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The optimal control problem with necessity condition for a viscous shallow water equation

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Abstract

The optimal control problem for a shallow water equation with a viscous term is analyzed. The existence of optimal control to the control problem is investigated. The necessity condition of optimal control is derived by using the first order Gâteaux derivative of cost functional and adjoint equation. The local uniqueness of the optimal control is established by means of the second order Gâteaux derivative of cost functional. The novelty of this paper is that the necessity condition and local uniqueness of optimal control to the problem are obtained with viscous coefficient $\varepsilon > 0$.

Keywords: Shallow water equation; Optimal control; Necessity optimal condition; Local uniqueness

1 Introduction

This paper is concerned with the optimal control problem for a shallow water equation with a viscous term,

$$u_t - u_{xxt} + 2ku_x - \varepsilon(u_{xx} - u_{xxxx}) + muu_x = au_xu_{xx} + buu_{xxx}, \tag{1.1}$$

where k is a constant, $m, a, b \in \mathbb{R}$, $\Omega = [0,1] \subset \mathbb{R}$, $(t,x) \in \mathbb{R}^+ \times \Omega$, $\varepsilon(u_{xx} - u_{xxxx})$ is the viscous term and $\varepsilon > 0$ is the viscous coefficient.

We give a brief overview of a variety of related work in the literature. Constantin [1] derived the shallow water equation

$$u_t - u_{xxt} + 2ku_x + muu_x = au_xu_{xx} + buu_{xxx}, \tag{1.2}$$

where $u(t,x)$ is the fluid velocity at time t in x direction and k is a constant related to the critical shallow water wave speed. They established the local well-posedness for the Cauchy problem of Eq. (1.2) and wave breaking phenomena of solutions. Lai [2] investigated the local well-posedness for the Cauchy problem of Eq. (1.2) in the Sobolev space $H^s(\mathbb{R})$ $(s > \frac{3}{2})$.

Taking $\varepsilon = 0, m = a + b$ in Eq. (1.1) yields a generalized shallow water equation. Lai [3] obtained the global existence of strong solutions and blow-up criterion of solutions to the Cauchy problem. For the case $b = 1$ in (1.1), Holm [4] not only studied the effects of balance parameter a and kernel function of solitary wave structures but also investigated their interactions analytically with $\varepsilon = 0$ and numerically with small viscosity $\varepsilon \neq 0$. Zhang [5] studied the optimal control problem for the generalized shallow water equation with a viscous term, which includes the viscous Camassa–Holm equation and viscous Degasperis–Procesi equation as special case. The optimal control and existence of optimal solution to the control problem are presented. Shen [6] investigated the optimal control problem for the θ-equation. The necessity optimal condition of optimal control to the control problem in fixed final horizon case is obtained by using functional analytical approach. In particular, taking $\varepsilon = 0, m = 3, a = 2, b = 1$ in (1.1), we obtain the classical Camassa–Holm equation, which models the propagation of shallow water waves. For the methods to establish local well-posedness for the Cauchy problem of Camassa–Holm equation and global existence of solutions, one may refer to [7–9] and the references therein. Tian [10] studied the optimal control problem for a generalized viscous Camassa–Holm equation. They established the existence and uniqueness of local weak solutions by using the Galerkin method. The optimal control and existence of optimal solution were obtained. Shen [11–13] studied the optimal control problem for a generalized viscous shallow water equation. Zong [14] investigated the boundary stabilization for the viscous Camassa–Holm equation and nonviscous Camassa–Holm equation. The nonlinear boundary control laws and global asymptotical stabilization for the control problem are analyzed. If we take $\varepsilon = 0, m = 4, a = 3, b = 1$ in (1.1), we obtain the classical Degasperis–Procesi equation. The local well-posedness for the Cauchy problem of the Degasperis–Procesi equation and blow-up mechanism of solutions were studied in [15, 16]. Tian [17] investigated the optimal control problem for a viscous Degasperis–Procesi equation by using the Galerkin method and optimal control theory of distributed parameter system.

However, the nonlinear partial differential equations created to model physical processes play an important role in almost all branches of mathematics. One may see more details in [18]. The well-posedness for the Cauchy problems and properties of solutions to the equations have been studied extensively. For example, Goubet and Hamraoui [19] investigated both numerically and theoretically the influence of a defect on the blow-up of radial solutions to the cubic nonlinear Schrodinger equation in two dimension. On the other hand, many researchers use the techniques in [20] to the study of the optimal control problems for fluids models [21–30]. The optimal control problems for the Dullin–Gottwald–Holm equation were studied in [31–33], which is similar in structure to the Camassa–Holm equation and the Degasperis–Procesi equation. Hwang [33] obtained the necessity optimal condition of optimal control to the control problem. The local uniqueness of optimal control was established by using the second order Gâteaux differentiability of cost functional. Zhao and Liu [22] investigated the existence of optimal control and optimal solution to the control problem for convective Cahn–Hilliard equation in three dimension. The first order necessity optimal condition of optimal control was presented. Leszczynski et al. [29] considered the optimal control problem for a general mathematical model of the drug treatment with a single agent. The sufficient condition for the strong local optimality of an extremal controlled trajectory was given. Papageorgiou et al. [30] presented the sensitivity analysis for the optimal control problems governed by nonlinear evolution inclusions. The non-emptiness of solution set and continuous selections of solution multifunction were investigated. In general, taking into account the viscous fluid is meaningful in physics. Castro [34] showed that the optimal control to the optimal control problem for the viscous Burgers equation converges to the nonviscous version as the viscosity coefficient tends to zero.

Motivated by the work in [6, 20, 29, 30, 32, 33], we studied the optimal control problem for the shallow water equation with a viscous term

$$\min J(v) = \left\| Cu(v) - z_d \right\|_M^2 + (Nv, v)_U, \quad \text{for all } v \in U, \tag{1.3}$$

where the control v and state $u(v)$ satisfy the distributed control system

$$\begin{cases} y_t(v) + 2ku_x(v) - \varepsilon y_{xx}(v) + (m-a-b)u(v)u_x(v) \\ \quad + au_x(v)y(v) + bu(v)y_x(v) = f + Bv, \quad (t,x) \in [0,T] \times \Omega, \\ u(v;t,x) = u_x(v;t,x) = u_{xx}(v;t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ y(v;0,x) = y_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

where $y_0(x) \in L^2(\Omega)$, $y(v) = u(v) - u_{xx}(v)$. $f(t,x) \in L^2([0,T];H^{-1}(\Omega))$ is the force, $v \in U_{\text{ad}} \subset U$ is the control. The solution $u(v)$ denotes the state of the control problem (1.4). $B \in \mathcal{L}(U, L^2([0,T];H^{-1}(\Omega)))$ is a controller. The observation is $z(v) = Cu(v)$, where $C \in \mathcal{L}(S(0,T), M)$ is an observation operator and M is a Hilbert space of observation variables. The target $z_d \in M$ is a desired value of $u(v)$. $N \in \mathcal{L}(U,U)$ is a self-adjoint, symmetric and positive operator, which satisfies

$$(Nv, v)_U = (v, Nv)_U \geq \nu \|v\|_U^2, \quad \text{for some } \nu > 0, \text{for all } v \in U.$$

Let U_{ad} be an admissible control set, which is a closed convex subset of U. The first term in the cost functional $J(v)$ in (1.3) measures physical objective and the second term is size of control. The control object is to match the desired target z_d by adjusting control v in control volume $[0,T] \times \Omega$. An element $v_0 \in U$ which attains the minimum of cost functional $J(v)$ over U_{ad} is called an optimal control to the optimal control problem (1.3).

Firstly, we consider the local well-posedness for the problem

$$\begin{cases} y_t + 2ku_x - \varepsilon y_{xx} + (m-a-b)uu_x \\ \quad + au_x y + buy_x = f, \quad (t,x) \in [0,T] \times \Omega, \\ u(t,x) = u_x(t,x) = u_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ y(0,x) = y_0(x), \quad x \in \Omega, \end{cases} \tag{1.5}$$

in the space $S(0,T)$, where $y = u - u_{xx}$ and $f(t,x)$ is the force function. Secondly, we consider the optimal control problem (1.3).

Notation. Let $V = H_0^1(\Omega), H = L^2(\Omega), H^* = L^2(\Omega), V^* = H^{-1}(\Omega)$. We have the embedding properties $V \hookrightarrow H = H^* \hookrightarrow V^*$ in which each embedding is dense. The inner product in V is $(\phi,\varphi)_V = (\phi_x, \varphi_x)_H$, for all $\phi, \varphi \in V$. For $a \lesssim b$, we mean that there exists a uniform constant C, which may be different on different lines such that $a \leq Cb$. The spaces $W([0,T];V) = \{f | f \in L^2([0,T];V), f_t \in L^2([0,T];V^*)\}$, $S(0,T) = \{f | f \in L^2([0,T];H_0^3(\Omega)), f_t \in L^2([0,T];H_0^1(\Omega))\}$ and $W(H_0^2, L^2) = \{f | f \in L^2([0,T];H_0^2(\Omega)), f_t \in$

$L^2([0,T];L^2(\Omega))\}$ are Hilbert spaces endowed with common inner product. Since the functions in all spaces are over Ω, we drop Ω if there is no ambiguity. As in the convergence case, the symbol $\rightharpoonup$ denotes the weak convergence.

2 Main results

The precise statements of the main results in this paper are listed.

Theorem 2.1 *Assume $f \in L^2([0,T];H^{-1}(\Omega)), u_0 \in H_0^2(\Omega)$. Then problem* (1.5) *admits a unique local solution $u \in S(0,T)$. The solution mapping $p = (u_0,f) \to u(p)$ from $P_0 = H_0^2(\Omega) \times L^2([0,T];H^{-1}(\Omega))$ into $S(0,T)$ is local Lipschitz continuous. For each $p_1 = (u_{10},f_1), p_2 = (u_{20},f_2) \in P_0$,*

$$\begin{aligned} &\|u_1(p_1) - u_2(p_2)\|_{S(0,T)} \\ &\quad \lesssim \|u_{10} - u_{20}\|_{H_0^2(\Omega)} + \|f_1 - f_2\|_{L^2([0,T];H^{-1}(\Omega))}. \end{aligned} \tag{2.1}$$

In addition, if $Bw \in L^2([0,T];H^{-1}(\Omega))$, there exists an optimal control v_0 to the optimal control problem (1.3).

Theorem 2.2 *Assume $Bw,f \in L^2([0,T];H^{-1}(\Omega)), u_0 \in H_0^2(\Omega)$. For the control problem* (1.4), *the solution mapping $v \to u(v)$ from U into $S(0,T)$ is Gâteaux differentiable at $v = v_0$. Let $z = Du(v_0)w$ be the Gâteaux direction derivative of $u(v)$ at $v = v_0$ in direction w, where $w = v - v_0$. Thus $z = Du(v_0)w$ is the unique solution to the problem*

$$\begin{cases} \pounds_t + 2kz_x - \varepsilon\pounds_{xx} + (m-a-b)(zu_x + uz_x) + a(z_x y + u_x\pounds) \\ \quad + b(zy_x + u\pounds_x) = Bw, \quad (t,x) \in [0,T] \times \Omega, \\ z(t,x) = z_x(t,x) = z_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ \pounds(0,x) = 0, \quad x \in \Omega, \end{cases} \tag{2.2}$$

where $y = u(v_0) - u_{xx}(v_0), \pounds = z - z_{xx}$.

Theorem 2.3 *Assume $Bw,f \in L^2([0,T];H^{-1}(\Omega)), u_0 \in H_0^2(\Omega)$. We have*:

(i) *the necessity condition of optimal control v to the optimal control problem* (5.7) *is characterized by* (3.17), (5.9) *and* (5.12);

(ii) *the necessity condition of optimal control v to the optimal control problem* (5.13) *is characterized by* (3.17), (5.15) *and* (5.17).

Theorem 2.4 *Assume $Bw,f \in L^2([0,T];H^{-1}(\Omega)), u_0 \in H_0^2(\Omega)$. If $T = T(v)$ is small, there exists a unique optimal control v to the optimal control problem* (5.13).

The remainder of this paper is organized as follows. The proofs of Theorems 2.1, 2.2, 2.3 and 2.4 are presented in Sects. 3, 4, 5 and 6, respectively. The conclusions in this paper are presented in Sect. 7.

3 Existence and uniqueness of weak solutions

We recall the definition of weak solutions and a related lemma.

Definition 3.1 Let $y_0(x) \in H$. The function $u(t,x) \in S(0,T)$ is a weak solution to problem (1.5) if $y(t,x) \in W([0,T];V)$ and $y(t,x)$ satisfies

$$\begin{aligned}
&\frac{d}{dt}(y,\varphi)_{(-1,1)} + 2k(u_x,\varphi)_2 + \varepsilon(y_x,\varphi_x)_2 + (m-a-b)(uu_x,\varphi)_2 \\
&\quad + a(u_x y,\varphi)_2 + b(uy_x,\varphi)_2 \\
&= (f,\varphi)_{(-1,1)}, \quad \text{for all } \varphi \in V, \text{ for a.e. } t \in [0,T].
\end{aligned} \tag{3.1}$$

Lemma 3.1 ([33]) *Let u satisfy the boundary conditions in* (1.5) *and assume $u - u_{xx} \in W([0,T];V)$. Then we have $\|u\|_{S(0,T)} \lesssim \|u - u_{xx}\|_{W([0,T];V)}$.*

Proof of Theorem 2.1 Using condition $p = (u_0, f) \in P_0$ and the Galerkin method as in [5, 13, 32] with suitable modifications, we deduce that problem (1.5) possesses a unique local solution $u \in S(0,T)$.

We are ready to present the detailed derivation for (2.1). Let $\phi = u_1 - u_2 = u(p_1) - u(p_2)$ and $\Phi = \phi - \phi_{xx}$. Then we have

$$\begin{cases}
\Phi_t + 2k\phi_x - \varepsilon\Phi_{xx} + (m-a-b)(\phi u_{1,x} + u_2\phi_x) + a(\phi_x y_1 + u_{2,x}\Phi) \\
\quad + b(\phi y_{1,x} + u_2\Phi_x) = f_1 - f_2, \quad (t,x) \in [0,T] \times \Omega, \\
\phi(t,x) = \phi_x(t,x) = \phi_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\
\Phi(0,x) = \Phi_0(x) = y_{10}(x) - y_{20}(x), \quad x \in \Omega.
\end{cases} \tag{3.2}$$

Multiplying (3.2) by Φ and integrating with respect to x and t on $[0,T] \times \Omega$ yield

$$\begin{aligned}
&\frac{1}{2}\|\Phi\|_{L^2}^2 + \varepsilon\int_0^T \|\Phi_x\|_{L^2}^2\,dt \\
&= \frac{1}{2}\|\Phi_0\|_{L^2}^2 - \int_0^T (2k\phi_x, \Phi)\,dt \\
&\quad - \int_0^T (m-a-b)(\phi u_{1,x} + u_2\phi_x, \Phi)\,dt - \int_0^T a(\phi_x y_1 + u_{2,x}\Phi, \Phi)\,dt \\
&\quad - \int_0^T b(\phi y_{1,x} + u_2\Phi_x, \Phi)\,dt + \int_0^T (f_1 - f_2, \Phi)_{(-1,1)}\,dt.
\end{aligned} \tag{3.3}$$

Using the fact $y_1, y_2 \in W([0,T];V)$ yields

$$\begin{aligned}
&\frac{1}{2}\|\Phi\|_{L^2}^2 + \varepsilon\int_0^T \|\Phi_x\|_{L^2}^2\,dt \\
&\le \frac{1}{2}\|\Phi_0\|_{L^2}^2 + C\int_0^T \|\Phi\|_{L^2}^2\bigl(1 + \|y_1\|_{H^1}\bigr)\,dt \\
&\quad + C\|f_1 - f_2\|_{L^2([0,T];V^*)}^2 + \frac{\varepsilon}{2}\|\Phi_x\|_{L^2}^2.
\end{aligned} \tag{3.4}$$

Applying the Gronwall inequality, we obtain

$$\|\Phi\|_{L^2}^2 + \int_0^T \|\Phi_x\|_{L^2}^2\,dt \lesssim \|\Phi_0\|_{L^2}^2 + \|f_1 - f_2\|_{L^2([0,T];V^*)}^2. \tag{3.5}$$

Using the first equation in (3.2) gives rise to

$$\|\Phi_t\|_{H^{-1}} \lesssim \|\Phi_x\|_{L^2} + C\|y_{1,x}\|_{L^2}\|\Phi\|_{L^2} + C\|f_1 - f_2\|_{V^*}. \tag{3.6}$$

Taking into account (3.5) and (3.6), we have

$$\|\Phi_t\|^2_{L^2([0,T];H^{-1})} \lesssim \|\Phi_0\|^2_{L^2} + \|f_1 - f_2\|^2_{L^2([0,T];V^*)}. \tag{3.7}$$

It follows that

$$\|\Phi\|_{W([0,T];V)} \lesssim \|\Phi_0\|_{L^2} + \|f_1 - f_2\|_{L^2([0,T];V^*)}. \tag{3.8}$$

Applying Lemma 3.1 yields

$$\|\phi\|_{S(0,T)} \lesssim \|u_{10} - u_{20}\|_{H^2_0} + \|f_1 - f_2\|_{L^2([0,T];V^*)}. \tag{3.9}$$

We prove the existence of optimal control v_0 to the optimal control problem (1.3).

Let $J = \inf_{v\in U_{\mathrm{ad}}} J(v)$. We bear in mind that U_{ad} is not empty. Then there exists a sequence $\{v_n\} \subset U$ such that

$$\inf_{v\in U_{\mathrm{ad}}} J(v) = \lim_{n\to\infty} J(v_n) = J. \tag{3.10}$$

Hence $\{J(v_n)\}$ is bounded. We deduce that there exists a constant $K_0 > 0$ such that

$$\nu\|v_n\|^2_U \le (Nv_n, v_n)_U \le J(v_n) \le K_0, \tag{3.11}$$

which derives that $\{v_n\}$ is bounded in U. Applying the property that U_{ad} is closed and convex, we choose a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that $v_n \rightharpoonup v_0$ in U as $n \to \infty$.

Let the state $u_n = u(v_n) \in S(0,T)$ corresponding to control v_n be solution to problem

$$\begin{cases} y_{n,t} + 2ku_{n,x} - \varepsilon y_{n,xx} + (m-a-b)u_n u_{n,x} + au_{n,x}y_n + bu_n y_{n,x} \\ \quad = f + Bv_n, \quad (t,x) \in [0,T] \times \Omega, \\ u_n(t,x) = u_{n,x}(t,x) = u_{n,xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ y_n(0,x) = y_{n0}(x), \quad x \in \Omega, \end{cases} \tag{3.12}$$

where $y_n = u_n - u_{n,xx}$. Using (3.12), we obtain

$$\begin{aligned} \|Bv_n\|_{L^2([0,T];V^*)} &\le \|B\|_{\mathcal{L}(U,L^2([0,T];V^*))}\|v_n\|_U \\ &\le \|B\|_{\mathcal{L}(U,L^2([0,T];V^*))}\sqrt{K_0\nu^{-1}}. \end{aligned} \tag{3.13}$$

Bearing in mind (3.8) gives rise to the inequality

$$\|y_n\|_{W([0,T];V)} \lesssim \|u_0\|_{H^2} + \|f\|_{L^2([0,T];V^*)} + K_1. \tag{3.14}$$

Applying Lemma 3.1 yields

$$\|u_n\|_{S(0,T)} \lesssim \|u_0\|_{H^2} + \|f\|_{L^2([0,T];V^*)} + K_1. \tag{3.15}$$

There exists a subsequence of $\{y_n\}$, denoted by $\{y_{n_k}\}$, and a function $y = u - u_{xx} \in W([0,T];V)$ such that $y_{n_k} \rightharpoonup y$ in $W([0,T];V)$. Using the fact that $H_0^1 \hookrightarrow L^2$ is compact, we deduce that there exists a subsequence of $\{y_n\}$, denoted by $\{y_{n_k}\}$, such that $y_{n_k} \to y$ in $L^2([0,T];L^2)$. Since the embedding $W([0,T];V) \hookrightarrow C([0,T];L^2)$ is compact, we deduce $u_n \in C([0,T];H_0^2)$. Then there exists a subsequence of $\{u_n\}$, denoted by $\{u_{n_k}\}$, such that $u_{n_k} \to u$ in H_0^2, for a.e. $t \in [0,T]$. Hence

$$\begin{aligned} &u_{n_k,x}y_{n_k} \to u_x y, \quad \text{in } L^2\big([0,T];L^2\big),\\ &u_{n_k}y_{n_k,x} \rightharpoonup u y_x, \quad \text{in } L^2\big([0,T];L^2\big),\\ &u_{n_k,x}u_{n_k} \to u_x u, \quad \text{in } L^2\big([0,T];L^2\big), \end{aligned} \tag{3.16}$$

as $k \to \infty$. We replace u_n, y_n by u_{n_k}, y_{n_k} in (3.12), respectively. Taking $k \to \infty$ shows that the limit function y satisfies

$$\begin{cases} y_t + 2ku_x - \varepsilon y_{xx} + (m-a-b)uu_x + au_x y + buy_x = f + Bv_0, \\ \quad (t,x) \in [0,T] \times \Omega, \\ u(t,x) = u_x(t,x) = u_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ y(0,x) = y_0(x), \quad x \in \Omega, \end{cases} \tag{3.17}$$

in the weak solution sense.

From Theorem 2.1, we obtain the uniqueness of weak solutions to problem (3.17). Then we deduce that $u = u(v_0) \in S(0,T)$ and $u(v_n) \rightharpoonup u(v_0)$ in $S(0,T)$. The operator C is continuous on $S(0,T)$ and $\|\cdot\|_M$ is lower semicontinuous. Hence

$$\big\|Cu(v_0) - z_d\big\|_M \le \liminf_{n\to\infty} \big\|Cu(v_n) - z_d\big\|_M. \tag{3.18}$$

It deduces from $\liminf_{n\to\infty} \|N^{\frac{1}{2}}v_n\|_U \ge \|N^{\frac{1}{2}}v_0\|_U$ that $\liminf_{n\to\infty}(Nv_n, v_n)_U \ge (Nv_0, v_0)_U$. Then $J = \liminf_{n\to\infty} J(v_n) \ge J(v_0)$. Meanwhile, from (3.10), we derive $J(v_0) \ge J$. Hence $J(v_0) = \inf_{v\in U_{\mathrm{ad}}} J(v)$. This completes the proof of Theorem 2.1. □

4 The proof of Theorem 2.2

From Theorem 2.1, we define the unique solution map $v \to u(v)$ from U into $S(0,T)$. Let $DJ(v_0)$ be the Gâteaux derivative of $J(v)$ defined in (1.3) at $v = v_0$. We intend to investigate the necessity optimal condition of optimal control

$$DJ(v_0)(v - v_0) \ge 0, \quad \text{for all } v \in U_{\mathrm{ad}}. \tag{4.1}$$

We use the adjoint equation (5.2) of (1.4) to give detail expression for (4.1).

The definition of the Gâteaux differentiability of solution mapping is presented.

Definition 4.1 ([33]) For the control problem (1.4), the solution map $v \to u(v)$ from U into $S(0,T)$ is Gâteaux differentiable at $v = v_0$, if for all $w \in U$, there exists $Du(v_0) \in \mathcal{L}(U, S(0,T))$ such that

$$\left\| \frac{u(v_0 + \lambda w) - u(v_0)}{\lambda} - Du(v_0)w \right\|_{S(0,T)} \to 0 \quad \text{as } \lambda \to 0. \tag{4.2}$$

The operator $Du(v_0)$ is the Gâteaux derivative of $u(v)$ at $v = v_0$ and the function $Du(v_0)w \in S(0,T)$ is the Gâteaux direction derivative of $u(v)$ at $v = v_0$ in direction $w \in U$.

Proof of Theorem 2.2 Let $\lambda \in (-1,0) \cup (0,1)$, $w = v - v_0$ and $z_\lambda = \frac{u(v_0+\lambda w)-u(v_0)}{\lambda}$. Using (1.4) and (3.17), we deduce that z_λ satisfies

$$\begin{cases} £_{\lambda,t} + 2kz_{\lambda,x} - \varepsilon £_{\lambda,xx} + (m-a-b)(z_\lambda u_{\lambda,x} + uz_{\lambda,x}) \\ \quad + a(z_{\lambda,x} y_\lambda + u_x £_\lambda) + b(z_\lambda y_{\lambda,x} + u £_{\lambda,x}) = Bw, \quad (t,x) \in [0,T] \times \Omega, \\ z_\lambda(t,x) = z_{\lambda,x}(t,x) = z_{\lambda,xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ £_\lambda(0,x) = 0, \quad x \in \Omega, \end{cases} \tag{4.3}$$

where $u_\lambda = u(v_0 + \lambda w)$, $y_\lambda = u_\lambda - u_{\lambda,xx}$ and $£_\lambda = z_\lambda - z_{\lambda,xx}$.

From Theorem 2.1, we have

$$\left\| u(v_0 + \lambda w) - u(v_0) \right\|_{S(0,T)} \le C_1 |\lambda| \|Bw\|_{L^2([0,T];V^*)}. \tag{4.4}$$

Hence

$$\|z_\lambda\|_{S(0,T)} \lesssim \|Bw\|_{L^2([0,T];V^*)} < \infty. \tag{4.5}$$

We deduce that there exist $z \in S(0,T)$ and a sequence $\{\lambda_k\} \subset (-1,1) \to 0$ such that $z_{\lambda_k} \rightharpoonup z$ in $S(0,T)$ as $k \to \infty$. Using the Aubin compact lemma gives rise to $z_{\lambda_k} \to z$ in H_0^2, for a.e. $t \in [0,T]$. From the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} & z_{\lambda_k} u_{\lambda_k,x} \to z u_x, \quad \text{in } L^2([0,T];L^2), \\ & z_{\lambda_k,x} y_{\lambda_k} \to z_x y, \quad \text{in } L^2([0,T];L^2), \\ & z_{\lambda_k} y_{\lambda_k,x} \to z y_x, \quad \text{in } L^2([0,T];L^2), \\ & £_{\lambda_k} \rightharpoonup £, \quad \text{in } L^2([0,T];H_0^1), \end{aligned} \tag{4.6}$$

as $k \to \infty$, where $y = u(v_0) - u_{xx}(v_0)$, $£ = z - z_{xx}$.

Therefore $£_{\lambda_k,t} \rightharpoonup £_t$ in $L^2([0,T];H^{-1})$. Then $z_\lambda \rightharpoonup z = Du(v_0)w$ in $S(0,T)$ as $\lambda \to 0$, where z is the solution to problem (2.2).

In what follows we present the derivation that $z_\lambda \to z = Du(v_0)w$ in $S(0,T)$ as $\lambda \to 0$.

Let $\phi_\lambda = z_\lambda - z$ and $\Phi_\lambda = \phi_\lambda - \phi_{\lambda,xx}$. From (2.2) and (4.3), we derive

$$\begin{cases} \Phi_{\lambda,t} + 2k\phi_{\lambda,x} - \varepsilon \Phi_{\lambda,xx} + (m-a-b)u\phi_{\lambda,x} + au_x \Phi_\lambda + bu\Phi_{\lambda,x} = \theta(\lambda), \\ \quad (t,x) \in [0,T] \times \Omega, \\ \phi_\lambda(t,x) = \phi_{\lambda,x}(t,x) = \phi_{\lambda,xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ \Phi_\lambda(0,x) = 0, \quad x \in \Omega, \end{cases} \tag{4.7}$$

where $\theta(\lambda) = -[(m-a-b)(z_\lambda u_{\lambda,x} - zu_x) + a(z_{\lambda,x}y_\lambda - z_x y) + b(z_\lambda y_{\lambda,x} - zy_x)]$.

Bearing in mind (4.6) shows that $\theta(\lambda) \to 0$ in $L^2([0,T];L^2)$ as $\lambda \to 0$.

We need to establish the estimates for ϕ_λ. Multiplying (4.7) by Φ_λ and using integration by parts, we obtain

$$\|\Phi_\lambda\|_{L^2}^2 + \int_0^T \|\Phi_{\lambda,x}\|_{L^2}^2\,dt \le C_2 \big\|\theta(\lambda)\big\|^2_{L^2([0,T];L^2)}. \tag{4.8}$$

Then we have

$$\Phi_\lambda \to 0 \quad \text{in } C\big([0,T];L^2\big) \cap L^2\big([0,T];H^1\big) \quad \text{as } \lambda \to 0. \tag{4.9}$$

Using (4.7) and (4.9) gives rise to

$$\Phi_{\lambda,t} \to 0 \quad \text{in } L^2\big([0,T];H^{-1}\big) \text{ as } \lambda \to 0. \tag{4.10}$$

Thus we obtain $\Phi_\lambda \to 0$ in $W([0,T];V)$. Applying Lemma 3.1 yields $z_\lambda \to z$ in $S(0,T)$. We complete the proof of Theorem 2.2. □

5 Necessity optimal condition of optimal control

We are in the position to present the necessity optimal condition of optimal control to the optimal control problem (1.3).

Theorem 2.2 implies that the cost functional $J(v)$ is Gâteaux differentiable at $v = v_0$ in the direction $v - v_0$. Using

$$DJ(v_0)w = \lim_{k\to 0} \frac{J(v_0 + kw) - J(v_0)}{k}$$

and $J(v_0) = (Cu(v_0) - z_d, Cu(v_0) - z_d)_M + (Nv_0, v_0)_U$, $w = v - v_0$, we have

$$\begin{aligned}
&DJ(v_0)w \\
&\quad= \lim_{k\to 0}\frac{1}{k}\Big[\Big(\big(Cu(v_0+kw) - z_d, Cu(v_0+kw) - z_d\big)_M \\
&\qquad + \big(N(v_0+kw), v_0+kw\big)_U\Big) \\
&\qquad - \Big(\big(Cu(v_0) - z_d, Cu(v_0) - z_d\big)_M + (Nv_0, v_0)_U\Big)\Big] \\
&\quad= \lim_{k\to 0}\frac{1}{k}\Big[\big(Cu(v_0+kw) - Cu(v_0), Cu(v_0+kw) - z_d\big)_M\Big] \\
&\qquad + \lim_{k\to 0}\frac{1}{k}\Big[\big(N(v_0+kw) - Nv_0, v_0\big)_U\Big] \\
&\qquad + \lim_{k\to 0}\frac{1}{k}\Big[\big(Cu(v_0), Cu(v_0+kw) - Cu(v_0)\big)_M\Big] \\
&\qquad + \lim_{k\to 0}\frac{1}{k}\Big[\big(v_0, N(v_0+kw) - Nv_0\big)_U\Big] \\
&\quad= 2\Big[\big(Cu(v_0) - z_d, C\big(Du(v_0)(v - v_0)\big)\big)_M + (Nv_0, v - v_0)_U\Big].
\end{aligned}$$

Let Λ be the isomorphism mapping from M onto M^*. Applying (4.1), we rewrite the necessity optimal condition of the optimal control as

$$\big(C^*\Lambda\big(Cu(v_0)-z_d\big), Du(v_0)(v-v_0)\big)_{(S^*(0,T),S(0,T))} + (Nv_0, v-v_0)_U \geq 0, \tag{5.1}$$

for all $v \in U_{\mathrm{ad}}$.

Similar to the methods in [20], we derive the necessity optimal condition via the adjoint equation,

$$\begin{cases} -P_t - 2kp_x - \varepsilon P_{xx} + (m-a-b)[pu_x - (up)_x] \\ \quad + a[-(py)_x + (1-\partial_x^2)(u_x p)] \\ \quad + b[py_x - (1-\partial_x^2)(up)_x] = f_3, \quad (t,x) \in [0,T] \times \Omega, \\ p(v_0;t,x) = p_x(v_0;t,x) = p_{xx}(v_0;t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ P(T,x) = 0, \quad x \in \Omega, \end{cases} \tag{5.2}$$

where $P = p(v_0;t,x) - p_{xx}(v_0;t,x)$, $f_3 = C^*\Lambda(Cu(v_0) - z_d)$.

The local well-posedness for problem (5.2) is given by the following lemma.

Lemma 5.1 *Assume $C^*\Lambda(Cu(v_0) - z_d) \in L^2([0,T];H^{-2}(\Omega))$ and reverse the direction of time $t \to T-t$ in* (5.2). *Problem* (5.2) *admits a unique solution $p(v_0)$ satisfying*

(1) $p(v_0) \in W\big(H_0^2, L^2\big)$;

(2) $$\begin{aligned} &\big(p_t - \varepsilon p_{xx} + (a-b)u_{2,x}p - bu_2 p_x, \Phi\big) \\ &\quad + \big(-2kp_x + (m-a-b)pu_{1,x} + bpy_{1,x}, \phi\big) \\ &\quad + \big((m-a-b)u_2 p + apy_1, \phi_x\big) \\ &= \big(C^*\Lambda\big(Cu(v_0)-z_d\big), \phi\big)_{(-2,2)}, \quad \textit{for all } \phi \in H_0^2; \end{aligned}$$

(3) $P(0,x) = 0$,

where $P = p(v_0) - p_{xx}(v_0)$.

Proof of Lemma 5.1 Let $p(v_0) = p$. By reversing the time $t \to T-t$, we change problem (5.2) into

$$\begin{cases} P_t - 2kp_x - \varepsilon P_{xx} + (m-a-b)[pu_x - (up)_x] \\ \quad + a[-(py)_x + (1-\partial_x^2)(u_x p)] \\ \quad + b[py_x - (1-\partial_x^2)(up)_x] = f_3, \quad (t,x) \in [0,T] \times \Omega, \\ p(t,x) = p_x(t,x) = p_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ P(0,x) = 0, \quad x \in \Omega. \end{cases} \tag{5.3}$$

We use the Galerkin method as in [13, 32] to establish the local well-posedness for problem (5.3). We present the main derivations.

Multiplying (5.3) by p and integrating by parts yield

$$\begin{aligned} &\|p\|_{L^2}^2 + \|p_x\|_{L^2}^2 + \int_0^T \left(\|p_x\|_{L^2}^2 + \|p_{xx}\|_{L^2}^2\right) dt \\ &\quad \lesssim \left(1 + \|y_x\|_{L^2}\right)\left(\|p\|_{L^2}^2 + \|p_x\|_{L^2}^2\right) + \|f_3\|_{H^{-2}}^2. \end{aligned} \tag{5.4}$$

Hence, the approximate solution sequence $\{p_n\}$ is uniformly bounded in $L^2([0,T];H_0^2)$. Using the property of operator $(1-\partial_x^2)^{-1}$, (5.3) and (5.4), we deduce that $\{p_{n,t}\}$ is bounded in $L^2([0,T];L^2)$. Thus $\{p_n\}$ is bounded in $W(H_0^2;L^2)$. Applying the Aubin compact lemma, we deduce that there exists a limit function $p \in W(H_0^2;L^2)$, which is the unique solution to problem (5.3). This completes the proof of Lemma 5.1. □

For simplicity, we consider the observations in the following two cases.

(1) Let $M = L^2([0,T]\times\Omega)$, $C_3 \in \mathcal{L}(S(0,T),M)$ and observation

$$z(v) = C_3 u(v) = u(v) \in L^2\left([0,T];L^2\right). \tag{5.5}$$

(2) Assume $M = L^2([0,T]\times\Omega)$, $C_4 \in \mathcal{L}(S(0,T),M)$ and observation

$$z(v) = C_4 u(v) = \left(1-\partial_x^2\right)u(v) = y(v) \in L^2\left([0,T];L^2\right). \tag{5.6}$$

Proof of Theorem 2.3 For the case of observation in (5.5), we consider the optimal control problem

$$\min J(v) = \int_0^T \left\|u(v) - z_d\right\|_{L^2}^2 dt + (Nv,v)_U, \quad \text{for all } v \in \mathcal{U}_{\text{ad}}, \tag{5.7}$$

where $u(v)$ is the state in (1.4).

Let v_0 be the optimal control to the optimal control problem (5.7). Then the necessity optimal condition (5.1) is rewritten into the form

$$\int_0^T \left(u(v_0) - z_d, z\right)_2 dt + (Nv_0, v - v_0)_U \geq 0, \quad \text{for all } v \in \mathcal{U}_{\text{ad}}. \tag{5.8}$$

We consider the adjoint system

$$\begin{cases} -P_t - 2kp_x - \varepsilon P_{xx} + (m-a-b)[pu_x - (up)_x] \\ \quad + a[-(py)_x + (1-\partial_x^2)(u_x p)] \\ \quad + b[py_x - (1-\partial_x^2)(up)_x] = u(v_0) - z_d, \quad (t,x) \in [0,T]\times\Omega, \\ p(t,x) = p_x(t,x) = p_{xx}(t,x) = 0, \quad (t,x) \in [0,T]\times\partial\Omega, \\ P(T,x) = 0, \quad x \in \Omega, \end{cases} \tag{5.9}$$

where $P = p(v_0) - p_{xx}(v_0)$, $y(v_0) = u(v_0) - u_{xx}(v_0)$.

Note that the observation $u(v_0) - z_d \in L^2([0,T]\times\Omega) \subset L^2([0,T];H^{-2})$. Using Lemma 5.1 shows that problem (5.9) admits a unique solution $p(v_0) \in W(H_0^2, L^2)$.

Multiplying (5.9) by $z(t,x)$ and integrating over $[0,T] \times \Omega$, we have

$$\begin{aligned}&\int_0^T \big(-P_t - 2kp_x - \varepsilon P_{xx} + (m-a-b)\big[pu_x - (up)_x\big] \\ &\quad + a\big[-(py)_x + \big(1-\partial_x^2\big)(u_x p)\big] + b\big[py_x - \big(1-\partial_x^2\big)(up)_x\big], z\big)\, dt \\ &= \int_0^T \big(u(v_0) - z_d, z\big)\, dt. \end{aligned} \tag{5.10}$$

Applying (2.2) and (5.10) yields

$$\begin{aligned}&\int_0^T \big(u(v_0) - z_d, z\big)\, dt \\ &= \int_0^T \big(p(v_0), £_t + 2kz_x - \varepsilon £_{xx} + (m-a-b)(u_x z + u z_x) \\ &\quad + a(yz_x + u_x £) + b(y_x z + u £_x)\big)\, dt \\ &= \int_0^T \big(p(v_0), B(v - v_0)\big)\, dt. \end{aligned} \tag{5.11}$$

From (5.10) and (5.11), we see that (5.8) is equivalent to

$$\int_0^T \big(p(v_0), B(v-v_0)\big)_2\, dt + (Nv_0, v - v_0)_U \geq 0, \quad \text{for all } v \in U_{\mathrm{ad}}. \tag{5.12}$$

We complete the proof of case (i) in Theorem 2.3.

For the observation in (5.6), we consider the optimal control problem

$$\min J(v) = \int_0^T \big\| y(v) - z_d \big\|_{L^2}^2\, dt + (Nv, v)_U, \quad \text{for all } v \in U_{\mathrm{ad}}, \tag{5.13}$$

where $y(v) = u(v) - u_{xx}(v)$, $u(v)$ is the state in (1.4).

Similar to (5.8), the necessity optimal condition (5.1) is rewritten as

$$\int_0^T \big(y(v_0) - y_d, £\big)_2\, dt + (Nv_0, v - v_0)_U \geq 0, \quad \text{for all } v \in U_{\mathrm{ad}}. \tag{5.14}$$

We consider the adjoint system

$$\begin{cases} -P_t - 2kp_x - \varepsilon P_{xx} + (m-a-b)[pu_x - (up)_x] \\ \qquad + a[-(py)_x + (1-\partial_x^2)(u_x p)] + b[py_x - (1-\partial_x^2)(up)_x] \\ \quad = (1-\partial_x^2)(y(v_0) - z_d), \quad (t,x) \in [0,T] \times \Omega, \\ p(v_0;t,x) = p_x(v_0;t,x) = p_{xx}(v_0;t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ P(T,x) = 0, \quad x \in \Omega, \end{cases} \tag{5.15}$$

where $P = p(v_0) - p_{xx}(v_0), y(v_0) = u(v_0) - u_{xx}(v_0)$.

Bearing in mind $(1-\partial_x^2)(y(v_0) - z_d) \in L^2([0,T]; H^{-2})$, we deduce from Lemma 5.1 that problem (5.15) admits a unique solution $p(v_0) \in W(H_0^2, L^2)$.

Multiplying (5.15) by $z(t,x)$ and integrating by parts, we obtain

$$\begin{aligned}&\int_0^T \big(-P_t - 2kp_x - \varepsilon P_{xx} + (m-a-b)\big[pu_x - (up)_x\big]\\&\quad + a\big[-(py)_x + \big(1-\partial_x^2\big)(u_x p)\big] + b\big[py_x - \big(1-\partial_x^2\big)(up)_x\big], z\big)\,dt\\&= \int_0^T \big(\big(1-\partial_x^2\big)\big(y(v_0) - z_d\big), z\big)\,dt.\end{aligned} \tag{5.16}$$

Thus, the necessity optimal condition (5.14) is equivalent to

$$\int_0^T \big(p(v_0), B(v - v_0)\big)_2\,dt + (Nv_0, v - v_0)_U \geq 0, \quad \text{for all } v \in U_{\mathrm{ad}}. \tag{5.17}$$

We complete the proof of case (ii) in Theorem 2.3. □

6 Local uniqueness of optimal control

Firstly, we give a lemma on the local uniqueness of optimal control to the optimal control problem (5.13).

Lemma 6.1 *For the control problem* (1.4), *the mapping $v \to u(v)$ from U into $S(0,T)$ is the second order Gâteaux differentiable at $v = v_0$. The second order Gâteaux direction derivative of $u(v)$ at $v = v_0$ in the direction $v - v_0 \in U$, say $g = D^2u(v_0)(v - v_0, v - v_0)$ is the unique solution to the problem*

$$\begin{cases} G_t + 2kg_x - \varepsilon G_{xx} + (m-a-b)(gu_x + 2zz_x + ug_x) \\ \quad + a(g_x y + 2z_x \pounds + u_x G) \\ \quad + b(gy_x + 2z\pounds_x + uG_x) = 0, \quad (t,x) \in [0,T] \times \Omega, \\ g(t,x) = g_x(t,x) = g_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ G(0,x) = 0, \quad x \in \Omega, \end{cases} \tag{6.1}$$

where $G(t,x) = g - g_{xx}$. And g satisfies the estimates

$$\|g\|_{S(0,T)} \lesssim \|v - v_0\|_U^2. \tag{6.2}$$

Proof of Lemma 6.1 The proof of that g is the unique solution to problem (6.1) is similar to the proof of Theorem 2.2. We omit the detail derivation. Using the fact that z is the solution to problem (2.2) gives rise to

$$\begin{aligned}\|\pounds\|_{W([0,T];V)} &\lesssim \big\|B(v - v_0)\big\|_{L^2([0,T];V^*)}\\ &\lesssim \|v - v_0\|_U.\end{aligned} \tag{6.3}$$

Hence

$$\begin{aligned}\|g\|_{S(0,T)} &\lesssim \|G\|_{W([0,T];V)}\\ &\lesssim \big\|2\big[(m-a-b)zz_x + az_x\pounds + bz\pounds_x\big)\big]\big\|_{L^2([0,T];V^*)}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\|2\left[(m-a-b)z\left(1-\partial_x^2\right)^{-1}£_x + az_x£ + bz£_x\right]\right\|_{L^2([0,T];L^2)} \\
&\lesssim \|£\|^2_{W([0,T];V)}.
\end{aligned} \tag{6.4}
$$

From (6.3) and (6.4), we obtain (6.2). □

Proof of Theorem 2.4 We only present the proof for the case of observation in (5.6). The similar result holds for (5.5). We establish the local uniqueness of optimal control by proving the strict convexity of map $v \in U_{\mathrm{ad}} \to J(v)$. Namely, for all $v_1, v_2 \in U_{\mathrm{ad}}$, let $w = v_2 - v_1$, then

$$
D^2 J(v_1 + \theta w)(w,w) > 0, \quad \theta \in (0,1). \tag{6.5}
$$

Let us denote $u(v_1 + \theta(v_2 - v_1)), z(v_1 + \theta(v_2 - v_1)), g(v_1 + \theta(v_2 - v_1))$ by $u(\theta), z(\theta), g(\theta)$, respectively. It follows

$$
\begin{aligned}
DJ(v_1 + \theta w)w &= \lim_{h_1 \to 0} \frac{J(v_1 + \theta w + h_1 w)w - J(v_1 + \theta w)w}{h_1} \\
&= 2\int_0^T \left(y(\theta) - z_d, £(\theta)\right)_2 dt + 2\left(N(v_1 + \theta w), w\right)_U,
\end{aligned} \tag{6.6}
$$

where $y(\theta) = u(\theta) - u_{xx}(\theta), £(\theta) = z(\theta) - z_{xx}(\theta)$.

Using (6.6), we obtain

$$
\begin{aligned}
D^2 J(v_1 + \theta w)(w,w) = {}& \lim_{h_2 \to 0} \frac{DJ(v_1 + \theta w + h_2 w)w - DJ(v_1 + \theta w)w}{h_2} \\
= {}& \lim_{h_2 \to 0} 2\int_0^T \frac{(y(v_1 + \theta w + h_2 w) - z_d, £(v_1 + \theta w + h_2 w))}{h_2} dt \\
& - \lim_{h_2 \to 0} 2\int_0^T \frac{(y(\theta) - z_d, £(\theta))}{h_2} dt \\
& + \lim_{h_2 \to 0} 2 \frac{(N(v_1 + \theta w + h_2 w), w)_U - (N(v_1 + \theta w), w)_U}{h_2} \\
= {}& 2\int_0^T \left(£(\theta), £(\theta)\right)_2 dt + 2\int_0^T \left(y(\theta) - z_d, G(\theta)\right)_2 dt + 2(Nw, w)_U \\
= {}& 2\int_0^T \left(\left(1-\partial_x^2\right)\left(y(\theta) - z_d\right), g(\theta)\right)_{(-2,2)} dt \\
& + 2\int_0^T \left\|£(\theta)\right\|^2_{L^2} dt + 2\left(N(v_2 - v_1), v_2 - v_1\right)_U.
\end{aligned}
$$

Applying Lemma 6.1, we have

$$
\begin{aligned}
& D^2 J(v_1 + \theta w)(w,w) \\
&\quad \geq 2\left[\nu - C_5 T^{\frac{1}{2}} \left\|y(\theta) - z_d\right\|_{L^2([0,T];L^2)}\right] \|v_2 - v_1\|^2_U \\
&\qquad + 2\int_0^T \left\|£(\theta)\right\|^2_{L^2} ds.
\end{aligned} \tag{6.7}
$$

If $T = T(\nu)$ is small, using (6.7) gives rise to (6.5). Hence, we obtain the strict convexity of cost functional $J(\nu)$, where $\nu \in U_{\mathrm{ad}}$. This completes the proof of Theorem 2.4. □

7 Conclusions

In this work, we studied the optimal control problem for a shallow water equation with a viscous term and viscous coefficient $\varepsilon > 0$. The existence of optimal control to the control problem is investigated. The necessity condition of optimal control is derived by using the first order Gâteaux derivative of the cost functional and the adjoint equation. The local uniqueness of optimal control is established by means of the second order Gâteaux derivative of the cost functional. Due to the independence of coefficients m, a and b in (1.4), the nonlinear term uu_x does not disappear after using the transformation $y = u - u_{xx}$, which leads to the difficulty of establishing the estimates for term uu_x. This is the major improvement in comparison with the results in the literature [5, 10, 17, 32], where the problems studied are special cases of the optimal control problem (1.3) in this paper. Moreover, we obtain the necessity condition and local uniqueness of optimal control to the optimal control problem (1.3) by using the Gâteaux derivative of cost functional. This is another novelty of our paper.

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