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \mu_2 \int_0^t \big\|\partial_y u_1(\tau)\big\|_{L^2}^2 \, d\tau + \kappa_1 \int_0^t \big\|\partial_x \theta(\tau)\big\|_{L^2}^2 \, d\tau \le C. \tag{2.3}$$

Proof Taking the L^2-inner product of the first three equations in (1.2) with u_1, u_2 and θ, respectively, integrating the resulting equations by parts over $\mathbb{R}^2$, and using the divergence-free condition $\partial_x u_1 + \partial_x u_2 = 0$, we find after adding them together that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \big(\big\|u_1(t)\big\|_{L^2}^2 + \big\|u_2(t)\big\|_{L^2}^2 + \big\|\theta(t)\big\|_{L^2}^2 \big) + \mu_2 \|\partial_y u_1\|_{L^2}^2 + \kappa_1 \|\partial_x \theta\|_{L^2}^2 \\ &\quad = 2 \iint_{\mathbb{R}^2} u_2 \theta \, dx \, dy \\ &\quad \le C\big(\|u_2\|_{L^2}^2 + \|\theta\|_{L^2}^2\big). \end{aligned} \tag{2.4}$$

This, together with the Gronwall's inequality, gives the desired estimates of (2.3), which implies

$$\|u_1, u_2, \theta\|_{L^\infty([0,T);L^2(\mathbb{R}^2))} + \mu_2\|\partial_y u_1\|_{L^2([0,T);L^2(\mathbb{R}^2))} + \kappa_1\|\partial_x\theta\|_{L^2([0,T);L^2(\mathbb{R}^2))} \le C. \tag{2.5}$$

We thus complete the proof of Lemma 2.4. □

To obtain the H^1-estimates for u_1, u_2 and θ, we consider the following equations:

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \mu_2\partial_{yyy}u_1 + \partial_x\theta, \\ \partial_t\partial_x\theta + \partial_x[(u\cdot\nabla)\theta] = \kappa_1\partial_{xxx}\theta + \partial_x u_2, \\ \partial_t\partial_y\theta + \partial_y[(u\cdot\nabla)\theta] = \kappa_1\partial_{xxy}\theta + \partial_y u_2, \end{cases} \tag{2.6}$$

where the vorticity is $\omega = \partial_x u_2 - \partial_y u_1$.

Lemma 2.5 *Consider* (1.2) *with* $\mu_1 = 0, \mu_2 > 0, \mu_3 = 0, \mu_4 = 0, \kappa_1 > 0, \kappa_2 = 0$. *Let* $u_1^0, u_2^0, \theta^0 \in H^1(\mathbb{R}^2)$, *then, for any* $T > 0$ *and* $0 < t < T$, *we have*

$$\begin{aligned} &\|\omega\|_{L^2}^2 + \|\partial_x\theta\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2 + \mu_2\int_0^t \big\|\partial_{xy}u_1(\tau)\big\|_{L^2}^2\,d\tau + \mu_2\int_0^t \big\|\partial_{yy}u_1(\tau)\big\|_{L^2}^2\,d\tau \\ &\quad + \kappa_1\int_0^t \big\|\partial_{xx}\theta(\tau)\big\|_{L^2}^2\,d\tau + \kappa_1\int_0^t \big\|\partial_{xy}\theta(\tau)\big\|_{L^2}^2\,d\tau \le C. \end{aligned} \tag{2.7}$$

Proof Multiplying the first equation of (2.6) by ω and integrating it over $\mathbb{R}^2$, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\|\omega(t)\big\|_{L^2}^2 + \mu_2\|\partial_{xy}u_1\|_{L^2}^2 + \mu_2\|\partial_{yy}u_1\|_{L^2}^2 \\ &\quad = \iint_{\mathbb{R}^2} \partial_x\theta\,\omega\,dx\,dy \\ &\quad \le C\big(\|\partial_x\theta\|_{L^2}^2 + \|\omega\|_{L^2}^2\big), \end{aligned} \tag{2.8}$$

and hence, using (2.3), (2.5) and Gronwall's inequality, we know that

$$\|\omega\|_{L^2}^2 + \mu_2\int_0^t \big\|\partial_{xy}u_1(\tau)\big\|_{L^2}^2\,d\tau + \mu_2\int_0^t \big\|\partial_{yy}u_1(\tau)\big\|_{L^2}^2\,d\tau \le C, \tag{2.9}$$

which gives

$$\|\omega\|_{L^\infty([0,T);L^2(\mathbb{R}^2))} + \mu_2\|\nabla\partial_y u_1\|_{L^2([0,T);L^2(\mathbb{R}^2))} \le C. \tag{2.10}$$

On the other hand, multiplying $(2.6)_2$ and $(2.6)_3$, respectively, by $\partial_x\theta$ and $\partial_y\theta$, and integrating by parts, yields

$$\frac{1}{2}\frac{d}{dt}\big(\big\|\partial_x\theta(t)\big\|_{L^2}^2 + \big\|\partial_y\theta(t)\big\|_{L^2}^2\big) + \kappa_1\|\partial_{xx}\theta\|_{L^2}^2 + \kappa_1\|\partial_{xy}\theta\|_{L^2}^2 = I = \sum_{l=1}^4 I_l, \tag{2.11}$$

where

$$I_1 = -\iint_{\mathbb{R}^2} \partial_x\big[(u\cdot\nabla)\theta\big]\partial_x\theta\,dx\,dy, \qquad I_2 = \iint_{\mathbb{R}^2} \partial_x u_2 \partial_x\theta\,dx\,dy,$$
$$I_3 = -\iint_{\mathbb{R}^2} \partial_y\big[(u\cdot\nabla)\theta\big]\partial_y\theta\,dx\,dy, \qquad I_4 = \iint_{\mathbb{R}^2} \partial_y u_2 \partial_y\theta\,dx\,dy.$$

We now estimate the right-hand side of (2.11) term by term. We first write I_1 as

$$\begin{aligned} I_1 &= -\iint_{\mathbb{R}^2} \partial_x\big[(u\cdot\nabla)\theta\big]\partial_x\theta\,dx\,dy \\ &= -\iint_{\mathbb{R}^2} \partial_x u_1 \partial_x\theta\partial_x\theta\,dx\,dy - \iint_{\mathbb{R}^2} \partial_x u_2 \partial_y\theta\partial_x\theta\,dx\,dy \\ &= I_{11} + I_{12}. \end{aligned} \tag{2.12}$$

Here, we have used the fact $\iint_{\mathbb{R}^2} u_1 \partial_{xx}\theta\partial_x\theta\,dx\,dy + \iint_{\mathbb{R}^2} u_2\partial_{xy}\theta\partial_x\theta\,dx\,dy = 0$, which can easily be obtained by integration by parts and the incompressible condition.

With the help of Lemma 2.1, the Cauchy–Schwarz inequality and Young's inequality, we infer that

$$\begin{aligned} I_{11} &= -\iint_{\mathbb{R}^2} \partial_x u_1 \partial_x\theta\partial_x\theta\,dx\,dy \\ &\le C\|\partial_x u_1\|_{L^2}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xx}\theta\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12} \\ &\le \frac{\kappa_1}{2}\|\partial_{xx}\theta\|_{L^2}^2 + \frac{\kappa_1}{8}\|\partial_{xy}\theta\|_{L^2}^2 + C\|\omega\|_{L^2}^2\|\partial_x\theta\|_{L^2}^2. \end{aligned} \tag{2.13}$$

Similarly, we obtain

$$\begin{aligned} I_{12} &= -\iint_{\mathbb{R}^2} \partial_x u_2 \partial_y\theta\partial_x\theta\,dx\,dy \\ &\le C\|\partial_x u_2\|_{L^2}\|\partial_y\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12} \\ &\le \frac{\kappa_1}{8}\|\partial_{xy}\theta\|_{L^2}^2 + C\|\omega\|_{L^2}^2\big(\|\partial_x\theta\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2\big). \end{aligned} \tag{2.14}$$

I_3 can be bounded by

$$\begin{aligned} I_3 &= -\iint_{\mathbb{R}^2} \partial_y\big[(u\cdot\nabla)\theta\big]\partial_y\theta\,dx\,dy \\ &= -\iint_{\mathbb{R}^2} \partial_y(u_1\partial_x\theta + u_2\partial_y\theta)\partial_y\theta\,dx\,dy \\ &= -\iint_{\mathbb{R}^2} \partial_y u_1\partial_x\theta\partial_y\theta\,dx\,dy - \iint_{\mathbb{R}^2} u_1\partial_{xy}\theta\partial_y\theta\,dx\,dy \\ &\quad - \iint_{\mathbb{R}^2} \partial_y u_2\partial_y\theta\partial_y\theta\,dx\,dy - \iint_{\mathbb{R}^2} u_2\partial_{yy}\theta\partial_y\theta\,dx\,dy \\ &= -\iint_{\mathbb{R}^2} \partial_y u_1\partial_x\theta\partial_y\theta\,dx\,dy - \iint_{\mathbb{R}^2} \partial_y u_2\partial_y\theta\partial_y\theta\,dx\,dy \\ &= -\iint_{\mathbb{R}^2} \partial_y u_1\partial_x\theta\partial_y\theta\,dx\,dy - 2\iint_{\mathbb{R}^2} u_1\partial_y\theta\partial_{xy}\theta\,dx\,dy \end{aligned}$$

$$
\begin{aligned}
&\le C\|\partial_y u_1\|_{L^2}\|\partial_x\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_{xy}\theta\|_{L^2}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\kappa_1}{4}\|\partial_{xy}\theta\|_{L^2}^2 + C\big(\|\partial_y u_1\|_{L^2}^2 + \|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2\big)\big(\|\partial_x\theta\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2\big). \qquad (2.15)
\end{aligned}
$$

In what follows, we estimate I_2 and I_4. Using the Cauchy inequality, we have

$$
I_2 = \iint_{\mathbb{R}^2} \partial_x u_2 \partial_x\theta \, dx\, dy \le C\big(\|\omega\|_{L^2}^2 + \|\partial_x\theta\|_{L^2}^2\big), \qquad (2.16)
$$

and

$$
I_4 = \iint_{\mathbb{R}^2} \partial_y u_2 \partial_y\theta \, dx\, dy \le C\big(\|\omega\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2\big). \qquad (2.17)
$$

Combining the above estimates, we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\big(\big\|\partial_x\theta(t)\big\|_{L^2}^2 + \big\|\partial_y\theta(t)\big\|_{L^2}^2\big) + \kappa_1\|\partial_{xx}\theta\|_{L^2}^2 + \kappa_1\|\partial_{xy}\theta\|_{L^2}^2 \\
&\quad \le C\big(\|\partial_y u_1\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2 + 1\big)\big(\|\partial_x\theta\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2\big) + C\|\omega\|_{L^2}^2. \quad (2.18)
\end{aligned}
$$

Then (2.18), together with Gronwall's inequality, immediately yields

$$
\|\nabla\theta\|_{L^\infty([0,T;L^2(\mathbb{R}^2))} + \kappa_1\|\nabla\partial_x\theta\|_{L^2([0,T);L^2(\mathbb{R}^2))} \le C. \qquad (2.19)
$$

Hence, we finish the proof of Lemma 2.5. □

We next prove Theorem 2.3 by using the method of vanishing viscosity. To this end, we consider the following regularized problem:

$$
\begin{cases}
\partial_t u_1^\varepsilon + (u^\varepsilon\cdot\nabla)u_1^\varepsilon + \partial_x\pi^\varepsilon = \varepsilon\partial_{xx}u_1^\varepsilon + \mu_2\partial_{yy}u_1^\varepsilon, \\
\partial_t u_2^\varepsilon + (u^\varepsilon\cdot\nabla)u_2^\varepsilon + \partial_y\pi^\varepsilon = \varepsilon\partial_{xx}u_2^\varepsilon + \varepsilon\partial_{yy}u_2^\varepsilon + \theta^\varepsilon, \\
\partial_t\theta^\varepsilon + (u^\varepsilon\cdot\nabla)\theta^\varepsilon = \kappa_1\partial_{xx}\theta^\varepsilon + \varepsilon\partial_{yy}\theta^\varepsilon + u_2^\varepsilon, \\
\nabla\cdot u^\varepsilon = 0,
\end{cases} \qquad (2.20)
$$

with the smooth initial data

$$
u_1^\varepsilon(x,y,0) = \varphi_\varepsilon * u_1^0, \qquad u_2^\varepsilon(x,y,0) = \varphi_\varepsilon * u_2^0, \qquad \theta^\varepsilon(x,y,0) = \varphi_\varepsilon * \theta^0, \qquad (2.21)
$$

where $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$, "$*$" is the usual convolution operator, and $\varphi_\varepsilon(x,y) = \varepsilon^{-2}\varphi(x/\varepsilon, y/\varepsilon)$ is the standard mollifier with width ε, which satisfying

$$
\varphi \ge 0, \quad \varphi \in C_0^\infty(\mathbb{R}^2), \quad \iint_{\mathbb{R}^2} \varphi \, dx\, dy = 1.
$$

Now, an application of the result of Sect. 3 in [32] in p. 934 shows that, for any $T > 0$, there exists a unique global smooth solution $(u_1^\varepsilon, u_2^\varepsilon, \theta^\varepsilon)$ of (2.20), (2.21) on $\mathbb{R}^2 \times (0,T)$ satisfying the global bounds stated in Lemmas 2.4–2.5 which are uniform in ε. So, by standard compactness arguments, we can extract a subsequence $(u_1^{\varepsilon_j}, u_2^{\varepsilon_j}, \theta^{\varepsilon_j})$ and pass to the limit as $j \to \infty$ to find that the limit function (u_1, u_2, θ) is indeed a global weak solution of the problem (1.2)–(1.3) with $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$. The uniqueness of the solution (u_1, u_2, θ) satisfying the condition stated in Theorem 2.3 can be proved in a very standard way, and for simplicity we omit the details here. The proof of Theorem 2.3 is therefore completed.

3 Global regularity criteria for weak solution of the Bénard system with vertical dissipation and horizontal thermal diffusivity

The issue of whether the 2D Bénard system always possesses global (in time) classical solutions can be difficult when there is only partial dissipation. Therefore, the goal of this section is to establish two global regularity criteria for the weak solution of the 2D Bénard system with vertical dissipation in the first component of velocity field and horizontal thermal diffusivity.

Theorem 3.1 *Let* $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$. *Suppose that* $u_1^0, u_2^0, \theta^0 \in H^2(\mathbb{R}^2)$ *and* $\partial_x u_1^0 + \partial_y u_2^0 = 0$. *If the condition holds that*

$$\int_0^T \left\| \partial_{xx} u_2(\tau) \right\|_{L^4(\mathbb{R}^2)} \, d\tau < \infty, \tag{3.1}$$

for any fixed $T > 0$, *then the problem* (1.2)–(1.3) *admits a global classical solution* (u_1, u_2, θ), *which obeys*

$$\begin{aligned} &u_1, u_2, \theta \in L^\infty\big([0,T); H^2(\mathbb{R}^2)\big), \\ &\quad \partial_{xyy} u_1 \in L^2\big([0,T); L^2(\mathbb{R}^2)\big), \qquad \partial_y u_1, \partial_x \theta \in L^2\big([0,T); H^2(\mathbb{R}^2)\big). \end{aligned}$$

Proof Applying ∇ to Eq. $(2.6)_1$ and taking the L^2-inner product with $\nabla\omega$, and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\| \nabla\omega(t) \right\|_{L^2}^2 + \mu_2 \|\partial_{xxy} u_1\|_{L^2}^2 + 2\mu_2 \|\partial_{xyy} u_1\|_{L^2}^2 + \mu_2 \|\partial_{yyy} u_1\|_{L^2}^2 \\ &\quad = -\iint_{\mathbb{R}^2} \nabla\big[(u\cdot\nabla)\omega\big] \cdot \nabla\omega \, dx\, dy + \iint_{\mathbb{R}^2} \nabla \partial_x \theta \nabla\omega \, dx\, dy. \end{aligned} \tag{3.2}$$

By the divergence-free condition, we further split the first term of (3.2) into four terms;

$$\begin{aligned} &-\iint_{\mathbb{R}^2} \nabla\big[(u\cdot\nabla)\omega\big] \cdot \nabla\omega \, dx\, dy \\ &\quad = -\iint_{\mathbb{R}^2} \partial_x u_1 (\partial_x\omega)^2 \, dx\, dy - \iint_{\mathbb{R}^2} \partial_x u_2 \partial_x \omega \partial_y \omega \, dx\, dy \\ &\qquad - \iint_{\mathbb{R}^2} \partial_y u_1 \partial_x \omega \partial_y \omega \, dx\, dy - \iint_{\mathbb{R}^2} \partial_y u_2 (\partial_y\omega)^2 \, dx\, dy. \end{aligned} \tag{3.3}$$

Differentiating Eqs. $(2.6)_2$ and $(2.6)_3$ with respect to x and y and multiplying the resulting equations by $\partial_{xx}\theta$ and $\partial_{yy}\theta$, respectively, we deduce after integrating by parts over $\mathbb{R}^2$

that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\partial_{xx}\theta(t)\right\|_{L^2}^2+\left\|\partial_{yy}\theta(t)\right\|_{L^2}^2\right)+\kappa_1\|\partial_{xxx}\theta\|_{L^2}^2+\kappa_1\|\partial_{xyy}\theta\|_{L^2}^2\\
&\quad=-\iint_{\mathbb{R}^2}\partial_{xx}\big[(u\cdot\nabla)\theta\big]\partial_{xx}\theta\,dx\,dy+\iint_{\mathbb{R}^2}\partial_{xx}u_2\partial_{xx}\theta\,dx\,dy\\
&\qquad-\iint_{\mathbb{R}^2}\partial_{yy}\big[(u\cdot\nabla)\theta\big]\partial_{yy}\theta\,dx\,dy+\iint_{\mathbb{R}^2}\partial_{yy}u_2\partial_{yy}\theta\,dx\,dy.
\end{aligned}\tag{3.4}$$

We now turn to the first and third terms of (3.4). Again we write them out explicitly as

$$\begin{aligned}
&-\iint_{\mathbb{R}^2}\partial_{xx}\big[(u\cdot\nabla)\theta\big]\partial_{xx}\theta\,dx\,dy\\
&\quad=\iint_{\mathbb{R}^2}\partial_{x}\big[(u\cdot\nabla)\theta\big]\partial_{xxx}\theta\,dx\,dy\\
&\quad=-\iint_{\mathbb{R}^2}\partial_x u_1\partial_x\theta\partial_{xxx}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_1\partial_{xx}\theta\partial_{xxx}\theta\,dx\,dy\\
&\qquad+\iint_{\mathbb{R}^2}\partial_x u_2\partial_y\theta\partial_{xxx}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_2\partial_{xy}\theta\partial_{xxx}\theta\,dx\,dy,
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
&-\iint_{\mathbb{R}^2}\partial_{yy}\big[(u\cdot\nabla)\theta\big]\partial_{yy}\theta\,dx\,dy\\
&\quad=\iint_{\mathbb{R}^2}\partial_{y}\big[(u\cdot\nabla)\theta\big]\partial_{yyy}\theta\,dx\,dy\\
&\quad=\iint_{\mathbb{R}^2}\partial_y u_1\partial_x\theta\partial_{yyy}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_1\partial_{xy}\theta\partial_{yyy}\theta\,dx\,dy\\
&\qquad+\iint_{\mathbb{R}^2}\partial_y u_2\partial_y\theta\partial_{yyy}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_2\partial_{yy}\theta\partial_{yyy}\theta\,dx\,dy.
\end{aligned}\tag{3.6}$$

Combining with (3.2)–(3.6) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla\omega(t)\right\|_{L^2}^2+\left\|\partial_{xx}\theta(t)\right\|_{L^2}^2+\left\|\partial_{yy}\theta(t)\right\|_{L^2}^2\right)+\mu_2\|\partial_{xxy}u_1\|_{L^2}^2+2\mu_2\|\partial_{xyy}u_1\|_{L^2}^2\\
&\qquad+\mu_2\|\partial_{yyy}u_1\|_{L^2}^2+\kappa_1\|\partial_{xxx}\theta\|_{L^2}^2+\kappa_1\|\partial_{xyy}\theta\|_{L^2}^2\\
&\quad=-\iint_{\mathbb{R}^2}\partial_x u_1(\partial_x\omega)^2\,dx\,dy-\iint_{\mathbb{R}^2}\partial_x u_2\partial_x\omega\partial_y\omega\,dx\,dy-\iint_{\mathbb{R}^2}\partial_y u_1\partial_x\omega\partial_y\omega\,dx\,dy\\
&\qquad-\iint_{\mathbb{R}^2}\partial_y u_2(\partial_y\omega)^2\,dx\,dy+\iint_{\mathbb{R}^2}\nabla\partial_x\theta\nabla\omega\,dx\,dy+\iint_{\mathbb{R}^2}\partial_x u_1\partial_x\theta\partial_{xxx}\theta\,dx\,dy\\
&\qquad+\iint_{\mathbb{R}^2}\partial_x u_2\partial_y\theta\partial_{xxx}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_2\partial_{xy}\theta\partial_{xxx}\theta\,dx\,dy+\iint_{\mathbb{R}^2}\partial_y u_1\partial_x\theta\partial_{yyy}\theta\,dx\,dy\\
&\qquad+\iint_{\mathbb{R}^2}u_1\partial_{xy}\theta\partial_{yyy}\theta\,dx\,dy+\iint_{\mathbb{R}^2}\partial_y u_2\partial_y\theta\partial_{yyy}\theta\,dx\,dy+\iint_{\mathbb{R}^2}\partial_{xx}u_2\partial_{xx}\theta\,dx\,dy\\
&\qquad+\iint_{\mathbb{R}^2}\partial_{yy}u_2\partial_{yy}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_1\partial_{xx}\theta\partial_{xxx}\theta\,dx\,dy+\iint_{\mathbb{R}^2}u_2\partial_{yy}\theta\partial_{yyy}\theta\,dx\,dy\\
&\quad=J=\sum_{l=1}^{15}J_l.
\end{aligned}\tag{3.7}$$

We now estimate J_1 through J_7. Applying Lemma 2.1, the Cauchy–Schwarz inequality and Young's inequality, we have

$$\begin{aligned}
J_1 &= -\iint_{\mathbb{R}^2} \partial_x u_1 (\partial_x \omega)^2 \, dx\, dy \\
&= -\iint_{\mathbb{R}^2} \partial_x u_1 (\partial_{xx} u_2)^2 \, dx\, dy - \iint_{\mathbb{R}^2} \partial_x u_1 (\partial_{xy} u_1)^2 \, dx\, dy \\
&\quad + 2\iint_{\mathbb{R}^2} \partial_x u_1 \partial_{xx} u_2 \partial_{xy} u_1 \, dx\, dy \\
&\le C\|\partial_x u_1\|_{L^4} \|\partial_{xx} u_2\|_{L^2} \|\partial_{xx} u_2\|_{L^4} \\
&\quad + C\|\partial_x u_1\|_{L^2} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xxy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy} u_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_x u_1\|_{L^4} \|\partial_{xx} u_2\|_{L^4} \|\partial_{xy} u_1\|_{L^2} \\
&\le C\|\partial_x u_1\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_x u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xx} u_2\|_{L^2} \|\partial_{xx} u_2\|_{L^4} \\
&\quad + C\|\omega\|_{L^2} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xxy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy} u_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_x u_1\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_x u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xx} u_2\|_{L^4} \|\partial_{xy} u_1\|_{L^2} \\
&\le C\|\omega\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{3}{2}} \|\partial_{xx} u_2\|_{L^4} + C\|\omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\partial_{xxy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy} u_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\omega\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{3}{2}} \|\partial_{xx} u_2\|_{L^4} \\
&\le \frac{\mu_2}{16} \|\partial_{xxy} u_1\|_{L^2}^2 + \frac{\mu_2}{2} \|\partial_{xyy} u_1\|_{L^2}^2 + C\big(\|\partial_{xx} u_2\|_{L^4} + \|\omega\|_{L^2}^2\big) \|\nabla \omega\|_{L^2}^2 \\
&\quad + C\|\partial_{xx} u_2\|_{L^4} \|\omega\|_{L^2}^2,
\end{aligned} \tag{3.8}$$

where we have utilized the inequality

$$\|f\|_{L^4} \le \|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}}. \tag{3.9}$$

Similarly, invoking Lemma 2.1 and (3.9), the Cauchy–Schwarz inequality and Young's inequality, the remainder terms can be estimated as follows:

$$\begin{aligned}
J_2 &= -\iint_{\mathbb{R}^2} \partial_x u_2 \partial_x \omega \partial_y \omega \, dx\, dy \\
&= -\iint_{\mathbb{R}^2} \partial_x u_2 \partial_{xx} u_2 \partial_{xy} u_2 \, dx\, dy + \iint_{\mathbb{R}^2} \partial_x u_2 \partial_{xy} u_1 \partial_{xy} u_2 \, dx\, dy \\
&\quad + \iint_{\mathbb{R}^2} \partial_x u_2 \partial_{xx} u_2 \partial_{yy} u_1 \, dx\, dy - \iint_{\mathbb{R}^2} \partial_x u_2 \partial_{xy} u_1 \partial_{yy} u_1 \, dx\, dy \\
&\le C\|\partial_x u_2\|_{L^4} \|\partial_{xx} u_2\|_{L^4} \|\partial_{xy} u_2\|_{L^2} \\
&\quad + C\|\partial_x u_2\|_{L^2} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xxy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xy} u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy} u_2\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_x u_2\|_{L^4} \|\partial_{xx} u_2\|_{L^4} \|\partial_{yy} u_1\|_{L^2} \\
&\quad + C\|\partial_x u_2\|_{L^2} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xxy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yyy} u_1\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_x u_2\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_x u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{xx} u_2\|_{L^4} \|\nabla \omega\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+ C\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xx}u_2\|_{L^4}\|\nabla\omega\|_{L^2} \\
&+ C\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
\leq{}& C\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\|\partial_{xx}u_2\|_{L^4} + C\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}\|\partial_{xxy}u_1\|_{L^2} \\
&+ C\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\|\partial_{xx}u_2\|_{L^4} + C\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
\leq{}& \frac{\mu_2}{8}\|\partial_{xxy}u_1\|_{L^2}^2 + \frac{\mu_2}{12}\|\partial_{yyy}u_1\|_{L^2}^2 + C\big(\|\partial_{xx}u_2\|_{L^4} + \|\omega\|_{L^2}^2\big)\|\nabla\omega\|_{L^2}^2 \\
&+ C\|\partial_{xx}u_2\|_{L^4}\|\omega\|_{L^2}^2, \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
J_3 ={}& -\iint_{\mathbb{R}^2}\partial_y u_1\partial_x\omega\partial_y\omega\,dx\,dy \\
={}& -\iint_{\mathbb{R}^2}\partial_y u_1\partial_{xx}u_2\partial_{xy}u_2\,dx\,dy + \iint_{\mathbb{R}^2}\partial_y u_1\partial_{xx}u_2\partial_{yy}u_1\,dx\,dy \\
&+ \iint_{\mathbb{R}^2}\partial_y u_1\partial_{xy}u_1\partial_{xy}u_2\,dx\,dy - \iint_{\mathbb{R}^2}\partial_y u_1\partial_{xy}u_1\partial_{yy}u_1\,dx\,dy \\
\leq{}& C\|\partial_y u_1\|_{L^4}\|\partial_{xx}u_2\|_{L^4}\|\partial_{xy}u_2\|_{L^2} + C\|\partial_y u_1\|_{L^4}\|\partial_{xx}u_2\|_{L^4}\|\partial_{yy}u_1\|_{L^2} \\
&+ C\|\partial_y u_1\|_{L^2}\|\partial_{xy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}u_2\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_y u_1\|_{L^2}\|\partial_{xy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
\leq{}& C\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xx}u_2\|_{L^4}\|\nabla\omega\|_{L^2} \\
&+ C\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xx}u_2\|_{L^4}\|\nabla\omega\|_{L^2} \\
&+ C\|\partial_y u_1\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_y u_1\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
\leq{}& C\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\|\partial_{xx}u_2\|_{L^4} + C\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\|\partial_{xx}u_2\|_{L^4} \\
&+ C\|\partial_y u_1\|_{L^2}\|\nabla\omega\|_{L^2}\|\partial_{xxy}u_1\|_{L^2} + C\|\partial_y u_1\|_{L^2}\|\nabla\omega\|_{L^2}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
\leq{}& \frac{\mu_2}{8}\|\partial_{xxy}u_1\|_{L^2}^2 + \frac{\mu_2}{12}\|\partial_{yyy}u_1\|_{L^2}^2 + C\big(\|\partial_{xx}u_2\|_{L^4} + \|\partial_y u_1\|_{L^2}^2\big)\|\nabla\omega\|_{L^2}^2 \\
&+ C\|\partial_{xx}u_2\|_{L^4}\|\partial_y u_1\|_{L^2}^2, \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
J_4 ={}& -\iint_{\mathbb{R}^2}\partial_y u_2(\partial_y\omega)^2\,dx\,dy \\
={}& 2\iint_{\mathbb{R}^2}u_2\partial_{xy}u_2\partial_{xyy}u_2\,dx\,dy + 2\iint_{\mathbb{R}^2}u_2\partial_{yy}u_1\partial_{yyy}u_1\,dx\,dy \\
&+ 2\iint_{\mathbb{R}^2}u_2\partial_{xyy}u_2\partial_{yy}u_1\,dx\,dy + 2\iint_{\mathbb{R}^2}u_2\partial_{xy}u_2\partial_{yyy}u_1\,dx\,dy \\
\leq{}& C\|\partial_{xyy}u_2\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}u_2\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_{yyy}u_1\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\|\partial_{xyy}u_2\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_{yyy}u_1\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}u_2\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_{xxy}u_1\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_{yyy}u_1\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_{xxy}u_1\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_{yyy}u_1\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{xxy}u_1\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{3\mu_2}{16}\|\partial_{xxy}u_1\|_{L^2}^2 + \frac{3\mu_2}{12}\|\partial_{yyy}u_1\|_{L^2}^2 + C\|u_2\|_{L^2}^2\|\omega\|_{L^2}^2\|\nabla\omega\|_{L^2}^2, \qquad (3.12)
\end{aligned}
$$

$$
J_5 = \iint_{\mathbb{R}^2} \nabla\partial_x\theta\nabla\omega\,dx\,dy \le C\bigl(\|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\bigr), \qquad (3.13)
$$

$$
\begin{aligned}
J_6 &= \iint_{\mathbb{R}^2} \partial_x u_1\partial_x\theta\partial_{xxx}\theta\,dx\,dy \\
&\le C\|\partial_{xxx}\theta\|_{L^2}\|\partial_x u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_x\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xx}\theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\kappa_1}{6}\|\partial_{xxx}\theta\|_{L^2}^2 + C\|\omega\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + C\|\partial_x\theta\|_{L^2}^2\|\partial_{xx}\theta\|_{L^2}^2, \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
J_7 &= \iint_{\mathbb{R}^2} \partial_x u_2\partial_y\theta\partial_{xxx}\theta\,dx\,dy \\
&\le C\|\partial_{xxx}\theta\|_{L^2}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xx}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\kappa_1}{6}\|\partial_{xxx}\theta\|_{L^2}^2 + C\|\omega\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + C\|\partial_y\theta\|_{L^2}^2\|\partial_{yy}\theta\|_{L^2}^2, \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
J_8 &= \iint_{\mathbb{R}^2} u_2\partial_{xy}\theta\partial_{xxx}\theta\,dx\,dy \\
&\le C\|\partial_{xxx}\theta\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\kappa_1}{6}\|\partial_{xxx}\theta\|_{L^2}^2 + \frac{\kappa_1}{14}\|\partial_{xyy}\theta\|_{L^2}^2 + C\|u_2\|_{L^2}^2\|\omega\|_{L^2}^2\bigl(\|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\bigr), \qquad (3.16)
\end{aligned}
$$

where we have used the fact $\|\partial_{xy}\theta\|_{L^2}^2 \le C(\|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2)$;

$$
\begin{aligned}
J_9 &= \iint_{\mathbb{R}^2} \partial_y u_1\partial_x\theta\partial_{yyy}\theta\,dx\,dy \\
&= -\iint_{\mathbb{R}^2} \partial_{yy}u_1\partial_x\theta\partial_{yy}\theta\,dx\,dy - \iint_{\mathbb{R}^2} \partial_y u_1\partial_{xy}\theta\partial_{yy}\theta\,dx\,dy \\
&\le C\|\partial_x\theta\|_{L^2}\|\partial_{yy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_y u_1\|_{L^2}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_x\theta\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_y u_1\|_{L^2}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\mu_2}{12}\|\partial_{yyy}u_1\|_{L^2}^2 + \frac{\kappa_1}{7}\|\partial_{xyy}\theta\|_{L^2}^2 \\
&\quad + C\bigl(\|\partial_x\theta\|_{L^2}^2 + \|\partial_y u_1\|_{L^2}^2\bigr)\bigl(\|\nabla\omega\|_{L^2}^2 + \|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\bigr), \qquad (3.17)
\end{aligned}
$$

$$\begin{aligned} J_{10} &= \iint_{\mathbb{R}^2} u_1 \partial_{xy}\theta \partial_{yyy}\theta \, dx\, dy \\ &= -\iint_{\mathbb{R}^2} \partial_y u_1 \partial_{xy}\theta \partial_{yy}\theta \, dx\, dy - \iint_{\mathbb{R}^2} u_1 \partial_{xyy}\theta \partial_{yy}\theta \, dx\, dy \\ &\le C\|\partial_y u_1\|_{L^2} \|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_{xyy}\theta\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_y u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\kappa_1}{7}\|\partial_{xyy}\theta\|_{L^2}^2 + C\big(\|\partial_y u_1\|_{L^2}^2 + \|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2\big)\big(\|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\big), \end{aligned} \tag{3.18}$$

$$\begin{aligned} J_{11} &= \iint_{\mathbb{R}^2} \partial_y u_2 \partial_y\theta \partial_{yyy}\theta \, dx\, dy \\ &= -\iint_{\mathbb{R}^2} \partial_{yy} u_2 \partial_y\theta \partial_{yy}\theta \, dx\, dy - \iint_{\mathbb{R}^2} \partial_y u_2 \partial_{yy}\theta \partial_{yy}\theta \, dx\, dy \\ &\le C\|\partial_y\theta\|_{L^2} \|\partial_{yy}u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{yyy}u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_{yy}\theta\|_{L^2} \|\partial_y u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le C\|\partial_y\theta\|_{L^2} \|\nabla\omega\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_{yy}\theta\|_{L^2} \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_{xy}u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\mu_2}{4}\|\partial_{xyy}u_1\|_{L^2}^2 + \frac{\kappa_1}{7}\|\partial_{xyy}\theta\|_{L^2}^2 \\ &\quad + C\big(\|\partial_y\theta\|_{L^2}^2 + \|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_{xy}u_1\|_{L^2}^{\frac{2}{3}}\big)\big(\|\nabla\omega\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\big), \end{aligned} \tag{3.19}$$

$$J_{12} = \iint_{\mathbb{R}^2} \partial_{xx} u_2 \partial_{xx}\theta \, dx\, dy \le C\big(\|\nabla\omega\|_{L^2}^2 + \|\partial_{xx}\theta\|_{L^2}^2\big), \tag{3.20}$$

$$J_{13} = \iint_{\mathbb{R}^2} \partial_{yy} u_2 \partial_{yy}\theta \, dx\, dy \le C\big(\|\nabla\omega\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\big), \tag{3.21}$$

and

$$\begin{aligned} J_{14} + J_{15} &= \iint_{\mathbb{R}^2} u_1 \partial_{xx}\theta \partial_{xxx}\theta \, dx\, dy + \iint_{\mathbb{R}^2} u_2 \partial_{yy}\theta \partial_{yyy}\theta \, dx\, dy \\ &= \iint_{\mathbb{R}^2} u_1 \partial_{xx}\theta \partial_{xxx}\theta \, dx\, dy - \iint_{\mathbb{R}^2} u_1 \partial_{yy}\theta \partial_{xyy}\theta \, dx\, dy \\ &\le C\|\partial_{xxx}\theta\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_y u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xx}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xxx}\theta\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_{xyy}\theta\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_y u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\kappa_1}{8}\|\partial_{xxx}\theta\|_{L^2}^2 + \frac{\kappa_1}{16}\|\partial_{xyy}\theta\|_{L^2}^2 \\ &\quad + C\|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2\big(\|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\big). \end{aligned} \tag{3.22}$$

Hence, inserting the estimates J_1–J_{15} into (3.7), we finally obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\big\|\nabla\omega(t)\big\|_{L^2}^2 + \big\|\partial_{xx}\theta(t)\big\|_{L^2}^2 + \big\|\partial_{yy}\theta(t)\big\|_{L^2}^2\big) + \mu_2\|\partial_{xxy}u_1\|_{L^2}^2 + \mu_2\|\partial_{xyy}u_1\|_{L^2}^2 \\ &\quad + \mu_2\|\partial_{yyy}u_1\|_{L^2}^2 + \kappa_1\|\partial_{xxx}\theta\|_{L^2}^2 + \kappa_1\|\partial_{xyy}\theta\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
&\leq C\big(\|\partial_y u_1\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\partial_x\theta\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2 + \|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2 + \|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_{xy}u_1\|_{L^2}^{\frac{2}{3}} \\
&\quad + \|u_2\|_{L^2}^2\|\omega\|_{L^2}^2 + \|\partial_{xx}u_2\|_{L^4} + 1\big)\big(\|\nabla\omega\|_{L^2}^2 + \|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2\big) \\
&\quad + C\|\partial_{xx}u_2\|_{L^4}\big(\|\partial_y u_1\|_{L^2}^2 + \|\omega\|_{L^2}^2\big). \qquad (3.23)
\end{aligned}$$

It thus follows from Gronwall's inequality that

$$\begin{aligned}
&\|\nabla\omega\|_{L^2}^2 + \|\partial_{xx}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2 + \mu_2\int_0^t \big\|\partial_{xxy}u_1(\tau)\big\|_{L^2}^2\,d\tau + \mu_2\int_0^t \big\|\partial_{xyy}u_1(\tau)\big\|_{L^2}^2\,d\tau \\
&\quad + \mu_2\int_0^t \big\|\partial_{yyy}u_1(\tau)\big\|_{L^2}^2\,d\tau + \kappa_1\int_0^t \big\|\partial_{xxx}\theta(\tau)\big\|_{L^2}^2\,d\tau + \kappa_1\int_0^t \big\|\partial_{xyy}\theta(\tau)\big\|_{L^2}^2\,d\tau \\
&\leq C, \qquad (3.24)
\end{aligned}$$

which yields

$$\begin{aligned}
&\|\nabla\omega\|_{L^\infty([0,T);L^2(\mathbb{R}^2))} + \|\triangle\theta\|_{L^\infty([0,T);L^2(\mathbb{R}^2))} + \mu_2\|\triangle\partial_y u_1\|_{L^2([0,T);L^2(\mathbb{R}^2))} \\
&\quad + \mu_2\|\partial_{xyy}u_1\|_{L^2([0,T);L^2(\mathbb{R}^2))} + \kappa_1\|\triangle\partial_x\theta\|_{L^2([0,T);L^2(\mathbb{R}^2))} \leq C. \qquad (3.25)
\end{aligned}$$

Therefore, we have completed the proof of Theorem 3.1. □

Furthermore, we establish another regularity criterion to the 2D Bénard system with vertical dissipation and horizontal thermal diffusivity.

Theorem 3.2 *Let* $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$. *Given a positive time* $T \in (0,\infty)$. *Assume that* $u_1^0, u_2^0, \theta^0 \in H^2(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *Let* (u_1, u_2, θ) *be the solution of* (1.2)–(1.3). *If*

$$\int_0^T \|\partial_x u\|_{L^\infty}^2\,dt < \infty \qquad (3.26)$$

for some $T > 0$, *then* $\|(u_1, u_2, \theta)\|_{H^2}$ *is finite on* $[0, T]$.

We now give the proof of Theorem 3.2, we first prove the global H^1 bound for (u_1, u_2, θ), we present the proof of the main theorem secondly.

• Global H^1-bound for (u_1, u_2, θ)

Proposition 3.3 *Let* $(u_1^0, u_2^0, \theta^0) \in H^2(\mathbb{R}^2)$ *and let* (u_1, u_2, θ) *be the corresponding solution of* (1.2)–(1.3). *Then* (u_1, u_2, θ) *obeys the following global* L^2*-bound*:

$$\begin{aligned}
&\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \mu_2\int_0^t \big\|\partial_y u_1(\tau)\big\|_{L^2}^2\,d\tau + \kappa_1\int_0^t \big\|\partial_x\theta(\tau)\big\|_{L^2}^2\,d\tau \\
&\leq C\big\|\big(u_0^1, u_2^0, \theta^0\big)\big\|_{L^2}^2 \qquad (3.27)
\end{aligned}$$

for any $t \geq 0$. *Here* C *is a constant depending only on* μ_2, κ_1, *and* T.

Proposition 3.4 *Assume that* $(u_1^0, u_2^0, \theta^0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u^0 = 0$. *Let* (u_1, u_2, θ) *be the corresponding solution of* (1.2)–(1.3). *Then, for any* $T > 0$ *and* $t \le T$,

$$\big\| \big(u_1(t), u_2(t), \theta(t)\big) \big\|_{H^1(\mathbb{R}^2)} \le C_1 e^{C_2 \int_0^t \|\partial_x u\|_{L^\infty}^2 \, d\tau}, \tag{3.28}$$

where C_1 *is a constant depending on* T *and initial data, and* C_2 *is a pure constant.*

Proof Taking the inner product of the first three equations in (1.2) with $-\Delta u_1$, $-\Delta u_2$ and $-\Delta\theta$, respectively, integrating with respect to space

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\big\|\nabla u_1(t)\big\|_{L^2}^2 + \big\|\nabla u_2(t)\big\|_{L^2}^2 + \big\|\nabla\theta(t)\big\|_{L^2}^2\big) + \mu_2\|\nabla\partial_y u_1\|_{L^2}^2 + \kappa_1\|\nabla\partial_x\theta\|_{L^2}^2 \\ &\quad = \underbrace{-\iint_{\mathbb{R}^2} \theta \cdot \Delta u_2 \, dx\, dy}_{K_1} + \underbrace{\iint_{\mathbb{R}^2} \big[(u \cdot \nabla)\theta\big] \cdot \Delta\theta \, dx\, dy}_{K_2} \underbrace{- \iint_{\mathbb{R}^2} u_2 \cdot \Delta\theta \, dx\, dy}_{K_3}; \end{aligned} \tag{3.29}$$

then, for notational convenience, we set

$$A(t) = \big\|\nabla u_1(\cdot, t)\big\|_{L^2}^2 + \big\|\nabla u_2(\cdot, t)\big\|_{L^2}^2 + \big\|\nabla\theta(\cdot, t)\big\|_{L^2}^2. \tag{3.30}$$

For K_1 and K_3, integrating by parts and applying the Hölder inequality gives

$$K_1 + K_3 \le \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2. \tag{3.31}$$

To estimate K_2 we write component-wise

$$\begin{aligned} K_2 &= -\iint_{\mathbb{R}^2} \nabla\theta \cdot \nabla u \cdot \nabla\theta \, dx\, dy \\ &= \underbrace{-\iint_{\mathbb{R}^2} \partial_x\theta\,\partial_x u_1\,\partial_x\theta \, dx\, dy}_{K_{21}} \underbrace{- \iint_{\mathbb{R}^2} \partial_x\theta\,\partial_x u_2\,\partial_y\theta \, dx\, dy}_{K_{22}} \underbrace{- \iint_{\mathbb{R}^2} \partial_y\theta\,\partial_y u_1\,\partial_x\theta \, dx\, dy}_{K_{23}} \\ &\quad \underbrace{- \iint_{\mathbb{R}^2} \partial_y\theta\,\partial_y u_2\,\partial_y\theta \, dx\, dy}_{K_{24}}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} K_{21} &= 2\iint_{\mathbb{R}^2} u_1\,\partial_x\theta\,\partial_{xx}\theta \, dx\, dy \\ &\le C\|\partial_{xx}\theta\|_{L^2}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_x\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xx}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\kappa_1}{6}\|\nabla\partial_x\theta\|_{L^2}^2 + C\|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2\|\nabla\theta\|_{L^2}^2, \end{aligned} \tag{3.33}$$

$$K_{22} \le C\|\partial_x u_2\|_{L^\infty}\|\nabla\theta\|_{L^2}^2, \tag{3.34}$$

$$\begin{aligned} K_{23} &\le C\|\partial_y u_1\|_{L^2}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_x\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\kappa_1}{6}\|\nabla\partial_x\theta\|_{L^2}^2 + C\|\partial_y u_1\|_{L^2}^2\|\nabla\theta\|_{L^2}^2, \end{aligned} \tag{3.35}$$

$$\begin{aligned} K_{24} &= -2\iint_{\mathbb{R}^2} u_1\partial_y\theta\partial_{xy}\theta\,dx\,dy \\ &\le C\|\partial_{xy}\theta\|_{L^2}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\kappa_1}{6}\|\nabla\partial_x\theta\|_{L^2}^2 + C\|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2\|\nabla\theta\|_{L^2}^2. \end{aligned} \tag{3.36}$$

After combining inequalities,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}A(t) + \mu_2\|\nabla\partial_y u_1\|_{L^2}^2 + \kappa_1\|\nabla\partial_x\theta\|_{L^2}^2 &\le \frac{\mu_2}{2}\|\nabla\partial_y u_1\|_{L^2}^2 + \frac{\kappa_1}{2}\|\nabla\partial_x\theta\|_{L^2}^2 \\ &\quad + C\|\partial_x u_2\|_{L^\infty}^2 A(t), \end{aligned} \tag{3.37}$$

after applying Gronwall's lemma, we get the H^1-norm for u_1, u_2, and θ. This completes the proof of Proposition 3.4. □

• H^2 bound

To estimate the H^2-norm of (u_1,u_2,θ), we consider the equation of $\omega = \nabla\times u$, $\nabla\theta$

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = -\mu_2\partial_{yyy}u_1 + \partial_x\theta, \\ \partial_t\nabla\theta + \nabla[(u\cdot\nabla)\theta] = \kappa_1\nabla\partial_{xx}\theta + \nabla u_2. \end{cases} \tag{3.38}$$

Proposition 3.5 *Assume that* $(u_1^0,u_2^0,\theta^0)\in H^2(\mathbb{R}^2)$, $\nabla\cdot u^0 = 0$ *and let* (u_1,u_2,θ) *be the solution of* (1.2)–(1.3). *Then* $(\omega,\nabla\theta)$ *satisfy*

$$\|\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 + \mu_2\int_0^T\|\Delta\partial_y u_1\|_{L^2}^2\,dt + \kappa_1\int_0^T\|\Delta\partial_x\theta\|_{L^2}^2\,dt \le C \tag{3.39}$$

if $\partial_x u\in L^2([0,T];L^\infty(\mathbb{R}^2))$.

Proof Taking the inner product of $(3.38)_1$ with $-\Delta\omega$ and $(3.38)_2$ with $-\Delta\nabla\theta$ in $L^2(\mathbb{R}^2)$, respectively, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\big) + \mu_2\|\Delta\partial_y u_1\|_{L^2}^2 + \kappa_1\|\Delta\partial_x\theta\|_{L^2}^2 \\ &\quad = \underbrace{\iint_{\mathbb{R}^2}(u\cdot\nabla)\omega\cdot\Delta\omega\,dx\,dy}_{L_1} + \underbrace{\iint_{\mathbb{R}^2}\nabla\partial_x\theta\cdot\nabla\omega\,dx\,dy}_{L_2} \\ &\qquad \underbrace{-\iint_{\mathbb{R}^2}\Delta\big[(u\cdot\nabla)\theta\big]\Delta\theta\,dx\,dy}_{L_3} + \underbrace{\iint_{\mathbb{R}^2}\Delta u_2\cdot\Delta\theta\,dx\,dy}_{L_4}. \end{aligned} \tag{3.40}$$

We now estimate L_1 through L_4. We firstly write the four terms in L_1 explicitly,

$$\begin{aligned} L_1 &= -\iint_{\mathbb{R}^2}\nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx\,dy \\ &= \underbrace{-\iint_{\mathbb{R}^2}\partial_x u_1\partial_x\omega\partial_x\omega\,dx\,dy}_{L_{11}} \underbrace{-\iint_{\mathbb{R}^2}\partial_x u_2\partial_x\omega\partial_y\omega\,dx\,dy}_{L_{12}} \\ &\quad \underbrace{-\iint_{\mathbb{R}^2}\partial_y u_1\partial_x\omega\partial_y\omega\,dx\,dy}_{L_{13}} \underbrace{-\iint_{\mathbb{R}^2}\partial_y u_2\partial_y\omega\partial_y\omega\,dx\,dy}_{L_{14}}. \end{aligned} \tag{3.41}$$

These terms can be bounded as follows:

$$L_{11} \le \|\partial_x u_1\|_{L^\infty} \|\partial_x \omega\|_{L^2}^2 \le \|\partial_x u_1\|_{L^\infty} \|\nabla\omega\|_{L^2}^2, \tag{3.42}$$

$$L_{12} \le \|\partial_x u_2\|_{L^\infty} \|\partial_x \omega\|_{L^2} \|\partial_y \omega\|_{L^2} \le \|\partial_x u_2\|_{L^\infty} \|\nabla\omega\|_{L^2}^2, \tag{3.43}$$

$$\begin{aligned} L_{13} &\le C\|\partial_x\omega\|_{L^2} \|\partial_y u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_y \omega\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\omega\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\mu_2}{6} \|\Delta\partial_y u_1\|_{L^2}^2 + C\|\partial_y u_1\|_{L^2}^{\frac{2}{3}} \|\nabla\partial_y u_1\|_{L^2}^{\frac{2}{3}} \|\nabla\omega\|_{L^2}^2, \end{aligned} \tag{3.44}$$

$$\begin{aligned} L_{14} &= 2\iint_{\mathbb{R}^2} u_2 \partial_y \omega \partial_{yy}\omega \, dx\, dy \\ &\le C\|\partial_{yy}\omega\|_{L^2} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_x u_2\|_{L^2}^{\frac{1}{2}} \|\partial_y \omega\|_{L^2}^{\frac{1}{2}} \|\partial_{yy}\omega\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\mu_2}{6} \|\Delta\partial_y u_1\|_{L^2}^2 + C\|u_2\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla\omega\|_{L^2}^2. \end{aligned} \tag{3.45}$$

The terms L_2 and L_4 can easily be bounded,

$$L_2 \le \underbrace{\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2}_{\|\nabla\partial_x\theta\|_{L^2} \le \|\Delta\theta\|_{L^2}}, \tag{3.46}$$

$$L_4 \le \underbrace{\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2}_{(\|\Delta u_2\|_{L^2} \le \|\nabla\omega\|_{L^2})}. \tag{3.47}$$

Finally, we deal with L_3.

$$L_3 = -\iint_{\mathbb{R}^2} \Delta(u_1\partial_x\theta + u_2\partial_y\theta)\Delta\theta \, dx\, dy \triangleq L_{31} + L_{32}. \tag{3.48}$$

We first split L_{31} and L_{32} each into two terms,

$$L_{31} = -\iint_{\mathbb{R}^2} \partial_{xx}(u_1\partial_x\theta + u_2\partial_y\theta)\Delta\theta \, dx\, dy \triangleq L_{311} + L_{312}, \tag{3.49}$$

$$\begin{aligned} L_{32} &= -\iint_{\mathbb{R}^2} \partial_{yy}(u_1\partial_x\theta + u_2\partial_y\theta)\partial_{xx}\theta \, dx\, dy - \iint_{\mathbb{R}^2} \partial_{yy}(u_1\partial_x\theta + u_2\partial_y\theta)\partial_{yy}\theta \, dx\, dy \\ &\triangleq L_{321} + L_{322}. \end{aligned} \tag{3.50}$$

These terms are bounded as follows:

$$\begin{aligned} L_{311} &= -\iint_{\mathbb{R}^2} \partial_{xx}(u_1\partial_x\theta)\Delta\theta \, dx\, dy \\ &= \iint_{\mathbb{R}^2} \partial_x u_1 \partial_x\theta \Delta\partial_x\theta \, dx\, dy + \iint_{\mathbb{R}^2} u_1 \partial_{xx}\theta \Delta\partial_x\theta \, dx\, dy \\ &\le C\|\Delta\partial_x\theta\|_{L^2} \|\partial_x u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xy} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_x\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xx}\theta\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\Delta\partial_x\theta\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_y u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{xx}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xxx}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\kappa_1}{18} \|\Delta\partial_x\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\nabla\omega\|_{L^2}^2 \\ &\quad + C\big(\|\nabla\theta\|_{L^2}^2 + \|u_1\|_{L^2}^2 \|\partial_y u_1\|_{L^2}^2\big) \|\Delta\theta\|_{L^2}^2, \end{aligned} \tag{3.51}$$

$$\begin{aligned}
L_{312} &= -\iint_{\mathbb{R}^2} \partial_{xx}(u_2\partial_y\theta)\Delta\theta\,dx\,dy \\
&= \iint_{\mathbb{R}^2} \partial_x u_2\partial_y\theta\,\Delta\partial_x\theta\,dx\,dy + \iint_{\mathbb{R}^2} u_2\partial_{xy}\theta\,\Delta\partial_x\theta\,dx\,dy \\
&\leq \|\partial_x u_2\|_{L^\infty}\|\partial_y\theta\|_{L^2}\|\Delta\partial_x\theta\|_{L^2} \\
&\quad + C\|\Delta\partial_x\theta\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\kappa_1}{18}\|\Delta\partial_x\theta\|_{L^2}^2 + C\|\partial_x u_2\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 + C\|u_2\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla\partial_x\theta\|_{L^2}^2,
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
L_{321} &= -\iint_{\mathbb{R}^2} \partial_{yy}(u_1\partial_x\theta + u_2\partial_y\theta)\partial_{xx}\theta\,dx\,dy \\
&= \iint_{\mathbb{R}^2} \partial_x u_1\partial_x\theta\,\partial_{xyy}\theta\,dx\,dy + \iint_{\mathbb{R}^2} u_1\partial_{xx}\theta\,\partial_{xyy}\theta\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^2} \partial_x u_2\partial_y\theta\,\partial_{xyy}\theta\,dx\,dy + \iint_{\mathbb{R}^2} u_2\partial_{xy}\theta\,\partial_{xyy}\theta\,dx\,dy \\
&\leq \frac{\kappa_1}{18}\|\Delta\partial_x\theta\|_{L^2}^2 + C\|\partial_x u\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 + C\|u_1\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\Delta\theta\|_{L^2}^2 \\
&\quad + C\|u_2\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla\partial_x\theta\|_{L^2}^2,
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
L_{322} &= -\iint_{\mathbb{R}^2} \partial_{yy}(u_1\partial_x\theta + u_2\partial_y\theta)\partial_{yy}\theta\,dx\,dy \\
&= -\iint_{\mathbb{R}^2} \partial_y(\partial_y u_1\partial_x\theta + u_1\partial_{xy}\theta)\partial_{yy}\theta\,dx\,dy \\
&\quad - \iint_{\mathbb{R}^2} \partial_y(\partial_y u_2\partial_y\theta + u_2\partial_{yy}\theta)\partial_{yy}\theta\,dx\,dy \\
&= -\iint_{\mathbb{R}^2} (\partial_{yy}u_1\partial_x\theta + 2\partial_y u_1\partial_{xy}\theta + u_1\partial_{xyy}\theta)\partial_{yy}\theta\,dx\,dy \\
&\quad - \iint_{\mathbb{R}^2} (\partial_{yy}u_2\partial_y\theta + 2\partial_y u_2\partial_{yy}\theta + u_2\partial_{yyy}\theta)\partial_{yy}\theta\,dx\,dy,
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
&\left|-\iint_{\mathbb{R}^2} \partial_{yy}u_1\partial_x\theta\,\partial_{yy}\theta\,dx\,dy\right| \\
&\quad\leq C\|\partial_x\theta\|_{L^2}\|\partial_{yy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yyy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad\leq \frac{\mu_2}{6}\|\Delta\partial_y u_1\|_{L^2}^2 + \frac{\kappa_1}{18}\|\Delta\partial_x\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^2}^2\big(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\big),
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
&\left|-2\iint_{\mathbb{R}^2} \partial_y u_1\partial_{xy}\theta\,\partial_{yy}\theta\,dx\,dy\right| \\
&\quad\leq C\|\partial_{yy}\theta\|_{L^2}\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad\leq \frac{\kappa_1}{18}\|\Delta\partial_x\theta\|_{L^2}^2 + C\|\partial_y u_1\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + C\|\nabla\partial_x\theta\|_{L^2}\|\Delta\theta\|_{L^2}^2,
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
&\left|-\iint_{\mathbb{R}^2} u_1\partial_{xyy}\theta\,\partial_{yy}\theta\,dx\,dy\right| \\
&\quad\leq C\|\partial_{xyy}\theta\|_{L^2}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_y u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{yy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad\leq \frac{\kappa_1}{18}\|\Delta\partial_x\theta\|_{L^2}^2 + C\|u_1\|_{L^2}^2\|\partial_y u_1\|_{L^2}^2\|\Delta\theta\|_{L^2}^2,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
&\left| -\iint_{\mathbb{R}^2} \partial_{yy} u_2 \partial_y \theta \partial_{yy} \theta \, dx \, dy \right| \\
&\quad \le C \|\partial_{yy} u_2\|_{L^2} \|\partial_y \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy} \theta\|_{L^2}^{\frac{1}{2}} \\
&\quad \le \frac{\kappa_1}{18} \|\Delta \partial_x \theta\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}^2 + C\|\nabla \partial_y u_1\|_{L^2}^2 \|\Delta \theta\|_{L^2}^2, \qquad (3.58)
\end{aligned}$$

$$\begin{aligned}
&\left| -2\iint_{\mathbb{R}^2} \partial_y u_2 \partial_{yy} \theta \partial_{yy} \theta \, dx \, dy \right| \\
&\quad = \left| -4 \iint_{\mathbb{R}^2} u_1 \partial_{yy} \theta \partial_{xyy} \theta \, dx \, dy \right| \\
&\quad \le C \|\partial_{xyy} \theta\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_y u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xyy} \theta\|_{L^2}^{\frac{1}{2}} \\
&\quad \le \frac{\kappa_1}{18} \|\Delta \partial_x \theta\|_{L^2}^2 + C\|u_1\|_{L^2}^2 \|\partial_y u_1\|_{L^2}^2 \|\Delta \theta\|_{L^2}^2, \qquad (3.59)
\end{aligned}$$

$$\begin{aligned}
&\left| -\iint_{\mathbb{R}^2} u_2 \partial_{yyy} \theta \partial_{yy} \theta \, dx \, dy \right| \\
&\quad = \left| \iint_{\mathbb{R}^2} u_1 \partial_{yy} \theta \partial_{xyy} \theta \, dx \, dy \right| \\
&\quad \le \frac{\kappa_1}{18} \|\Delta \partial_x \theta\|_{L^2}^2 + C\|u_1\|_{L^2}^2 \|\partial_y u_1\|_{L^2}^2 \|\Delta \theta\|_{L^2}^2. \qquad (3.60)
\end{aligned}$$

After combining all inequalities, together with Gronwall's inequality, we obtain the H^1-bound for ω, $\nabla\theta$. Therefore, we obtain the global H^2 bound for (u,θ) for the 2D Bénard system (1.2)–(1.3) with vertical dissipation in the horizontal velocity equation and horizontal dissipation in the temperature equation. We thus complete the proof of Theorem 3.2. □

4 Global existence and regularity criteria of weak solution for Bénard system with other partial dissipation

We devote this section to showing the global weak solution and regularity criteria for the 2D Bénard system with other cases for partial viscosity and thermal diffusivity. More precisely, we shall show the following cases and theorems.

Theorem 4.1 *Let* (i) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 > 0$, $\mu_4 = 0$, $\kappa_1 = 0$, $\kappa_2 > 0$; (ii) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 > 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$. *Suppose that* $u_1^0, u_2^0, \theta^0 \in H^1(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *Then the problem* (1.2)–(1.3) *with* (i) *and* (ii) *admits a global weak solution* (u_1, u_2, θ), *which obeys*

$$\begin{aligned}
&u_1, u_2, \theta \in L^\infty\big([0,T]; H^1\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_2, \partial_y \theta \in L^2\big([0,T]; H^1\big(\mathbb{R}^2\big)\big); \\
&u_1, u_2, \theta \in L^\infty\big([0,T]; H^1\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_2, \partial_x \theta \in L^2\big([0,T]; H^1\big(\mathbb{R}^2\big)\big)
\end{aligned}$$

for any $T > 0$, *respectively. Moreover, suppose that* $u_1^0, u_2^0, \theta^0 \in H^2(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *If one of the following two conditions holds*:

$$\int_0^T \big\|\partial_{yy} u_1(\tau)\big\|_{L^4(\mathbb{R}^2)} \, d\tau < \infty; \qquad (4.1)$$

$$\int_0^T \big\|\partial_y u_1(\tau)\big\|_{L^\infty(\mathbb{R}^2)} \, d\tau < \infty \qquad (4.2)$$

for any fixed $T > 0$, *then the problem* (1.2)–(1.3) *with* (i) *and* (ii) *admits a global classical solution* (u_1, u_2, θ), *which, respectively, obeys*

$$\begin{aligned}
&u_1, u_2, \theta \in L^\infty\big([0,T); H^2\big(\mathbb{R}^2\big)\big),\\
&\partial_{xxy} u_2 \in L^2\big([0,T); L^2\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_2, \partial_y \theta \in L^2\big([0,T); H^2\big(\mathbb{R}^2\big)\big);\\
&u_1, u_2, \theta \in L^\infty\big([0,T); H^2\big(\mathbb{R}^2\big)\big),\\
&\partial_{xxy} u_2 \in L^2\big([0,T); L^2\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_2, \partial_x \theta \in L^2\big([0,T); H^2\big(\mathbb{R}^2\big)\big).
\end{aligned}$$

Theorem 4.2 *Let* (iii) $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 > 0$, $\kappa_2 = 0$; (iv) $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 = 0$, $\kappa_2 > 0$. *Suppose that* $u_1^0, u_2^0, \theta^0 \in H^1(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *Then the problem* (1.2)–(1.3) *with* (iii) *and* (iv) *admits a global weak solution* (u_1, u_2, θ), *which obeys*

$$\begin{aligned}
&u_1, u_2, \theta \in L^\infty\big([0,T); H^1\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_1, \partial_x \theta \in L^2\big([0,T); H^1\big(\mathbb{R}^2\big)\big);\\
&u_1, u_2, \theta \in L^\infty\big([0,T); H^1\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_1, \partial_y \theta \in L^2\big([0,T); H^1\big(\mathbb{R}^2\big)\big)
\end{aligned}$$

for any $T > 0$, *respectively. Furthermore, suppose that* $u_1^0, u_2^0, \theta^0 \in H^2(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *If one of the following three conditions holds*:

$$\int_0^T \big\|\partial_{yy} u_2(\tau)\big\|_{L^4(\mathbb{R}^2)}\, d\tau < \infty; \tag{4.3}$$

$$\int_0^T \big\|\partial_{xy} u_1(\tau)\big\|_{L^4(\mathbb{R}^2)}\, d\tau < \infty; \tag{4.4}$$

$$\int_0^T \big\|\partial_{y} u_2(\tau)\big\|_{L^\infty(\mathbb{R}^2)}\, d\tau < \infty \tag{4.5}$$

for any fixed $T > 0$, *then the problem* (1.2)–(1.3) *with* (iii) *and* (iv) *admits a global classical solution* (u_1, u_2, θ), *which, respectively, obeys*

$$\begin{aligned}
&u_1, u_2, \theta \in L^\infty\big([0,T); H^2\big(\mathbb{R}^2\big)\big),\\
&\partial_{xxy} u_1 \in L^2\big([0,T); L^2\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_1, \partial_x \theta \in L^2\big([0,T); H^2\big(\mathbb{R}^2\big)\big);\\
&u_1, u_2, \theta \in L^\infty\big([0,T); H^2\big(\mathbb{R}^2\big)\big),\\
&\partial_{xxy} u_1 \in L^2\big([0,T); L^2\big(\mathbb{R}^2\big)\big), \qquad \partial_x u_1, \partial_y \theta \in L^2\big([0,T); H^2\big(\mathbb{R}^2\big)\big).
\end{aligned}$$

Theorem 4.3 *Let* (v) $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\kappa_1 = 0$, $\kappa_2 > 0$. *Suppose that* $u_1^0, u_2^0, \theta^0 \in H^1(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *Then the problem* (1.2)–(1.3) *with* (v) *admits a global weak solution* (u_1, u_2, θ), *which obeys*

$$u_1, u_2, \theta \in L^\infty\big([0,T); H^1\big(\mathbb{R}^2\big)\big), \qquad \partial_y u_1, \partial_y \theta \in L^2\big([0,T); H^1\big(\mathbb{R}^2\big)\big)$$

for any $T > 0$. *Moreover, suppose that* $u_1^0, u_2^0, \theta^0 \in H^2(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *If one of the following two conditions holds*:

$$\int_0^T \big\|\partial_{xx} u_2(\tau)\big\|_{L^4(\mathbb{R}^2)}\, d\tau < \infty; \tag{4.6}$$

$$\int_0^T \big\|\partial_{x} u_2(\tau)\big\|_{L^\infty(\mathbb{R}^2)}\, d\tau < \infty \tag{4.7}$$

for any fixed $T > 0$, *then the problem* (1.2)–(1.3) *with* (v) *admits a global classical solution* (u_1, u_2, θ), *which obeys*

$$u_1, u_2, \theta \in L^\infty\big([0,T); H^2(\mathbb{R}^2)\big),$$
$$\partial_{xyy} u_1 \in L^2\big([0,T); L^2(\mathbb{R}^2)\big), \qquad \partial_y u_1, \partial_y \theta \in L^2\big([0,T); H^2(\mathbb{R}^2)\big).$$

Theorem 4.4 *Let* (vi) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 > 0$, $\kappa_1 > 0$, $\kappa_2 = 0$ *and* (vii) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 > 0$, $\kappa_1 = 0$, $\kappa_2 > 0$. *Suppose that* $u_1^0, u_2^0, \theta^0 \in H^1(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *Then the problem* (1.2)–(1.3) *with* (vi) *and* (vii) *admits a global weak solution* (u_1, u_2, θ), *which obeys*

$$u_1, u_2, \theta \in L^\infty\big([0,T); H^1(\mathbb{R}^2)\big), \qquad \partial_y u_2, \partial_x \theta \in L^2\big([0,T); H^1(\mathbb{R}^2)\big);$$
$$u_1, u_2, \theta \in L^\infty\big([0,T); H^1(\mathbb{R}^2)\big), \qquad \partial_y u_2, \partial_y \theta \in L^2\big([0,T); H^1(\mathbb{R}^2)\big)$$

for any $T > 0$, *respectively. Furthermore, suppose that* $u_1^0, u_2^0, \theta^0 \in H^2(\mathbb{R}^2)$ *and* $\nabla \cdot u^0 = 0$. *If one of the following three conditions holds*:

$$\int_0^T \big\|\partial_{xx} u_1(\tau)\big\|_{L^4(\mathbb{R}^2)}\, d\tau < \infty; \tag{4.8}$$

$$\int_0^T \big\|\partial_{xy} u_2(\tau)\big\|_{L^4(\mathbb{R}^2)}\, d\tau < \infty; \tag{4.9}$$

$$\int_0^T \big\|\partial_x u_1(\tau)\big\|_{L^\infty(\mathbb{R}^2)}\, d\tau < \infty \tag{4.10}$$

for any fixed $T > 0$, *then the problem* (1.2)–(1.3) *admits a global classical solution* (u_1, u_2, θ), *which, respectively, obeys*

$$u_1, u_2, \theta \in L^\infty\big([0,T); H^2(\mathbb{R}^2)\big),$$
$$\partial_{xyy} u_2 \in L^2\big([0,T); L^2(\mathbb{R}^2)\big), \qquad \partial_y u_2, \partial_x \theta \in L^2\big([0,T); H^2(\mathbb{R}^2)\big);$$
$$u_1, u_2, \theta \in L^\infty\big([0,T); H^2(\mathbb{R}^2)\big),$$
$$\partial_{xyy} u_2 \in L^2\big([0,T); L^2(\mathbb{R}^2)\big), \qquad \partial_y u_2, \partial_y \theta \in L^2\big([0,T); H^2(\mathbb{R}^2)\big).$$

Due to those theorems' proofs being similar to the results of Sect. 3, we can leave the proofs of those theorems to the interested readers.

5 Conclusion

The Bénard fluid problem is a very classical problem in the fluid dynamics area. The global existence or non-existence of the classical solution to an inviscid Bénard system is an open and challenging problem, even in the two-dimensional case. Therefore, it is of interest to consider the Bénard system with partial viscosity. Inspired by recent work [13, 31, 32], we first consider the global weak solution for the 2D Bénard system with partial dissipation. Secondly, we establish some regularity criteria for the corresponding system.

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List of abbreviations
We denote by $L^p = L^p(\mathbb{R}^2)$ the usual Lebesgue space, and by $H^n = \{u \in L^2(\mathbb{R}^2) | \nabla^n u \in L^2\}$ the usual Sobolev space.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LM conceived and designed the work. LM drafted the manuscript. LM revised the manuscript. LM and LZ read and approved the final version.

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