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Existence of fast homoclinic solutions for a class of second-order damped vibration systems

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Abstract

By applying the mountain pass theorem in critical point theory, the existence of fast homoclinic solutions is obtained for the following second-order damped vibration system:

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0,$$

where $p \in (2, +\infty)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $L(t)$ is a positive definite symmetric matrix-valued function for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is not periodic in t , $a(t)$ is a continuous, positive function on \mathbb{R} and $q: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $Q(t) = \int_0^t q(s) ds$ with $\lim_{|t| \rightarrow +\infty} Q(t) = +\infty$.

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1 Introduction

Consider fast homoclinic solutions of the following second-order system:

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0, \quad (1.1)$$

where $p \in (2, +\infty)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $L(t)$ is a positive definite symmetric matrix-valued function for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is not periodic in t , $a(t)$ is a continuous, positive function on \mathbb{R} , and $q: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $Q(t) = \int_0^t q(s) ds$ with

$$\lim_{|t| \rightarrow +\infty} Q(t) = +\infty. \quad (1.2)$$

When $q(t) \equiv 0$ and $L(t) \equiv 0$, problem (1.1) reduces to the following special second-order Hamiltonian system:

$$\ddot{u}(t) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}. \quad (1.3)$$

In [1, 2], and [3], the authors considered homoclinic solutions for the special Hamiltonian system (1.3) in a weighted Sobolev space and obtained some results by using the mountain pass theorem in critical point theory. For the applications of mountain pass theorem, please see the references [4] and [5]. In [6], Benci and Fortunato investigated a class of nonlinear Dirichlet problems in a weighted Sobolev space.

When $p = 2$ and $L(t) \equiv 0$, problem (1.1) reduces to the following nonlinear second-order damped vibration problem:

$$\ddot{u}(t) + q(t)\dot{u}(t) - a(t)u(t) + \nabla W(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}. \tag{1.4}$$

When $a(t) \equiv 0$ and $q(t) \equiv 0$, problem (1.1) becomes the following second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}. \tag{1.5}$$

As we known, the existence of homoclinic orbits is very important in the study of the behavior of dynamical systems. The first work about homoclinic orbits was done by Poincaré [7].

In the past years, the existence and multiplicity of homoclinic solutions for system (1.5) were investigated by many researchers by using critical point theory. For example, see [8–19], and the references cited therein. For the existence of homoclinic solutions for damped vibration problem (1.4), please see the literature [20–25], and the references cited therein. For other kinds of damped vibration problem, please see the literature [26] and [27]. Besides, by applying the mountain pass theorem and symmetric mountain pass theorem in critical point theory, Zhang and Li [28] investigated the existence and multiplicity of fast homoclinic solutions for a class of nonlinear second-order nonautonomous systems and obtained some results, which generalized and improved problem (1.4).

Motivated mainly by the above mentioned works, we will investigate fast homoclinic solutions for problem (1.1) and establish some results. In the following, we first state some properties of the weighted Sobolev space E and then introduce the concept of fast homoclinic solutions for problem (1.1). On the weighted Sobolev space E , a certain variational functional associated with (1.1) is defined and fast homoclinic solutions are the critical points of the certain functional.

Let

$$X = \left\{ u \in W^{1,2}(\mathbb{R}, \mathbb{R}^N) \mid \int_{\mathbb{R}} e^{Q(t)} [|\dot{u}(t)|^2 + (L(t)u(t), u(t))] dt < +\infty \right\},$$

where $Q(t)$ is defined in (1.2). Then X is a weighted Sobolev space with the norm given by

$$\|u\| = \left(\int_{\mathbb{R}} e^{Q(t)} [|\dot{u}(t)|^2 + (L(t)u(t), u(t))] dt \right)^{1/2}, \quad u, v \in X.$$

It is obvious that

$$X \subset L^2(e^{Q(t)})$$

with the embedding being continuous. Here $L^q(e^{Q(t)})$ ($2 \leq q < +\infty$) denotes the Banach spaces of functions on \mathbb{R} with values in \mathbb{R}^N under the norm

$$\|u\|_q := \left\{ \int_{\mathbb{R}} e^{Q(t)} |u(t)|^q dt \right\}^{1/q}.$$

If σ is a positive, continuous function on \mathbb{R} and $1 < s < +\infty$, let

$$L^s_{\sigma}(e^{Q(t)}) = \left\{ u \in L^1_{loc}(e^{Q(t)}) \mid \int_{\mathbb{R}} \sigma(t) e^{Q(t)} |u(t)|^s dt < +\infty \right\}.$$

L^s_{σ} equipped with the norm

$$\|u\|_{s,\sigma} = \left(\int_{\mathbb{R}} \sigma(t) e^{Q(t)} |u(t)|^s dt \right)^{1/s}$$

is a reflexive Banach space.

Set $E = X \cap L^p_a(e^{Q(t)})$, where a is the function given in condition (A). Then E with its standard norm $\|\cdot\|$ is a reflexive Banach space. Similar to [20–25], the definition of fast homoclinic solutions is given in the following.

Definition 1.1 If (1.2) holds, a solution of (1.1) is called a fast homoclinic solution if $u \in E$.

The main results are given in the following.

Theorem 1.1 *Suppose that q satisfies (1.2), a, L , and W satisfy the following conditions:*

(A) *Let $p > 2$, $a(t)$ is a continuous, positive function on \mathbb{R} such that, for all $t \in \mathbb{R}$,*

$$a(t) \geq \vartheta |t|^\beta, \quad \vartheta > 0, \beta > (p - 2)/2.$$

(L) *$L \in C(\mathbb{R}, \mathbb{R}^N \times \mathbb{R}^N)$ is an $M^N(\mathbb{R})$ -valued continuous function of $t \in \mathbb{R}$, and there exists a constant $\gamma > 0$ such that*

$$(L(t)x, x) \geq \gamma |x|^2, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

(W1) *$W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and there exists a constant $R > 0$ such that*

$$|\nabla W(t, x)| = o(|x|^{p-1}) \quad \text{as } x \rightarrow 0$$

uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$.

(W2) *There is a constant $\mu > p$ such that*

$$0 < \mu W_1(t, x) \leq (\nabla W_1(t, x), x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}.$$

(W3) *$W_2(t, 0) = 0$ and there exists a constant $\varrho \in (p, \mu)$ such that*

$$W_2(t, x) \geq 0, \quad (\nabla W_2(t, x), x) \leq \varrho W_2(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then problem (1.1) has at least one nontrivial fast homoclinic solution.

Theorem 1.2 *Suppose that $q, a, L,$ and W satisfy (1.2), (A), (L), (W2), and the following conditions:*

$$(W1)' \quad W(t, x) = W_1(t, x) - W_2(t, x), \quad W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}), \text{ and}$$

$$|\nabla W(t, x)| = o(|x|^{p-1}) \quad \text{as } x \rightarrow 0$$

uniformly in $t \in \mathbb{R}$.

$$(W3)' \quad W_2(t, 0) = 0 \text{ and there exists a constant } \varrho \in (p, \mu) \text{ such that}$$

$$(\nabla W_2(t, x), x) \leq \varrho W_2(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then problem (1.1) has at least one nontrivial fast homoclinic solution.

The rest of this paper is organized as follows. In Sect. 2, some preliminaries are presented. In Sect. 3, the proofs of the main results are given. In Sect. 4, two examples are given to illustrate the main results.

2 Preliminaries

The functional φ corresponding to (1.1) on E is given by

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}} e^{Q(t)} \left\{ \frac{1}{2} [|\dot{u}(t)|^2 + (L(t)u(t), u(t))] + \frac{a(t)}{p} |u(t)|^p - W(t, u(t)) \right\} dt, \\ u &\in E. \end{aligned} \tag{2.1}$$

Clearly, it follows from (A), (L), (W1), or (W1)' that $\varphi : E \rightarrow \mathbb{R}$. By Theorem 2.1 of [6], we can deduce that the map

$$u \rightarrow a(t)e^{Q(t)}|u(t)|^{p-2}u(t)$$

is continuous from $L_a^p(e^{Q(t)})$ in the dual space $L_{a^{-1/(p-1)}}^{p_1}(e^{Q(t)})$, where $p_1 = \frac{p}{p-1}$. As the embeddings $E \subset X \subset L^\kappa(e^{Q(t)})$ for all $\kappa \geq 2$ are continuous, if (A), (L) and (W1) or (W1)' hold, then $\varphi \in C^1(E, \mathbb{R})$ and one can easily check that for $u \in E$

$$\begin{aligned} \langle \varphi'(u), v \rangle &= \int_{\mathbb{R}} e^{Q(t)} [(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) + a(t)|u(t)|^{p-2}(u(t), v(t)) \\ &\quad - (\nabla W(t, u(t)), v(t))] dt. \end{aligned} \tag{2.2}$$

Furthermore, the critical points of φ in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

Let E and $\|\cdot\|$ be given in Sect. 1. The following lemmas are important.

Lemma 2.1 ([1]) *For any $u \in E$,*

$$\|u\|_\infty \leq \frac{1}{\sqrt{2e_0\sqrt{\gamma}}} \|u\| = \frac{1}{\sqrt{2e_0\sqrt{\gamma}}} \left\{ \int_{\mathbb{R}} e^{Q(s)} [|\dot{u}(s)|^2 + (L(s)u(s), u(s))] ds \right\}^{1/2}, \tag{2.3}$$

$$|u(t)| \leq \frac{1}{\sqrt[4]{\gamma}} \left\{ \int_t^{+\infty} e^{-Q(s)} e^{Q(s)} [|\dot{u}(s)|^2 + (L(s)u(s), u(s))] ds \right\}^{1/2} \tag{2.4}$$

and

$$|u(t)| \leq \frac{1}{\sqrt[4]{\gamma}} \left\{ \int_{-\infty}^t e^{-Q(s)} e^{Q(s)} [|\dot{u}(s)|^2 + (L(s)u(s), u(s))] ds \right\}^{1/2}, \tag{2.5}$$

where $\|u\|_\infty = \text{ess sup}_{t \in \mathbb{R}} |u(t)|$, $e_0 = e^{\min\{Q(t):t \in \mathbb{R}\}}$.

Lemma 2.2 ([24]) *If a satisfies assumption (A), then*

$$\text{the embedding } L_a^p(e^{Q(t)}) \subset L^2(e^{Q(t)}) \text{ is continuous.} \tag{2.6}$$

Moreover, there exists a Hilbert space Z such that

$$\text{the embeddings } L_a^p(e^{Q(t)}) \subset Z \subset L^2(e^{Q(t)}) \text{ are continuous,} \tag{2.7}$$

$$\text{the embedding } X \cap Z \subset L^2(e^{Q(t)}) \text{ is compact.} \tag{2.8}$$

The following lemma is the mountain pass theorem which is very useful in the proofs of our theorems.

Lemma 2.3 ([29]) *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. Suppose $I(0) = 0$ and*

- (i) *There exist constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho(0)} \geq \alpha$.*
- (ii) *There exists $e \in E \setminus \bar{B}_\rho(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ which can be characterized as $c = \inf_{h \in \Phi} \max_{s \in [0,1]} I(h(s))$, where $\Phi = \{h \in C([0,1], E) | h(0) = 0, h(1) = e\}$ and $B_\rho(0)$ is an open ball in E of radius ρ centered at 0.

Lemma 2.4 *Assume that (W2) and (W3) or (W3)' hold. Then, for every $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,*

- (i) *$s^{-\mu} W_1(t, sx)$ is nondecreasing on $(0, +\infty)$;*
- (ii) *$s^{-\nu} W_2(t, sx)$ is nonincreasing on $(0, +\infty)$.*

The proof of Lemma 2.4 is routine and we omit it.

3 Proofs of theorems

Proof of Theorem 1.1 The proof of Theorem 1.1 is divided into three steps. Step 1. We will prove that the functional φ satisfies the (PS)-condition. Let $\{u_n\} \subset E$ satisfying $\varphi(u_n)$ be bounded and $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then there exists a constant $C_1 > 0$ such that

$$|\varphi(u_n)| \leq C_1, \quad \|\varphi'(u_n)\|_{E^*} \leq \mu C_1. \tag{3.1}$$

From (2.1), (2.2), (3.1), (W2), and (W3), we have

$$2C_1 + 2C_1 \|u_n\| \geq 2\varphi(u_n) - \frac{2}{\mu} \langle \varphi'(u_n), u_n \rangle$$

$$\begin{aligned}
 &= \frac{\mu - 2}{\mu} \|u_n\|^2 + \left(\frac{2}{p} - \frac{2}{\mu}\right) \int_{\mathbb{R}} e^{Q(t)} a(t) |u_n(t)|^p dt \\
 &\quad - 2 \int_{\mathbb{R}} e^{Q(t)} \left[W_1(t, u_n(t)) - \frac{1}{\mu} (\nabla W_1(t, u_n(t)), u_n(t)) \right] dt \\
 &\quad + 2 \int_{\mathbb{R}} e^{Q(t)} \left[W_2(t, u_n(t)) - \frac{1}{\mu} (\nabla W_2(t, u_n(t)), u_n(t)) \right] dt \\
 &\geq \frac{\mu - 2}{\mu} \|u_n\|^2 + \left(\frac{2}{p} - \frac{2}{\mu}\right) \|u_n\|_{p,a}^p.
 \end{aligned}$$

It follows from Lemma 2.2, $\mu > p > 2$, and the above inequalities that there exists a constant $C_2 > 0$ such that

$$\|u_n\| \leq C_2, \quad n \in \mathbb{N}. \tag{3.2}$$

Now we prove that $u_n \rightarrow u_0$ in E . Passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u_0$ in E . For any given number $\varepsilon > 0$, from (W1), we can choose $\xi > 0$ such that

$$|\nabla W(t, x)| \leq \varepsilon \gamma |x|^{p-1} \quad \text{for } |t| \geq R \text{ and } |x| \leq \xi. \tag{3.3}$$

Since $Q(t) \rightarrow \infty$ as $|t| \rightarrow \infty$, we can take $T > R$ such that

$$Q(t) \geq \ln\left(\frac{C_2^2}{\sqrt{\gamma} \xi^2}\right) \quad \text{for } |t| \geq T. \tag{3.4}$$

It follows from (2.4), (3.2), and (3.4) that

$$\begin{aligned}
 |u_n(t)|^2 &\leq \frac{1}{\sqrt{\gamma}} \int_t^{+\infty} e^{-Q(s)} e^{Q(s)} [|\dot{u}_n(s)|^2 + (L(s)u_n(s), u_n(s))] ds \\
 &\leq \frac{\xi^2}{C_2^2} \|u_n\|^2 \leq \xi^2 \quad \text{for } t \geq T \text{ and } n \in \mathbb{N}.
 \end{aligned} \tag{3.5}$$

Similarly, from (2.5), (3.2), and (3.4), we have

$$|u_n(t)|^2 \leq \xi^2 \quad \text{for } t \leq -T \text{ and } n \in \mathbb{N}. \tag{3.6}$$

Since $u_n \rightharpoonup u_0$ in E , it is easy to verify that $u_n(t)$ converges to $u_0(t)$ pointwise for all $t \in \mathbb{R}$. Hence, it follows from (3.5) and (3.6) that

$$|u_0(t)| \leq \xi \quad \text{for } t \in (-\infty, -T] \cup [T, +\infty). \tag{3.7}$$

Since $e^{Q(t)} \geq e_0 > 0$ on $[-T, T] = J$, the operator defined by $S : E \rightarrow H^1(J) : u \rightarrow u|_J$ is a linear continuous map. Hence $u_n \rightarrow u_0$ in $H^1(J)$. The Sobolev theorem implies that $u_n \rightarrow u_0$ uniformly on J , hence there is $n_0 \in \mathbb{N}$ such that

$$\int_{-T}^T e^{Q(t)} |\nabla W(t, u_n(t)) - \nabla W(t, u_0(t))| |u_n(t) - u_0(t)| dt < \varepsilon \quad \text{for } n \geq n_0. \tag{3.8}$$

It follows from (L), (3.2), (3.3), (3.5), (3.6), (3.7), and Young’s inequality that

$$\begin{aligned}
 & \int_{\mathbb{R} \setminus [-T, T]} e^{Q(t)} |\nabla W(t, u_n(t)) - \nabla W(t, u_0(t))| |u_n(t) - u_0(t)| dt \\
 & \leq \int_{\mathbb{R} \setminus [-T, T]} e^{Q(t)} (|\nabla W(t, u_n(t))| + |\nabla W(t, u_0(t))|) (|u_n(t)| + |u_0(t)|) dt \\
 & \leq \varepsilon \int_{\mathbb{R} \setminus [-T, T]} e^{Q(t)} \gamma (|u_n(t)|^{p-1} + |u_0(t)|^{p-1}) (|u_n(t)| + |u_0(t)|) dt \\
 & \leq 2\varepsilon \int_{\mathbb{R} \setminus [-T, T]} e^{Q(t)} \gamma (|u_n(t)|^p + |u_0(t)|^p) dt \\
 & \leq 2\xi^{p-2} \varepsilon \int_{\mathbb{R} \setminus [-T, T]} e^{Q(t)} \gamma (|u_n(t)|^2 + |u_0(t)|^2) dt \\
 & \leq 2\xi^{p-2} \varepsilon \int_{\mathbb{R} \setminus [-T, T]} e^{Q(t)} [(L(t)u_n(t), u_n(t)) + (L(t)u_0(t), u_0(t))] dt \\
 & \leq 2\xi^{p-2} \varepsilon (\|u_n\|^2 + \|u_0\|^2) \\
 & \leq 2\xi^{p-2} \varepsilon (C_2^2 + \|u_0\|^2), \quad n \in \mathbb{N}.
 \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\int_{\mathbb{R}} e^{Q(t)} |\nabla W(t, u_n(t)) - \nabla W(t, u_0(t))| |u_n(t) - u_0(t)| dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.10}$$

From (2.2), as $n \rightarrow \infty$, we have

$$\begin{aligned}
 0 & \leftarrow \langle \varphi'(u_n) - \varphi'(u_0), u_n - u_0 \rangle \\
 & = \|u_n - u_0\|^2 + \int_{\mathbb{R}} e^{Q(t)} a(t) (|u_n(t)|^{p-2} u_n(t) - |u_0(t)|^{p-2} u_0(t)) (u_n(t) - u_0(t)) dt \\
 & \quad - \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_n(t)) - \nabla W(t, u_0(t)), u_n(t) - u_0(t)) dt.
 \end{aligned} \tag{3.11}$$

It is easy to see that, for any $\varsigma > 1$, there exists a constant $C_3 > 0$ such that

$$(|x|^{\varsigma-1} x - |y|^{\varsigma-1} y)(x - y) \geq C_3 |x - y|^{\varsigma+1}, \quad \forall x, y \in \mathbb{R}. \tag{3.12}$$

Hence, there exists a constant $C_4 > 0$ such that

$$\begin{aligned}
 & \int_{\mathbb{R}} e^{Q(t)} a(t) (|u_n(t)|^{p-2} u_n(t) - |u_0(t)|^{p-2} u_0(t)) (u_n(t) - u_0(t)) dt \\
 & \geq C_4 \int_{\mathbb{R}} e^{Q(t)} a(t) |u_n(t) - u_0(t)|^p dt.
 \end{aligned} \tag{3.13}$$

It follows from (3.10), (3.11), and (3.13) that

$$\|u_n\| \rightarrow \|u_0\| \quad \text{as } n \rightarrow \infty \tag{3.14}$$

and

$$\int_{\mathbb{R}} e^{Q(t)} a(t) |u_n(t)|^p dt \rightarrow \int_{\mathbb{R}} e^{Q(t)} a(t) |u_0(t)|^p dt \quad \text{as } n \rightarrow \infty. \tag{3.15}$$

Hence, it follows from (3.14) and (3.15) that $u_n \rightarrow u_0$ in E . This shows that φ satisfies the (PS)-condition.

Step 2. From (W1), there exists $\delta \in (0, 1)$ such that

$$|\nabla W(t, x)| \leq \frac{\gamma}{2} |x|^{p-1} \quad \text{for } |t| \geq R, |x| \leq \delta. \tag{3.16}$$

By (3.16), we have

$$|W(t, x)| \leq \frac{\gamma}{2p} |x|^p \quad \text{for } |t| \geq R, |x| \leq \delta. \tag{3.17}$$

Let

$$C_5 = \sup \left\{ \frac{W_1(t, x)}{\gamma} \mid t \in [-R, R], x \in \mathbb{R}, |x| = 1 \right\}. \tag{3.18}$$

Set $\sigma = \min\{1/(2pC_5 + 1)^{1/(\mu-2)}, \delta\}$ and $\|u\| = \sqrt{2e_0\sqrt{\gamma}\sigma} := \rho$, it follows from Lemma 2.1 that $|u(t)| \leq \sigma \leq \delta < 1$ for $t \in \mathbb{R}$. From Lemma 2.4(i), (L), and (3.18), we have

$$\begin{aligned} \int_{-R}^R e^{Q(t)} W_1(t, u(t)) dt &\leq \int_{\{t \in [-R, R]: u(t) \neq 0\}} e^{Q(t)} W_1\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^\mu dt \\ &\leq C_5 \gamma \int_{-R}^R e^{Q(t)} |u(t)|^\mu dt \\ &\leq C_5 \sigma^{\mu-2} \int_{-R}^R e^{Q(t)} \gamma |u(t)|^2 dt \\ &\leq C_5 \sigma^{\mu-2} \int_{-R}^R e^{Q(t)} (L(t)u(t), u(t)) dt \\ &\leq \frac{1}{2p} \int_{-R}^R e^{Q(t)} (L(t)u(t), u(t)) dt. \end{aligned} \tag{3.19}$$

By Lemma 2.2, (L), (W3), (3.17), and (3.19), we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \|u\|^2 + \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} a(t) |u(t)|^p dt - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{1}{p} \|u(t)\|_{p,a}^p - \int_{\mathbb{R} \setminus [-R, R]} e^{Q(t)} W(t, u(t)) dt - \int_{-R}^R e^{Q(t)} W_1(t, u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{C_6}{p} \|u\|_2^p - \frac{1}{2p} \int_{\mathbb{R} \setminus [-R, R]} \gamma e^{Q(t)} |u(t)|^p dt - \frac{1}{2p} \int_{-R}^R e^{Q(t)} (L(t)u(t), u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{C_6}{p} \|u\|_2^p - \frac{\delta^{p-2}}{2p} \int_{\mathbb{R} \setminus [-R, R]} e^{Q(t)} (L(t)u(t), u(t)) dt \\ &\quad - \frac{1}{2p} \int_{-R}^R e^{Q(t)} (L(t)u(t), u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{C_6}{p} \|u\|_2^p - \frac{1}{2p} \int_{\mathbb{R} \setminus [-R, R]} e^{Q(t)} (L(t)u(t), u(t)) dt \\ &\quad - \frac{1}{2p} \int_{-R}^R e^{Q(t)} (L(t)u(t), u(t)) dt \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \|u\|^2 + \frac{C_6}{p} \|u\|_2^p - \frac{1}{2p} \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t), u(t)) dt \\ &\geq \frac{p-1}{2p} \|u\|^2, \end{aligned}$$

where C_6 is a positive constant. Therefore, we can choose a constant $\alpha > 0$ depending on ρ such that $\varphi(u) \geq \alpha$ for any $u \in E$ with $\|u\| = \rho$.

Step 3. From Lemma 2.4(ii) and (2.3), we have, for any $u \in E$,

$$\begin{aligned} &\int_{-4}^4 e^{Q(t)} W_2(t, u(t)) dt \\ &= \int_{\{t \in [-4,4]; |u(t)| > 1\}} e^{Q(t)} W_2(t, u(t)) dt + \int_{\{t \in [-4,4]; |u(t)| \leq 1\}} e^{Q(t)} W_2(t, u(t)) dt \\ &\leq \int_{\{t \in [-4,4]; |u(t)| > 1\}} e^{Q(t)} W_2\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^q dt + \int_{-4}^4 e^{Q(t)} \max_{|x| \leq 1} W_2(t, x) dt \\ &\leq \|u\|_{\infty}^q \int_{-4}^4 e^{Q(t)} \max_{|x|=1} W_2(t, x) dt + \int_{-4}^4 e^{Q(t)} \max_{|x| \leq 1} W_2(t, x) dt \\ &\leq \left(\frac{1}{\sqrt{2e_0\sqrt{\gamma}}}\right)^q \|u\|^q \int_{-4}^4 e^{Q(t)} \max_{|x|=1} W_2(t, x) dt + \int_{-4}^4 e^{Q(t)} \max_{|x| \leq 1} W_2(t, x) dt \\ &= C_7 \|u\|^q + C_8, \end{aligned} \tag{3.20}$$

where $C_7 = \left(\frac{1}{\sqrt{2e_0\sqrt{\gamma}}}\right)^q \int_{-4}^4 e^{Q(t)} \max_{|x|=1} W_2(t, x) dt$, $C_8 = \int_{-4}^4 e^{Q(t)} \max_{|x| \leq 1} W_2(t, x) dt$. Take $\omega \in E$ such that

$$|\omega(t)| = \begin{cases} 1 & \text{for } |t| \leq 2, \\ 0 & \text{for } |t| \geq 4, \end{cases} \tag{3.21}$$

and $|\omega(t)| \leq 1$ for $|t| \in (2, 4]$. For $s > 1$, from Lemma 2.4(i) and (3.21), we get

$$\int_{-2}^2 e^{Q(t)} W_1(t, s\omega(t)) dt \geq s^\mu \int_{-2}^2 e^{Q(t)} W_1(t, \omega(t)) dt = C_9 s^\mu, \tag{3.22}$$

where $C_9 = \int_{-2}^2 e^{Q(t)} W_1(t, \omega(t)) dt > 0$. From (W3), (2.1), (3.20), (3.21), and (3.22), we get for $s > 1$

$$\begin{aligned} \varphi(s\omega) &= \frac{s^2}{2} \|\omega\|^2 + \frac{s^p}{p} \int_{\mathbb{R}} e^{Q(t)} a(t) |\omega(t)|^p dt + \int_{\mathbb{R}} e^{Q(t)} [W_2(t, s\omega(t)) - W_1(t, s\omega(t))] dt \\ &\leq \frac{s^2}{2} \|\omega\|^2 + \frac{s^p}{p} \|\omega\|_{p,a}^p + \int_{-4}^4 e^{Q(t)} W_2(t, s\omega(t)) dt - \int_{-2}^2 e^{Q(t)} W_1(t, s\omega(t)) dt \\ &\leq \frac{s^2}{2} \|\omega\|^2 + \frac{s^p}{p} \|\omega\|_{p,a}^p + C_7 s^q \|\omega\|^q + C_8 - C_9 s^\mu. \end{aligned} \tag{3.23}$$

Since $\mu > q > p > 2$ and $C_9 > 0$, it follows from (3.23) that there exists $s_1 > 1$ such that $\|s_1\omega\| > \rho$ and $\varphi(s_1\omega) < 0$. Set $e = s_1\omega(t)$, then $e \in E$, $\|e\| = \|s_1\omega\| > \rho$ and $\varphi(e) = \varphi(s_1\omega) < 0$. It is easy to see that $\varphi(0) = 0$. From Lemma 2.3, φ has a critical value $c > \alpha$ given by

$$c = \inf_{g \in \Phi} \max_{s \in [0,1]} \varphi(g(s)), \tag{3.24}$$

where

$$\Phi = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Hence, there exists $u^* \in E$ such that

$$\varphi(u^*) = c, \quad \varphi'(u^*) = 0.$$

The function u^* is a desired solution of problem (1.1). Since $c > 0$, u^* is a nontrivial fast homoclinic solution. The proof is complete. \square

Proof of Theorem 1.2 From the proof of Theorem 1.1, we know that the condition $W_2(t, x) \geq 0$ in (W3) is only used in the proofs of (3.2) and Step 2. Hence, we only need to prove that (3.2) and Step 2 still hold if we use (W1)' and (W3)' instead of (W1) and (W3), respectively. We first prove that (3.2) holds. From (W2), (W3)', (2.1), (2.2), (3.1), and Lemma 2.2, we have

$$\begin{aligned} & 2C_1 + \frac{2C_1\mu}{\varrho} \|u_n\| \\ & \geq 2\varphi(u_n) - \frac{2}{\varrho} \langle \varphi'(u_n), u_n \rangle \\ & = \frac{(\varrho - 2)}{\varrho} \|u_n\|^2 + 2 \int_{\mathbb{R}} e^{Q(t)} \left[W_2(t, u_n(t)) - \frac{1}{\varrho} \langle \nabla W_2(t, u_n(t)), u_n(t) \rangle \right] dt \\ & \quad - 2 \int_{\mathbb{R}} e^{Q(t)} \left[W_1(t, u_n(t)) - \frac{1}{\varrho} \langle \nabla W_1(t, u_n(t)), u_n(t) \rangle \right] dt \\ & \quad + 2 \left(\frac{1}{p} - \frac{1}{\varrho} \right) \int_{\mathbb{R}} e^{Q(t)} a(t) |u_n(t)|^p dt \\ & \geq \frac{\varrho - 2}{\varrho} \|u_n\|^2 + 2C_6 \left(\frac{1}{p} - \frac{1}{\varrho} \right) \|u_n\|_2^p. \end{aligned}$$

It follows from $\varrho > p > 2$ and the above inequalities that there exists a constant $C_2 > 0$ such that (3.2) holds. Next, we will prove that Step 2 still holds. From (W1); there exists $\delta \in (0, 1)$ such that

$$|\nabla W(t, x)| \leq \frac{\gamma}{2} |x|^{p-1} \quad \text{for } t \in \mathbb{R}, |x| \leq \delta. \tag{3.25}$$

By (3.25), we have

$$|W(t, x)| \leq \frac{\gamma}{2p} |x|^p \quad \text{for } t \in \mathbb{R}, |x| \leq \delta. \tag{3.26}$$

Let $\|u\| = \sqrt{2e_0\sqrt{\gamma}\delta} := \rho$, it follows from Lemma 2.1 that $|u(t)| \leq \delta$. From (L), (2.1), and (3.26) we have that

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \|u\|^2 + \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} a(t) |u(t)|^p dt - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{1}{p} \|u\|_{p,a}^p - \frac{1}{2p} \int_{\mathbb{R}} \gamma e^{Q(t)} |u(t)|^p dt \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1}{2} \|u\|^2 + \frac{1}{p} \|u\|_{p,a}^p - \frac{\delta^{p-2}}{2p} \int_{\mathbb{R}} \gamma e^{Q(t)} |u(t)|^2 dt \\
 &\geq \frac{1}{2} \|u\|^2 + \frac{C_6}{p} \|u\|_2^p - \frac{\delta^{p-2}}{2p} \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t), u(t)) dt \\
 &\geq \frac{1}{2} \|u\|^2 + \frac{C_6}{p} \|u\|_2^p - \frac{1}{2p} \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t), u(t)) dt \\
 &\geq \frac{p-1}{2p} \|u\|^2.
 \end{aligned}$$

Therefore, we can choose a constant $\alpha > 0$ depending on ρ such that $\varphi(u) \geq \alpha$ for any $u \in E$ with $\|u\| = \rho$. The proof of Theorem 1.2 is complete. \square

4 Examples

Example 4.1 Consider the following system:

$$\ddot{u}(t) + t\dot{u}(t) - L(t)u(t) - |t| |u(t)| u(t) + \nabla W(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, \tag{4.1}$$

where $q(t) = t, p = 3, a = |t|, t \in \mathbb{R}, u \in \mathbb{R}^N$. Let

$$W(t, x) = \sum_{i=1}^m a_i |x|^{\mu_i} - \sum_{j=1}^n b_j |x|^{\varrho_j}, \quad L(t) = \text{diag}(1 + t^2, \dots, 1 + t^2),$$

where $\mu_1 > \mu_2 > \dots > \mu_m > \varrho_1 > \varrho_2 > \dots > \varrho_n > 3, a_i, b_j > 0, i = 1, \dots, m, j = 1, \dots, n$. Let

$$W_1(t, x) = \sum_{i=1}^m a_i |x|^{\mu_i}, \quad W_2(t, x) = \sum_{j=1}^n b_j |x|^{\varrho_j}.$$

Then it is easy to check that all the conditions of Theorem 1.1 are satisfied with $\mu = \mu_m$ and $\varrho = \varrho_1$. Hence, problem (4.1) has at least one nontrivial fast homoclinic solution.

Example 4.2 Consider the following system:

$$\ddot{u}(t) + (t + t^3)\dot{u}(t) - L(t)u(t) - |t|^3 |u(t)|^2 u(t) + \nabla W(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, \tag{4.2}$$

where $q(t) = t + t^3, p = 4, a = |t|^3, t \in \mathbb{R}, u \in \mathbb{R}^N$. Let L be the same in Example 4.1 and

$$W(t, x) = a_1 |x|^{\mu_1} + a_2 |x|^{\mu_2} + a_3 |x|^{\mu_3} - b_1 (\sin t) |x|^{\varrho_1} - b_2 (\cos t) |x|^{\varrho_2} - b_3 |x|^{\varrho_3},$$

where $\mu_1 > \mu_2 > \mu_3 > \varrho_1 > \varrho_2 > \varrho_3 > 4, a_1, a_2, a_3 > 0, b_1, b_2, b_3 > 0$. Let

$$\begin{aligned}
 W_1(t, x) &= a_1 |x|^{\mu_1} + a_2 |x|^{\mu_2} + a_3 |x|^{\mu_3}, \\
 W_2(t, x) &= b_1 (\sin t) |x|^{\varrho_1} + b_2 (\cos t) |x|^{\varrho_2} + b_3 |x|^{\varrho_3}.
 \end{aligned}$$

Then it is easy to check that all the conditions of Theorem 1.2 are satisfied with $\mu = \mu_3$ and $\varrho = \varrho_1$. Hence, problem (4.2) has at least one nontrivial fast homoclinic solution.

5 Conclusions

In this paper, we study fast homoclinic solutions for a type of second-order damped vibration system. The difference from other papers is that our system has both damped vibration and $L(t)u(t)$. Besides, we consider the term $a(t)|u(t)|^{p-2}u(t)$ in the system. So the system is more general than the other papers and the results obtained are more general. From this point, our work is valued.

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Availability of data and materials

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Authors' contributions

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