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Commutator of fractional integral with Lipschitz functions associated with Schrödinger operator on local generalized Morrey spaces

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, where $n \geq 3$ and the nonnegative potential V belongs to the reverse Hölder class RH_{q_1} for some $q_1 > n/2$. Let b belong to a new Campanato space $\Lambda_\nu^\theta(\rho)$ and $\mathcal{I}_\beta^L$ be the fractional integral operator associated with L. In this paper, we study the boundedness of the commutators $[b, \mathcal{I}_\beta^L]$ with $b \in \Lambda_\nu^\theta(\rho)$ on local generalized Morrey spaces $LM_{p,\varphi}^{\alpha,V,\{x_0\}}$, generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}$ and vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}$ associated with Schrödinger operator, respectively. When b belongs to $\Lambda_\nu^\theta(\rho)$ with $\theta > 0$, $0 < \nu < 1$ and (φ_1, φ_2) satisfies some conditions, we show that the commutator operator $[b, \mathcal{I}_\beta^L]$ are bounded from $LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}$ to $LM_{q,\varphi_2}^{\alpha,V,\{x_0\}}$, from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p - 1/q = (\beta + \nu)/n$.

MSC: 42B35; 35J10; 47H50

Keywords: Schrödinger operator; Fractional integral; Commutator; Lipschitz function; Local generalized Morrey space

1 Introduction and main results

Let us consider the Schrödinger operator

$$L = -\Delta + V \quad \text{on } \mathbb{R}^n, n \geq 3,$$

where V is a nonnegative, $V \not\equiv 0$, and belongs to the reverse Hölder class RH_q for some $q \geq n/2$, i.e., there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y)\, dy \right)^{1/q} \leq \frac{C}{|B(x,r)|} \int_{B(x,r)} V(y)\, dy \tag{1.1}$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V \in RH_\infty$.

As in [29], for a given potential $V \in RH_q$ with $q \geq n/2$, we define the auxiliary function

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1\right\}.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Let $\theta > 0$ and $0 < \nu < 1$, in view of [22], the Campanato class, associated with the Schrödinger operator $\Lambda_\nu^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\nu/n}}\int_{B(x,r)}\left|b(y) - b_B\right|dy \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\nu^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in the inequality above.

Note that if $\theta = 0$, $\Lambda_\nu^\theta(\rho)$ is the classical Campanato space; if $\nu = 0$, $\Lambda_\nu^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [5].

We now present the definition of generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ (including the weak version) associated with a Schrödinger operator, which was introduced by the first author in [18].

The classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey in [24] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the reader to [9–12, 24, 35]. The generalized Morrey spaces are defined with r^λ replaced by a general nonnegative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [15, 23, 25, 30]).

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x,r) := \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p}\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^{W,\alpha,V}(f;x,r) := \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p}\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))}.$$

Definition 1.1 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\alpha,V,\{x_0\}} = LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ the local generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{LM_{p,\varphi}^{\alpha,V,\{x_0\}}} = \sup_{r>0}\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x_0,r).$$

Also $WLM_{p,\varphi}^{\alpha,V,\{x_0\}} = WLM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ we denote the weak local generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ with

$$\|f\|_{WLM_{p,\varphi}^{\alpha,V,\{x_0\}}} = \sup_{r>0}\mathfrak{A}_{p,\varphi}^{W,\alpha,V}(f;x_0,r) < \infty.$$

The local spaces $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ and $WLM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{LM_{p,\varphi}^{\alpha,V,\{x_0\}}} = \sup_{r>0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x_0,r), \qquad \|f\|_{WLM_{p,\varphi}^{\alpha,V,\{x_0\}}} = \sup_{r>0} \mathfrak{A}_{p,\varphi}^{W,\alpha,V}(f;x_0,r),$$

respectively.

Remark 1.1

(i) When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ is the local (central) Morrey space $LM_{p,\lambda}^{\{0\}}(\mathbb{R}^n)$ studied in [4].

(ii) When $\alpha = 0$, $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ is the local generalized Morrey space $VM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ were introduced by the first author in [13]; see also [14, 16, 21] etc.

Definition 1.2 The vanishing generalized Morrey space associated with the Schrödinger operator $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x,r) = 0. \tag{1.3}$$

The vanishing weak generalized Morrey space associated with the Schrödinger operator $VWM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi}^{W,\alpha,V}(f;x,r) = 0.$$

The vanishing spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and $VWM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM_{p,\varphi}^{\alpha,V}} \equiv \|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x,r),$$

$$\|f\|_{VWM_{p,\varphi}^{\alpha,V}} \equiv \|f\|_{WM_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{W,p,\varphi}^{\alpha,V}(f;x,r),$$

respectively.

In the case $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$ $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [33], where applications to PDE were considered.

We refer to [3, 20, 27, 28] for some properties of vanishing generalized Morrey spaces.

Definition 1.3 Let $L = -\Delta + V$ with $V \in RH_{q_1}$, $q_1 > n/2$. The fractional integral associated with L is defined by

$$\mathcal{I}_\beta^L f(x) = L^{-\beta/2} f(x) = \int_0^\infty e^{-tL}(f)(x) t^{\beta/2-1}\,dt$$

for $0 < \beta < n$. The commutator of $\mathcal{I}_\beta^L$ is defined by

$$\bigl[b, \mathcal{I}_\beta^L\bigr] f(x) = b(x)\mathcal{I}_\beta^L f(x) - \mathcal{I}_\beta^L(bf)(x).$$

Note that, if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $\mathcal{I}_\beta^L$ and $[b,\mathcal{I}_\beta^L]$ are the Riesz potential I_β and the commutator of the Riesz potential $[b,I_\beta]$, respectively, that is,

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n}\,dy, \qquad [b,I_\beta]f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^n} f(y)\,dy.$$

When $b \in BMO$, Chanillo proved in [8] that $[b,I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$. When b belongs to the Campanato space Λ_ν, $0 < \nu < 1$, Paluszynski in [26] showed that $[b,I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - (\beta+\nu)/n$, $1 < p < n/(\beta+\nu)$. When $b \in BMO_\theta(\rho)$, Bui in [6] obtained the boundedness of $[b,\mathcal{I}_\beta^L]$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$.

Inspired by the above results, we are interested in the boundedness of $[b,\mathcal{I}_\beta^L]$ on generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$, when b belongs to the new Campanato class $\Lambda_\nu^\theta(\rho)$.

In this paper, we consider the boundedness of the commutator of $\mathcal{I}_\beta^L$ on the local generalized Morrey spaces $LM_{p,\varphi}^{\alpha,V,\{x_0\}}$, the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$. When b belongs to the new Campanato space $\Lambda_\nu^\theta(\rho)$, $0 < \nu < 1$, we show that $[b,\mathcal{I}_\beta^L]$ are bounded from $LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}$ to $LM_{q,\varphi_2}^{\alpha,V,\{x_0\}}$, from $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and from $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $VM_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$ with $1/q = 1/p - (\beta+\nu)/n$, $1 < p < n/(\beta+\nu)$.

Our main results are as follows.

Theorem 1.1 *Let $x_0 \in \mathbb{R}^n$, $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > n/2$, $0 < \nu < 1$, $\alpha \geq 0$, $1 \leq p < n/(\beta+\nu)$, $1/q = 1/p - (\beta+\nu)/n$ and let $\varphi_1, \varphi_2 \in \Omega_{p,loc}^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leq c_0 \varphi_2(x_0,r), \tag{1.4}$$

where c_0 does not depend on r. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}$ to $LM_{q,\varphi_2}^{\alpha,V,\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\alpha,V,\{x_0\}}$ to $WLM_{\frac{n}{n-\beta-\nu},\varphi_2}^{\alpha,V,\{x_0\}}$. Moreover, for $p > 1$

$$\big\| [b,\mathcal{I}_\beta^L]f \big\|_{LM_{q,\varphi_2}^{\alpha,V,\{x_0\}}} \leq C[b]_\nu^\theta \|f\|_{LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}},$$

and for $p = 1$

$$\big\| [b,\mathcal{I}_\beta^L]f \big\|_{WLM_{\frac{n}{n-\beta-\nu},\varphi_2}^{\alpha,V,\{x_0\}}} \leq C[b]_\nu^\theta \|f\|_{LM_{1,\varphi_1}^{\alpha,V,\{x_0\}}},$$

where C does not depend on f.

Corollary 1.1 *Let $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > n/2$, $0 < \nu < 1$, $\alpha \geq 0$, $1 \leq p < n/(\beta+\nu)$, $1/q = 1/p - (\beta+\nu)/n$ and let $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leq c_0 \varphi_2(x,r), \tag{1.5}$$

where c_0 does not depend on x and r. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ for $p>1$ and from $M^{\alpha,V}_{1,\varphi_1}$ to $WM^{\alpha,V}_{\frac{n}{n-\beta-\nu},\varphi_2}$. Moreover, for $p>1$

$$\big\|\big[b,\mathcal{I}_\beta^L\big]f\big\|_{M^{\alpha,V}_{q,\varphi_2}} \le C[b]_\theta\|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

and for $p=1$

$$\big\|\big[b,\mathcal{I}_\beta^L\big]f\big\|_{WM^{\alpha,V}_{\frac{n}{n-\beta-\nu},\varphi_2}} \le C\|f\|_{M^{\alpha,V}_{1,\varphi_1}},$$

where C does not depend on f.

Theorem 1.2 *Let $b\in\Lambda_\nu^\theta(\rho)$, $V\in RH_{q_1}$, $q_1>n/2$, $0<\nu<1$, $\alpha\ge 0$, $b\in\Lambda_\nu^\theta(\rho)$, $1<p<n/(\beta+\nu)$, $1/q=1/p-(\beta+\nu)/n$, and let $\varphi_1\in\Omega^{\alpha,V}_{p,1}$, $\varphi_2\in\Omega^{\alpha,V}_{q,1}$ satisfy the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\,\frac{dt}{t}<\infty$$

for every $\delta>0$, and

$$\int_r^\infty \varphi_1(x,t)\,\frac{dt}{t^{1-\beta-\nu}} \le C_0\varphi_2(x,r), \tag{1.6}$$

where C_0 does not depend on $x\in\mathbb{R}^n$ and $r>0$. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$ for $p>1$ and from $VM^{\alpha,V}_{1,\varphi_1}$ to $VWM^{\alpha,V}_{\frac{n}{n-\beta-\nu},\varphi_2}$.

Remark 1.2 Note that, in the case of $V\equiv 0$, $\nu=0$ Corollary 1.1 and Theorem 1.2 were proved in [19, Corollary 5.5 and 7.5] and in the case of $\varphi(x,r)=r^{(\lambda-n)/p}$, $\nu=0$ in [32, Theorems 1.3 and 1.4].

In this paper, we shall use the symbol $A\lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A\le CB$. $A\approx B$ means that $A\lesssim B$ and $B\lesssim A$.

2 Some technical lemmas and propositions

We would like to recall the important properties concerning the critical function.

Lemma 2.1 ([29]) *Let $V\in RH_{q_1}$ with $q_1>n/2$. For the associated function ρ there exist C and $k_0\ge 1$ such that*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \rho(y)\le C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}} \tag{2.1}$$

for all $x,y\in\mathbb{R}^n$.

Lemma 2.2 ([2]) *Suppose $x\in B(x_0,r)$. Then for $k\in N$ we have*

$$\frac{1}{(1+\frac{2^k r}{\rho(x)})^N} \lesssim \frac{1}{(1+\frac{2^k r}{\rho(x_0)})^{N/(k_0+1)}}.$$

According to [5], the new BMO space $BMO_\theta(\rho)$ with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \left|b(y)-b_B\right| dy \leq C\left(1+\frac{r}{\rho(x)}\right)^{\theta}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)\,dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

Let $\theta > 0$ and $0 < \nu < 1$, a seminorm on the Campanato class $\Lambda_\nu^\theta(\rho)$ is denoted by $[b]_\nu^\theta$,

$$[b]_\nu^\theta := \sup_{x\in\mathbb{R}^n, r>0} \frac{\frac{1}{|B(x,r)|^{1+\nu/n}}\int_{B(x,r)} |b(y)-b_B|\,dy}{(1+\frac{r}{\rho(x)})^\theta} < \infty.$$

The Lipschitz space, associated with the Schrödinger operator (see [22]), consists of the functions f satisfying

$$\|f\|_{\mathrm{Lip}_\nu^\theta(\rho)} := \sup_{x\in\mathbb{R}^n, r>0} \frac{|f(x)-f(y)|}{|x-y|^\nu(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)})^\theta} < \infty.$$

It is easy to see that this space is exactly the Lipschitz space when $\theta = 0$.

Note that if $\theta = 0$ in (1.2), $\Lambda_\nu^\theta(\rho)$ is exactly the classical Campanato space; if $\nu = 0$, $\Lambda_\nu^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$; if $\theta = 0$ and $\nu = 0$, it is exactly the John–Nirenberg space BMO.

The following relations between $\mathrm{Lip}_\nu^\theta(\rho)$ and $\Lambda_\nu^\theta(\rho)$ were proved in [22, Theorem 5].

Lemma 2.3 ([22]) *Let* $\theta > 0$ *and* $0 < \nu < 1$. *Then following embedding is valid*:

$$\Lambda_\nu^\theta(\rho) \subseteq \mathrm{Lip}_\nu^\theta(\rho) \subseteq \Lambda_\nu^{(k_0+1)\theta}(\rho),$$

where k_0 *is the constant appearing in Lemma* 2.1.

We give some inequalities about the Campanato space, associated with the Schrödinger operator $\Lambda_\nu^\theta(\rho)$.

Lemma 2.4 ([22]) *Let* $\theta > 0$ *and* $1 \leq s < \infty$. *If* $b \in \Lambda_\nu^\theta(\rho)$, *then there exists a positive constant C such that*

$$\left(\frac{1}{|B|}\int_B \left|b(y)-b_B\right|^s dy\right)^{1/s} \leq C[b]_\nu^\theta r^\nu\left(1+\frac{r}{\rho(x)}\right)^{\theta'}$$

for all $B = B(x,r)$, *with* $x \in \mathbb{R}^n$ *and* $r > 0$, *where* $\theta' = (k_0+1)\theta$ *and* k_0 *is the constant appearing in* (2.1).

Let K_β be the kernel of $\mathcal{I}_\beta^L$. The following result gives the estimate on the kernel $K_\beta(x,y)$.

Lemma 2.5 ([6]) *If $V \in RH_{q_1}$ with $q_1 > n/2$, then, for every N, there exists a constant C such that*

$$\left|K_\beta(x,y)\right| \le \frac{C}{(1+\frac{|x-y|}{\rho(x)})^N} \frac{1}{|x-y|^{n-\beta}}. \tag{2.2}$$

Finally, we recall a relationship between an essential supremum and an essential infimum.

Lemma 2.6 ([34]) *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\Big(\operatorname*{ess\,inf}_{x\in E} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

It is natural, first of all, to find conditions ensuring that the spaces $LM_{p,\varphi}^{\alpha,V,\{x_0\}}$ and $M_{p,\varphi}^{\alpha,V}$ are nontrivial, that is, consist not only of functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.7 *Let $x_0 \in \mathbb{R}^n$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. If*

$$\sup_{t<r<\infty} \left(1+\frac{r}{\rho(x_0)}\right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(x_0,r)} = \infty \quad \textit{for some } t>0, \tag{2.3}$$

then $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Proof Let (2.4) be satisfied and f be not equivalent to zero. Then $\|f\|_{L_p(B(x_0,t))} > 0$, hence

$$\begin{aligned} \|f\|_{LM_{p,\varphi}^{\alpha,V,\{x_0\}}} &\ge \sup_{t<r<\infty} \left(1+\frac{r}{\rho(x_0)}\right)^\alpha \varphi(x_0,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,r))} \\ &\ge \|f\|_{L_p(B(x_0,t))} \sup_{t<r<\infty} \left(1+\frac{r}{\rho(x_0)}\right)^\alpha \varphi(x_0,r)^{-1} r^{-\frac{n}{p}}. \end{aligned}$$

Therefore $\|f\|_{LM_{p,\varphi}^{\alpha,V,\{x_0\}}} = \infty$. □

Remark 2.1 We denote by $\Omega_{p,loc}^{\alpha,V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that, for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \left(1+\frac{r}{\rho(x)}\right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty.$$

In what follows, keeping in mind Lemma 2.7, for the non-triviality of the space $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,loc}^{\alpha,V}$.

Lemma 2.8 ([2]) *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$.*

(i) *If*

$$\sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \textit{for some } t>0 \textit{ and for all } x\in\mathbb{R}^n, \tag{2.4}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau > 0 \textit{ and for all } x \in \mathbb{R}^n, \tag{2.5}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.2 We denote by $\Omega^{\alpha,V}_{p}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that, for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n}\left\|\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{-\frac{n}{p}}}{\varphi(x,r)}\right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and}$$

$$\sup_{x\in\mathbb{R}^n}\left\|\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}\right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.8, for the non-triviality of the space $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega^{\alpha,V}_{p}$.

Remark 2.3 We denote by $\Omega^{\alpha,V}_{p,1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n}\inf_{r>\delta}\left(1+\frac{r}{\rho(x)}\right)^{-\alpha}\varphi(x,r) > 0, \quad \text{for some } \delta > 0, \tag{2.6}$$

and

$$\lim_{r\to 0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{n/p}}{\varphi(x,r)} = 0.$$

For the non-triviality of the space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega^{\alpha,V}_{p,1}$.

3 Proof of Theorem 1.1

We first prove the following conclusions.

Lemma 3.1 *Let* $0 < \nu < 1$, $0 < \beta + \nu < n$ *and* $b \in \Lambda^{\theta}_{\nu}(\rho)$, *then the following pointwise estimate holds*:

$$\left|\left[b,\mathcal{I}^L_\beta\right]f(x)\right| \lesssim [b]^{\theta}_{\nu} I_{\beta+\nu}\left(|f|\right)(x).$$

Proof Note that

$$\begin{aligned}\left[b,\mathcal{I}^L_\beta\right]f(x) &= b(x)\mathcal{I}^L_\beta(f)(x) - \mathcal{I}^L_\beta(bf)(x)\\ &= \int_{\mathbb{R}^n}\left[b(x)-b(y)\right]K_\beta(x,y)f(y)\,dy.\end{aligned}$$

If $b \in \Lambda_\nu^\theta(\rho)$, then from Lemma 2.5 we have

$$\begin{aligned}\left|\left[b,\mathcal{I}_\beta^L\right]f(x)\right| &\leq \int_{\mathbb{R}^n}\left|b(x)-b(y)\right|\left|K_\beta(x,y)\right|\left|f(y)\right|dy \\ &\lesssim [b]_\nu^\theta \int_{\mathbb{R}^n}|x-y|^\nu\left|K_\beta(x,y)\right|\left|f(y)\right|dy \\ &= [b]_\nu^\theta I_{\beta+\nu}\left(|f|\right)(x).\end{aligned}$$

□

From Lemma 3.1 we get the following.

Corollary 3.1 *Suppose $V \in RH_{q_1}$ with $q_1 > n/2$ and $b \in \Lambda_\nu^\theta(\rho)$ with $0 < \nu < 1$. Let $0 < \beta+\nu < n$ and let $1 \leq p < q < \infty$ satisfy $1/q = 1/p - (\beta+\nu)/n$. Then for all f in $L_p(\mathbb{R}^n)$ we have*

$$\left\|\left[b,\mathcal{I}_\beta^L\right]f\right\|_{L_q(\mathbb{R}^n)} \lesssim \|f\|_{L_p(\mathbb{R}^n)}$$

when $p > 1$, and also

$$\left\|\left[b,\mathcal{I}_\beta^L\right]f\right\|_{WL_q(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}$$

when $p = 1$.

In order to prove Theorem 1.1, we need the following.

Theorem 3.1 *Suppose $V \in RH_{q_1}$ with $q_1 > n/2$, $b \in \Lambda_\nu^\theta(\rho)$, $\theta > 0$, $0 < \nu < 1$. Let $0 < \beta+\nu < n$ and let $1 \leq p < q < \infty$ satisfy $1/q = 1/p - (\beta+\nu)/n$ then the inequality*

$$\begin{aligned}\left\|\left[b,\mathcal{I}_\beta^L\right]f\right\|_{L_q(B(x_0,r))} &\lesssim \left\|I_{\beta+\nu}\left(|f|\right)\right\|_{L_q(B(x_0,r))} \\ &\lesssim r^{\frac{n}{q}}\int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\end{aligned}$$

holds for any $f \in L_{loc}^p(\mathbb{R}^n)$. Moreover, for $p = 1$ the inequality

$$\begin{aligned}\left\|\left[b,\mathcal{I}_\beta^L f\right]\right\|_{WL_{\frac{n}{n-\beta-\nu}}(B(x_0,r))} &\lesssim \left\|I_{\beta+\nu}\left(|f|\right)\right\|_{WL_{\frac{n}{n-\beta-\nu}}(B(x_0,r))} \\ &\lesssim r^{n-\beta}\int_{2r}^\infty \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta-\nu}}\frac{dt}{t}\end{aligned}$$

holds for any $f \in L_{loc}^1(\mathbb{R}^n)$.

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ and $\lambda B = B(x_0,\lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\begin{aligned}\left\|\left[b,\mathcal{I}_\beta^L\right]f\right\|_{L_q(B(x_0,r))} &\lesssim \left\|I_{\beta+\nu}\left(|f|\right)\right\|_{L_q(B(x_0,r))} \\ &\leq \|I_{\beta+\nu}f_1\|_{L_q(B(x_0,r))} + \|I_{\beta+\nu}f_2\|_{L_q(B(x_0,r))}.\end{aligned}$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of $I_{\beta+\nu}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [31]) it follows that

$$\begin{aligned}\|I_{\beta+\nu}f_1\|_{L_q(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,2r))} \\ &\lesssim r^{\frac{n}{q}}\|f\|_{L_p(B(x_0,2r))}\int_{2r}^{\infty}\frac{dt}{t^{\frac{n}{q}+1}} \lesssim r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \end{aligned} \tag{3.1}$$

To estimate $\|I_{\beta+\nu}f_2\|_{L_p(B(x_0,r))}$, the obverse of $x \in B$, $y \in (2B)^c$ implies $|x-y| \approx |x_0-y|$. Then by (2.2) we have

$$\sup_{x\in B}\left|I_{\beta+\nu}f_2(x)\right| \lesssim \int_{(2B)^c}\frac{|f(y)|}{|x_0-y|^{n-\beta-\nu}}\,dy \lesssim \sum_{k=1}^{\infty}\left(2^{k+1}r\right)^{-n+\beta}\int_{2^{k+1}B}\left|f(y)\right|dy.$$

By Hölder's inequality we get

$$\begin{aligned}\sup_{x\in B}\left|I_{\beta+\nu}f_2(x)\right| &\lesssim \sum_{k=1}^{\infty}\|f\|_{L_p(2^{k+1}B)}\left(2^{k+1}r\right)^{-1-\frac{n}{p}+\beta}\int_{2^k r}^{2^{k+1}r}dt \\ &\lesssim \sum_{k=1}^{\infty}\int_{2^k r}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} \lesssim \int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \end{aligned} \tag{3.2}$$

Then

$$\|I_{\beta+\nu}f_2\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} \tag{3.3}$$

holds for $1 \le p < n/\beta$. Therefore, by (3.1) and (3.3) we get

$$\left\|I_{\beta+\nu}\left(|f|\right)\right\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} \tag{3.4}$$

for $1 < p < n/\beta$.

When $p = 1$, by the boundedness of $I_{\beta+\nu}$ from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\beta-\nu}}(\mathbb{R}^n)$, we get

$$\|I_{\beta+\nu}f_1\|_{WL_{\frac{n}{n-\beta-\nu}}(B(x_0,r))} \lesssim \|f\|_{L_1(B(x_0,2r))} \lesssim r^{n-\beta-\nu}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta-\nu}}\frac{dt}{t}.$$

By (3.3) we have

$$\begin{aligned}\|I_{\beta+\nu}f_2\|_{WL_{\frac{n}{n-\beta-\nu}}(B(x_0,r))} &\le \|I_{\beta+\nu}f_2\|_{L_{\frac{n}{n-\beta-\nu}}(B(x_0,2r))} \\ &\lesssim r^{n-\beta-\nu}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta-\nu}}\frac{dt}{t}. \end{aligned}$$

Then

$$\left\|I_{\beta+\nu}\left(|f|\right)\right\|_{WL_{\frac{n}{n-\beta-\nu}}(B(x_0,r))} \lesssim r^{n-\beta-\nu}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta-\nu}}\frac{dt}{t}.$$

□

Proof of Theorem 1.1 From Lemma 2.6, we have

$$\frac{1}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}=\operatorname*{ess\,sup}_{t<s<\infty}\frac{1}{\varphi_1(x,s)s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecreasing function of t, and $f\in M^{\alpha,V}_{p,\varphi_1}$, then

$$\begin{aligned}\frac{(1+\frac{t}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}&\lesssim \operatorname{ess\,sup}_{t<s<\infty}\frac{(1+\frac{t}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x,s)s^{\frac{n}{p}}}\\&\lesssim \sup_{0<s<\infty}\frac{(1+\frac{s}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,s))}}{\varphi_1(x,s)s^{\frac{n}{p}}}\lesssim\|f\|_{M^{\alpha,V}_{p,\varphi_1}}.\end{aligned}$$

Since $\alpha\ge 0$, and (φ_1,φ_2) satisfies the condition (1.5),

$$\begin{aligned}&\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\&\quad=\int_{2r}^{\infty}\frac{(1+\frac{t}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{(1+\frac{t}{\rho(x_0)})^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\&\quad\lesssim\|f\|_{M^{\alpha,V}_{p,\varphi_1}}\int_{2r}^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{(1+\frac{t}{\rho(x_0)})^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\&\quad\lesssim\|f\|_{M^{\alpha,V}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t}\\&\quad\lesssim\|f\|_{M^{\alpha,V}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).\end{aligned}\tag{3.5}$$

Then by Theorem 3.1 we get

$$\begin{aligned}\big\|\big[b,\mathcal{I}^L_\beta\big]f\big\|_{M^{\alpha,V}_{q,\varphi_2}}&\lesssim\big\|I_{\beta+\nu}\big(|f|\big)\big\|_{M^{\alpha,V}_{q,\varphi_2}}\\&\lesssim\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/q}\big\|I_{\beta+\nu}\big(|f|\big)\big\|_{L_p(B(x_0,r))}\\&\lesssim\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\&\lesssim\|f\|_{M^{\alpha,V}_{p,\varphi_1}}.\end{aligned}$$

Let $q=\frac{n}{n-\beta-\nu}$, similar to the estimates of (3.5) we have

$$\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta-\nu}}\frac{dt}{t}\lesssim\|f\|_{M^{\alpha,V}_{1,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).$$

Thus by Theorem 3.1 we get

$$\begin{aligned}\big\|\big[b,\mathcal{I}^L_\beta\big]f\big\|_{WM^{\alpha,V}_{\frac{n}{n-\beta-\nu},\varphi_2}}&\lesssim\big\|I_{\beta+\nu}\big(|f|\big)\big\|_{WM^{\alpha,V}_{\frac{n}{n-\beta-\nu},\varphi_2}}\\&\lesssim\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}r^{\beta-n}\big\|I_{\beta+\nu}\big(|f|\big)\big\|_{WL_{\frac{n}{n-\beta-\nu}}(B(x_0,r))}\end{aligned}$$

$$\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta-\nu}} \frac{dt}{t}$$
$$\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}. \qquad \square$$

4 Proof of Theorem 1.2

The statement is derived from the estimate (3.4). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows by Theorem 1.1. So we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r) = 0 \quad \Rightarrow \quad \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{q,\varphi_2}\left(\left[b,\mathcal{I}^L_\beta\right]f;x,r\right) = 0 \tag{4.1}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;x,r) = 0 \quad \Rightarrow \quad \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{n/(n-\beta),\varphi_2}\left(\left[b,\mathcal{I}^L_\beta\right]f;x,r\right) = 0. \tag{4.2}$$

To show that $\sup_{x\in\mathbb{R}^n}(1+\frac{r}{\rho(x)})^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathcal{I}^L_\beta]f\|_{L_q(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (3.4):

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \left\|\left[b,\mathcal{I}^L_\beta\right]f\right\|_{L_q(B(x,r))} \le C\left[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)\right], \tag{4.3}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{(1+\frac{r}{\rho(x)})^{\alpha}}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\frac{n}{q}-1}\|f\|_{L_p(B(x,t))}\,dt$$

and

$$J_{\delta_0}(x,r) := \frac{(1+\frac{r}{\rho(x)})^{\alpha}}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} t^{-\frac{n}{q}-1}\|f\|_{L_p(B(x,t))}\,dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^{\alpha} \varphi_1(x,t)^{-1} t^{-n/p} \|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (1.6) and (4.3). This allows one to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (2.6) we have

$$J_{\delta_0}(x,r) \le c_{\sigma_0} \frac{(1+\frac{r}{\rho(x)})^{\alpha}}{\varphi_1(x,r)} \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1.3). Then by (2.6) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{(1+\frac{r}{\rho(x)})^{\alpha}}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (4.1).

The proof of (4.2) is similar to the proof of (4.1).

5 Conclusions

In this paper, we study the boundedness of the commutators $[b,\mathcal{I}^L_\beta]$ with $b\in\Lambda^\theta_\nu(\rho)$ on local generalized Morrey spaces $LM^{\alpha,V,\{x_0\}}_{p,\varphi}$, generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}$ and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}$ associated with the Schrödinger operator, respectively. When b belongs to $\Lambda^\theta_\nu(\rho)$ with $\theta>0$, $0<\nu<1$ and (φ_1,φ_2) satisfies some conditions, we show that the commutator operator $[b,\mathcal{I}^L_\beta]$ are bounded from $LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}$ to $LM^{\alpha,V,\{x_0\}}_{q,\varphi_2}$, from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$, $1/p-1/q=(\beta+\nu)/n$.

Our results about the boundedness of $[b,\mathcal{I}^L_\beta]$ with $b\in\Lambda^\theta_\nu(\rho)$ from $LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}$ to $LM^{\alpha,V,\{x_0\}}_{q,\varphi_2}$ (Theorem 1.1) are based on the local estimate for the commutators $[b,\mathcal{I}^L_\beta]$ (Theorem 3.1).

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Authors' contributions
This work was carried out in collaboration between all authors. VSG raised these interesting problems in the research. VSG and AA proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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References

1. Akbulut, A., Eroglu, A., Najafov, A.M.: Some embedding theorems on the Nikolskii–Morrey type spaces. Adv. Anal. **1**(1), 18–26 (2016)
2. Akbulut, A., Guliyev, V.S., Omarova, M.N.: Marcinkiewicz integrals associated with Schrödinger operators and their commutators on vanishing generalized Morrey spaces. Bound. Value Probl. **2017**, 121 (2017)
3. Akbulut, A., Kuzu, O.: Marcinkiewicz integrals associated with Schrödinger operator on generalized Morrey spaces. J. Math. Inequal. **8**(4), 791–801 (2014)
4. Alvarez, J., Lakey, J., Guzman-Partida, M.: Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures. Collect. Math. **51**(1), 1–47 (2000)

5. Bongioanni, B., Harboure, E., Salinas, O.: Commutators of Riesz transforms related to Schödinger operators. J. Fourier Anal. Appl. **17**(1), 115–134 (2011)
6. Bui, T.: Weighted estimates for commutators of some singular integrals related to Schrödinger operator. Bull. Sci. Math. **138**(2), 270–292 (2014)
7. Cao, X., Chen, D.: The boundedness of Toeplitz-type operators on vanishing-Morrey spaces. Anal. Theory Appl. **27**(4), 309–319 (2011)
8. Chanillo, S.: A note on commutators. Indiana Univ. Math. J. **31**(1), 7–16 (1982)
9. Chiarenza, F., Frasca, M.: Morrey spaces and Hardy–Littlewood maximal function. Rend. Mat. **7**, 273–279 (1987)
10. Di Fazio, G., Ragusa, M.A.: Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. J. Funct. Anal. **112**, 241–256 (1993)
11. Fan, D., Lu, S., Yang, D.: Boundedness of operators in Morrey spaces on homogeneous spaces and its applications. Acta Math. Sin. New Ser. **14**, 625–634 (1998)
12. Giannetti, F., Passarelli di Napoli, A., Ragusa, M.A., Tachikawa, A.: Partial regularity for minimizers of a class of non autonomous functionals with nonstandard growth. Calc. Var. Partial Differ. Equ. **56**(6), Article ID 153 (2017)
13. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. Doctor's degree dissertation. Mat. Inst. Steklov, Moscow (1994) 329 pp. (in Russian)
14. Guliyev, V.S.: Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications. Casioglu, Baku (1999) 332 pp. (in Russian)
15. Guliyev, V.S.: Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal. Appl. **2009**, Article ID 503948 (2009)
16. Guliyev, V.S.: Local generalized Morrey spaces and singular integrals with rough kernel. Azerb. J. Math. **3**(2), 79–94 (2013)
17. Guliyev, V.S.: Generalized local Morrey spaces and fractional integral operators with rough kernel. J. Math. Sci. (N.Y.) **193**(2), 211–227 (2013)
18. Guliyev, V.S.: Function spaces and integral operators associated with Schrödinger operators: an overview. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40**, 178–202 (2014)
19. Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.: Boundedness of sublinear operators and commutators on generalized Morrey spaces. Integral Equ. Oper. Theory **71**(3), 327–355 (2011)
20. Guliyev, V.S., Guliyev, R.V., Omarova, M.N.: Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces. Appl. Comput. Math. **17**(1), 56–71 (2018)
21. Guliyev, V.S., Omarova, M.N., Ragusa, M.A., Scapellato, A.: Commutators and generalized local Morrey spaces. J. Math. Anal. Appl. **457**(2), 1388–1402 (2018)
22. Liu, Y., Sheng, J.: Some estimates for commutators of Riesz transforms associated with Schrödinger operators. J. Math. Anal. Appl. **419**, 298–328 (2014)
23. Mizuhara, T.: Boundedness of some classical operators on generalized Morrey spaces. In: Igari, S. (ed.) Harmonic Analysis. ICM 90 Satellite Proceedings, pp. 183–189. Springer, Tokyo (1991)
24. Morrey, C.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**, 126–166 (1938)
25. Nakai, E.: Hardy–Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. Math. Nachr. **166**, 95–103 (1994)
26. Paluszyński, M.: Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. Indiana Univ. Math. J. **44**(1), 1–17 (1995)
27. Ragusa, M.A.: Commutators of fractional integral operators on vanishing-Morrey spaces. J. Glob. Optim. **40**(1–3), 361–368 (2008)
28. Samko, N.: Maximal, potential and singular operators in vanishing generalized Morrey spaces. J. Glob. Optim. **57**(4), 1385–1399 (2013)
29. Shen, Z.: L_p estimates for Schrödinger operators with certain potentials. Ann. Inst. Fourier (Grenoble) **45**(2), 513–546 (1995)
30. Softova, L.: Singular integrals and commutators in generalized Morrey spaces. Acta Math. Sin. Engl. Ser. **22**(3), 757–766 (2006)
31. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Univ. Press, Princeton (1993)
32. Tang, L., Dong, J.: Boundedness for some Schrödinger type operator on Morrey spaces related to certain nonnegative potentials. J. Math. Anal. Appl. **355**, 101–109 (2009)
33. Vitanza, C.: Functions with vanishing Morrey norm and elliptic partial differential equations. In: Proceedings of Methods of Real Analysis and Partial Differential Equations, Capri, pp. 147–150. Springer, Berlin (1990)
34. Wheeden, R., Zygmund, A.: Measure and Integral, an Introduction to Real Analysis. Pure and Applied Mathematics, vol. 43. Dekker, New York (1977)
35. Xiao, J.: A new perspective on the Riesz potential. Adv. Nonlinear Anal. **6**(3), 317–326 (2017)