Kuang and Yang *Boundary Value Problems* (2018) 2018:86
https://doi.org/10.1186/s13661-018-1006-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Variational approach to anti-periodic boundary value problems involving the discrete p-Laplacian

Juhong Kuang[1] and Youyuan Yang[2*]

[*]Correspondence:
yingluoer-06@163.com
[2]School of Financial Mathematics and Statistics, Guangdong University of Finance, Guangzhou, P.R. China
Full list of author information is available at the end of the article

Abstract
Using critical point theory, we obtain the existence and multiplicity of nonzero solutions to anti-periodic boundary value problems with p-Laplacian in the case where the nonlinearities are p-sublinear at zero. Some examples are given to illustrate the results.

MSC: 37J45; 39A12; 39A70

Keywords: Anti-periodic boundary value problem; Discrete p-Laplacian; Critical points theorem

1 Introduction

Difference equations occur in many fields [1, 20], such as economics, discrete optimization, computer science. In the past decade, discrete p-Laplacian problems and difference equations have become a hot topic; see [11–19, 21, 22] and [25, 26]. Among the methods used are the method of upper and lower solutions, fixed point theory, Leray–Schauder degree, mountain pass lemma and the linking theorem. Recently, a lot of new results [5–11, 16, 23, 24] have been established by using variational methods.

In these last years, the existence and multiplicity of solutions for nonlinear discrete problems subject to various boundary value conditions have been widely studied by using different methods (see, e.g. [2–4] and [12–19, 21]). Bai et al. [2, 3] studied the second-order difference equations with Neumann boundary value conditions. D'Aguì et al. [16] investigated the existence of positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian in the case where the nonlinear term is p-sublinear at zero. However, little work has been done that has referred to anti-periodic boundary value problems with the discrete p-Laplacian operators in the case where the nonlinearities are p-sublinear at zero.

The idea of this paper comes from the method in [6, 9, 16]. One obtained two distinct critical points for functionals unbounded from below without p-superlinear nonlinearities at zero. The loss of p-superlinear condition at zero puts some critical points theorems cannot be immediately used. Therefore, In this paper, we mainly deal with the existence and multiplicity of solutions for anti-periodic boundary value problems

$$\begin{cases} -\Delta[a(k-1)\phi_p(\Delta u(k-1))] = \lambda f(k,u(k)), \\ u(0) = -u(T), \qquad u(1) = -u(T+1), \end{cases} \tag{1.1}$$

for $k \in [1,T]$, where $p > 1$ is a fixed real number and $\phi_p(t) = |t|^{p-2}t$ for all $t \in R$. $a(k) > 0$ and $a(0) = a(T)$, $f : [1,T] \times R \to R$, is continuous and is p-sublinear at zero in the second variable for all $k \in [1,T]$. Moreover, Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 u(k) = \Delta(\Delta u(k))$.

The rest of this paper is organized as follows. In Sect. 2, we establish the variational structure associated with (1.1), and provide some preliminary results. In Sect. 3, we state our main results and give examples. In Sect. 4, we provide the proofs of the main results.

2 Variational structure and some preliminaries

In this section, we establish a variational structure which reduces the existence of solutions for (1.1) to the existence of critical points of the corresponding functional.

Throughout this paper, we always assume that the following conditions are satisfied:

(a) $a(k) > 0$ for all $k \in [1,T]$ and $a(0) = a(T)$. Let $\bar{a}$ and a_* be the maximum and minimum of $\{a(k)\}$, respectively.

(f) $f(k,u)$ is continuous in u and $F(k,u) = \int_0^u f(k,s)\,ds$ for $u \in R$ and $k \in [1,T]$.

We define the set E as

$$E = \bigl\{u = \bigl\{u(k)\bigr\} | u(T+1) = -u(1), u(k) \in R \text{ for } k \in [1, T+1]\bigr\}.$$

Then E is a vector space with $au + bv = \{au(k) + bv(k)\}$ for $u, v \in E$ and $a, b \in R$. Obviously, E is isomorphic to R^T and hence E can be equipped with the norm $\|\cdot\|_p$ as

$$\|u\|_p = \left(\sum_{k=1}^{T} \bigl|u(k)\bigr|^p\right)^{\frac{1}{p}} \quad \text{for } u \in E.$$

We also define norms $\|\cdot\|_\infty$ and $\|\cdot\|$ in E by

$$\|u\|_\infty = \max\bigl\{\bigl|u(k)\bigr| : 1 \le k \le T\bigr\}$$

and

$$\|u\| = \left[a(T)\bigl|u(1) + u(T)\bigr|^p + \sum_{k=1}^{T-1} a(k)\bigl|\Delta u(k)\bigr|^p\right]^{1/p}$$

respectively. Consider the functionals $\Phi(u)$, $\Psi(u)$ and $I_\lambda(u)$on E defined by

$$\Phi(u) = \frac{a(T)}{p}\bigl|u(1) + u(T)\bigr|^p + \sum_{k=1}^{T-1} \frac{a(k)}{p}\bigl|\Delta u(k)\bigr|^p, \tag{2.1}$$

$$\Psi(u) = \sum_{k=1}^{T} F\bigl(k, u(k)\bigr) \tag{2.2}$$

and

$$\begin{aligned} I_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &= \frac{a(T)}{p}\left|u(1)+u(T)\right|^p + \sum_{k=1}^{T-1}\frac{a(k)}{p}\left|\Delta u(k)\right|^p - \lambda\sum_{k=1}^{T}F\bigl(k,u(k)\bigr). \end{aligned} \tag{2.3}$$

Then the partial derivatives of $\Phi(u)$ are given by

$$\begin{cases} \frac{\partial\Phi(u)}{\partial u(1)} = -a(1)\phi_p(\Delta u(1)) + a(T)\phi_p(u(1)+u(T)), \\ \frac{\partial\Phi(u)}{\partial u(2)} = a(1)\phi_p(\Delta u(1)) - a(2)\phi_p(\Delta u(2)), \\ \dots, \\ \frac{\partial\Phi(u)}{\partial u(T-1)} = a(T-2)\phi_p(\Delta u(T-2)) - a(T-1)\phi_p(\Delta u(T-1)), \\ \frac{\partial\Phi(u)}{\partial u(T)} = a(T-1)\phi_p(\Delta u(T-1)) + a(T)\phi_p(u(1)+u(T)). \end{cases} \tag{2.4}$$

This, combined with $a(0) = a(T)$, $u(0) = -u(T)$ and $u(1) = -u(T+1)$, gives us

$$\begin{cases} \frac{\partial\Phi(u)}{\partial u(1)} = -\Delta[a(0)\phi_p(\Delta u(0))], \\ \frac{\partial\Phi(u)}{\partial u(2)} = -\Delta[a(1)\phi_p(\Delta u(1))], \\ \dots, \\ \frac{\partial\Phi(u)}{\partial u(T-1)} = -\Delta[a(T-2)\phi_p(\Delta u(T-2))], \\ \frac{\partial\Phi(u)}{\partial u(T)} = -\Delta[a(T-1)\phi_p(\Delta u(T-1))]. \end{cases} \tag{2.5}$$

Then Φ has continuous Gâteaux derivatives in finite dimensional space and $\Phi \in C^1(E,R)$, the Fréchet derivative is given by

$$\bigl\langle\Phi'(u),v\bigr\rangle = -\sum_{k=1}^{T}\Delta\bigl[a(k-1)\phi_p\bigl(\Delta u(k-1)\bigr)v(k)\bigr] \tag{2.6}$$

for $u, v \in E$. By direct computation, we have

$$\bigl\langle\Phi'(u),v\bigr\rangle = \sum_{k=1}^{T}\bigl[a(k)\phi_p\bigl(\Delta u(k)\bigr)\bigr]\Delta v(k). \tag{2.7}$$

Similarly, we have $\Psi \in C^1(E,R)$. The Fréchet derivative is given by

$$\bigl\langle\Psi'(u),v\bigr\rangle = \sum_{k=1}^{T}f\bigl(k,u(k)\bigr)v(k) \tag{2.8}$$

for $u, v \in E$. Therefore, $I_\lambda \in C^1(E,R)$, the Fréchet derivative is given by

$$\bigl\langle I_\lambda'(u),v\bigr\rangle = \bigl\langle\Phi'(u),v\bigr\rangle - \lambda\bigl\langle\Psi'(u),v\bigr\rangle.$$

The partial derivatives of I_λ are given by

$$\begin{cases} \frac{\partial I_\lambda}{\partial u(1)} = -\Delta[a(0)\phi_p(\Delta u(0))] - \lambda f(1,u(1)), \\ \frac{\partial I_\lambda}{\partial u(2)} = -\Delta[a(1)\phi_p(\Delta u(1))] - \lambda f(2,u(2)), \\ \dots, \\ \frac{\partial I_\lambda}{\partial u(T-1)} = -\Delta[a(T-2)\phi_p(\Delta u(T-2))] - \lambda f(T-1,u(T-1)), \\ \frac{\partial I_\lambda}{\partial u(T)} = -\Delta[a(T-1)\phi_p(\Delta u(T-1))] - \lambda f(T,u(T)). \end{cases} \tag{2.9}$$

Equations (2.3) and (2.9) imply that a nonzero critical point of the functional I_λ on E is a nontrivial solution of (1.1).

Definition 2.1 Let $I \in C^1(H,R)$. A sequence $\{x_j\} \subset H$ is called a Palais–Smale sequence (P.S. sequence) for I if $\{I(x_j)\}$ is bounded and $I'(x_j) \to 0$ as $j \to +\infty$. We say I satisfies the Palais–Smale condition (P.S. condition) if any P.S. sequence for I possesses a convergent subsequence.

Our main tool is taken from [9], which we recall here for the reader's convenience.

Theorem 2.1 ([9]) *Let X be a real Banach space and let $\Phi, \Psi : X \to R$ be two functionals of class C^1 such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in R$ and $u^* \in X$, with $0 < \Phi(u^*) < r$, such that*

$$\frac{1}{r} \sup_{u \in \Phi^{-1}(-\infty,r]} \Psi(u) < \frac{\Psi(u^*)}{\Phi(u^*)}, \tag{2.10}$$

and for each

$$\lambda \in \Lambda = \left(\frac{\Phi(u^*)}{\Psi(u^*)}, r\Bigl(\sup_{u \in \Phi^{-1}(-\infty,r]} \Psi(u) \Bigr)^{-1} \right),$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the P.S. condition and it is unbounded from below. Then for each I_λ it admits at least two nonzero critical points $u_{\lambda 1}$, $u_{\lambda 2}$ such that $I_\lambda(u_{\lambda 1}) < 0 < I_\lambda(u_{\lambda 2})$.

3 Main results and examples

Theorem 3.1 *Assume that the conditions* (a) *and* (f) *hold. There exist two positive constants b and ρ such that*

$$F(k,u) \ge b|u|^p \quad \text{for } k \in [1,T] \text{ and } |u| \ge \rho. \tag{3.1}$$

There also exist two positive constants c_* and d_* with

$$d_* < \left(\frac{a_*}{\bar{a}}\right)^{1/p} \left(\frac{1}{T}\right)^{1/q} c_*$$

such that

$$\frac{pT^{p-1}}{a_*(2c_*)^p} \sum_{k=1}^{T} \Bigl(\max_{|\xi| \le c_*} F(k,\xi) \Bigr) < \min\left\{ \frac{p}{\bar{a}(2d_*)^p} \sum_{k=1}^{T} F(k,d_*), \frac{pb}{\bar{a}2^{(p+1)}} \right\}, \tag{3.2}$$

where $1/p + 1/q = 1$. *Then, for each* $\lambda \in \Lambda$ *with*

$$\Lambda = \left(\max\left\{ \frac{\bar{a}2^{(p+1)}}{pb}, \frac{\bar{a}(2d_*)^p}{p}\left[\sum_{k=1}^{T} F(k,d_*)\right]^{-1} \right\}, \frac{a_*(2c_*)^p}{pT^{p-1}}\left[\sum_{k=1}^{T}\Bigl(\max_{|\xi|\le c_*} F(k,\xi)\Bigr)\right]^{-1} \right),$$

(1.1) *admits at least two nonzero solutions* $u_{\lambda 1}$, $u_{\lambda 2}$ *such that* $I_\lambda(u_{\lambda 1}) < 0 < I_\lambda(u_{\lambda 2})$.

Remark 3.1 If all the conditions of Theorem 3.1 are satisfied and $f(k,u)$ is odd in u for each $k \in [1,T]$, then (1.1) admits at least four nonzero solutions $\pm u_{\lambda 1}$, $\pm u_{\lambda 2}$ such that $I_\lambda(-u_{\lambda 1}) = I_\lambda(u_{\lambda 1}) < 0 < I_\lambda(u_{\lambda 2}) = I_\lambda(-u_{\lambda 2})$.

Corollary 3.1 *Assume that the conditions* (a) *and* (f) *hold. If* $f(k,u)$ *is odd in* u *for each* $k \in [1,T]$, *and*

$$\lim_{|s|\to+\infty} \frac{F(k,s)}{|s|^p} = +\infty, \qquad \lim_{s\to 0^+} \frac{F(k,s)}{s^p} = +\infty \tag{3.3}$$

for all $k \in [1,T]$, *then, for each* $\lambda \in \Lambda^*$ *with*

$$\Lambda^* = \left(0, \frac{a_*(2c_*)^p}{pT^{p-1}}\left[\sum_{k=1}^{T}\Bigl(\max_{|\xi|\le c_*} F(k,\xi)\Bigr)\right]^{-1}\right),$$

(1.1) *admits at least four nonzero solutions* $\pm u_{\lambda 1}$ *and* $\pm u_{\lambda 2}$.

Example 3.1 Let $p = 4$, $T = 2$, $a(k) = k$ and

$$f(k,u) = u^5 + u$$

for all $k \in [1,T]$. Then, for each $\lambda \in (0, 3/8)$, it is easy to check that all the conditions of Corollary 3.1 are satisfied, (1.1) admits at least four nonzero solutions.

Theorem 3.2 *Assume that the conditions* (a) *and* (f) *hold,* $T = 2$, $a_* = \bar{a} = a$, $f(k,x) \ge 0$ *for all* $x < 0$, $k \in [1,T]$. *Put*

$$L_\infty^+(k) = \lim_{s\to+\infty} \frac{F(k,s)}{s^p}, \qquad L_\infty^+ = \min_{k\in[1,T]} L_\infty^+(k). \tag{3.4}$$

If $L_\infty^+ > 0$ *and there exist two positive constants* c_* *and* d_* *with*

$$d_* < \left(\frac{1}{2}\right)^{1/q} c_*$$

such that

$$\frac{p}{2ac_*^p}\sum_{k=1}^{T}\Bigl(\max_{|\xi|\le c_*} F(k,\xi)\Bigr) < \min\left\{ \frac{p}{a(2d_*)^p}\sum_{k=1}^{T} F(k,d_*), \frac{pL_\infty^+}{a2^{(p+1)}} \right\}, \tag{3.5}$$

where $1/p + 1/q = 1$. *Then, for each* $\lambda \in \Lambda$ *with*

$$\Lambda = \left(\max\left\{ \frac{a2^{(p+1)}}{pL_\infty^+}, \frac{a(2d_*)^p}{p}\left[\sum_{k=1}^{T} F(k,d_*)\right]^{-1} \right\}, \frac{2ac_*^p}{p}\left[\sum_{k=1}^{T}\Bigl(\max_{|\xi|\le c_*} F(k,\xi)\Bigr)\right]^{-1} \right),$$

(1.1) *admits at least two nonzero solutions* $u_{\lambda 1}$, $u_{\lambda 2}$ *such that* $I_\lambda(u_{\lambda 1}) < 0 < I_\lambda(u_{\lambda 2})$.

Corollary 3.2 *Assume that the conditions* (a) *and* (f) *hold,* $T = 2$, $a_* = \bar{a} = a$, $f(k,x) \ge 0$ *for all* $x < 0$, $k \in [1,T]$. *If*

$$\lim_{s\to+\infty} \frac{F(k,s)}{s^p} = +\infty \quad \textit{and} \quad \lim_{s\to 0^+} \frac{F(k,s)}{s^p} = +\infty, \tag{3.6}$$

for all $k \in [1,T]$, *then, for each* $\lambda \in \Lambda^*$ *with*

$$\Lambda^* = \left(0, \frac{2ac_*^p}{p}\left[\sum_{k=1}^{T}\Bigl(\max_{|\xi|\le c_*} F(k,\xi)\Bigr)\right]^{-1}\right),$$

(1.1) *admits at least two nonzero solutions.*

Example 3.2 Let $p = 2$, $T = 2$, $a(k) = 4$ and

$$f(k,u) = e^u$$

for $k \in [1,T]$. Then, for each $\lambda \in (0, \frac{2}{e-1})$, it is easy to check that all the conditions of Corollary 3.2 are satisfied, (1.1) admits at least two nonzero solutions.

4 Proofs of main results

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 4.1 *If* $u \in E$ *and* $p > 1$, *then*

$$\frac{a_*}{p}\left(\frac{2}{T}\right)^p \sum_{k=1}^{T} \bigl|u(k)\bigr|^p \le \Phi(u) \le \frac{\bar{a}2^{(p+1)}}{p}\sum_{k=1}^{T}\bigl|u(k)\bigr|^p$$

and

$$\frac{2}{T} a_*^{1/p} \|u\|_p \le \|u\| \le 2(2\bar{a})^{1/p}\|u\|_p.$$

Proof On the one hand,

$$\begin{aligned} \Phi(u) &= \frac{a(T)}{p}\bigl|u(1)+u(T)\bigr|^p + \sum_{k=1}^{T-1}\frac{a(k)}{p}\bigl|\Delta u(k)\bigr|^p \\ &\le \frac{\bar{a}}{p}\left[2^p\bigl(\bigl|u(1)\bigr|^p + \bigl|u(T)\bigr|^p\bigr) + \sum_{k=1}^{T-1} 2^p\bigl(\bigl|u(k)\bigr|^p + \bigl|u(k+1)\bigr|^p\bigr)\right] \\ &\le \frac{\bar{a}}{p}2^p\sum_{k=1}^{T}2\bigl|u(k)\bigr|^p = \frac{\bar{a}2^{(p+1)}}{p}\sum_{k=1}^{T}\bigl|u(k)\bigr|^p. \end{aligned} \tag{4.1}$$

On the other hand, $u(1) = -u(T+1)$, for each $k \in [1, T]$,

$$\begin{aligned}
2u(k) &= u(2) - u(1) + \cdots + u(k) - u(k-1) + u(k) - u(k+1) + \cdots + u(T) - u(T+1) \\
&\leq \bigl|u(2) - u(1)\bigr| + \cdots + \bigl|u(k) - u(k-1)\bigr| + \bigl|u(k) - u(k+1)\bigr| + \cdots + \bigl|u(T) + u(1)\bigr| \\
&= \bigl|u(2) - u(1)\bigr| + \cdots + \bigl|u(k) - u(k-1)\bigr| + \bigl|u(k+1) - u(k)\bigr| + \cdots + \bigl|u(T) + u(1)\bigr| \\
&\leq \left[\bigl|u(T) + u(1)\bigr|^p + \sum_{k=1}^{T-1}\bigl|\Delta u(k)\bigr|^p\right]^{1/p} T^{1/q},
\end{aligned} \tag{4.2}$$

where $1/p + 1/q = 1$, that is,

$$\|u\|_\infty \leq \frac{1}{2}\left[\bigl|u(T) + u(1)\bigr|^p + \sum_{k=1}^{T-1}\bigl|\Delta u(k)\bigr|^p\right]^{1/p} T^{1/q}. \tag{4.3}$$

Since

$$\sum_{k=1}^{T}\bigl|u(k)\bigr|^p \leq T\bigl(\|u\|_\infty\bigr)^p,$$

this, combined with (4.3), gives us

$$\sum_{k=1}^{T}\bigl|u(k)\bigr|^p \leq \left(\frac{T}{2}\right)^p\left[\bigl|u(T) + u(1)\bigr|^p + \sum_{k=1}^{T-1}\bigl|\Delta u(k)\bigr|^p\right]$$

and

$$\Phi(u) \geq \frac{a_*}{p}\left[\bigl|u(T) + u(1)\bigr|^p + \sum_{k=1}^{T-1}\bigl|\Delta u(k)\bigr|^p\right] \geq \frac{a_*}{p}\left(\frac{2}{T}\right)^p\sum_{k=1}^{T}\bigl|u(k)\bigr|^p.$$

The proof is complete. □

Lemma 4.2 *If the condition* (3.1) *holds, then the functional I_λ satisfies the P.S. condition and it is unbounded from below for all $\lambda \in (\frac{\bar{a}2^{(p+1)}}{pb}, +\infty)$.*

Proof Let $\{I_\lambda(u_j)\}$ be a bounded sequence and $\{u_j\}$ be a sequence in E, i.e., there exists a positive constant M such that

$$\bigl|I_\lambda(u_j)\bigr| \leq M \quad \text{for } j \in Z^+.$$

Let

$$M_\rho = \max_{1\leq k\leq T}\bigl\{\bigl|F(k,u) - b|u|^p\bigr| : |u| \leq \rho\bigr\}.$$

It is easy to check that

$$F(k,u) \geq b|u|^p - M_\rho \quad \text{for } k \in [1, T] \text{ and } u \in R. \tag{4.4}$$

By (4.4) and (4.1), we have

$$I_\lambda(u_j) = \Phi(u_j) - \lambda\Psi(u_j) \le \left(\frac{\bar{a}2^{(p+1)}}{p} - \lambda b\right)\|u_j\|_p^p + T\lambda M_\rho \tag{4.5}$$

for $j \in Z^+$. Now, we claim $\{u_j\}$ is bounded. In fact, $\|u_j\| \to +\infty$, $\|u_j\|_p \to +\infty$ and $\frac{\bar{a}2^{(p+1)}}{p} - \lambda b < 0$, one has $I_\lambda(u_j) \to -\infty$ and this is absurd. Hence, I_λ satisfies the P.S. condition. Next, we prove that I_λ is unbounded from below. By (4.5), we have $I_\lambda(u_n) \to -\infty$ as $\|u_n\| \to +\infty$. □

Proof of Theorem 3.1 Put Φ and Ψ as in (2.1) and (2.2), it is easily checked that Φ and Ψ satisfy all regularity assumptions required in Theorem 2.1. So, our end is to verify condition (2.10) in Theorem 2.1. Let $u \in \Phi^{-1}(-\infty, r]$; this means that

$$\frac{a_*}{p}\left[\left|u(T) + u(1)\right|^p + \sum_{k=1}^{T-1}\left|\Delta u(k)\right|^p\right] \le \frac{a(T)}{p}\left|u(1) + u(T)\right|^p + \sum_{k=1}^{T-1}\frac{a(k)}{p}\left|\Delta u(k)\right|^p \le r,$$

this, combined with (4.2), gives us

$$\left|u(k)\right| \le \frac{1}{2}\left(\frac{rp}{a_*}\right)^{1/p} T^{1/q}$$

for $k \in [1, T]$. Let

$$c_* = \frac{1}{2}\left(\frac{rp}{a_*}\right)^{1/p} T^{1/q},$$

then

$$r = \frac{a_*(2c_*)^p}{pT^{p-1}}$$

and

$$\begin{aligned}\frac{1}{r}\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) &\le \frac{1}{r}\sum_{k=1}^{T}\max_{|\xi|\le c_*}F(k,\xi)\\ &= \frac{pT^{p-1}}{a_*(2c_*)^p}\sum_{k=1}^{T}\max_{|\xi|\le c_*}F(k,\xi).\end{aligned} \tag{4.6}$$

Now, we define $u_* \in E$ by $u_* = \{u_*(k)\}$ and

$$u_*(k) = d_* < \left(\frac{a_*}{\bar{a}}\right)^{1/p}\left(\frac{1}{T}\right)^{1/q} c_*$$

for $k \in [1, T]$. It is easy to check that $\Phi(u_*) < r$ and

$$\frac{\Psi(u_*)}{\Phi(u_*)} \ge \frac{p}{\bar{a}(2d_*)^p}\sum_{k=1}^{T}F(k, d_*).$$

This, combined with (4.6) and (3.2), produces at once (2.10). Therefore, Theorem 2.1 ensures that (1.1) has at least two nonzero critical points $u_{\lambda 1}$ and $u_{\lambda 2}$. The proof is complete. □

Proof of Theorem 3.2 Let $\{u_j\}$ be a sequence in E such that $\{I_\lambda(u_j)\}$ is bounded and $I'_\lambda(u_j) \to 0$ as $j \to +\infty$. Put $u_j^+(k) = \max\{u_j(k), 0\}$ and $u_j^-(k) = \max\{-u_j(k), 0\}$ for all $k \in [1, T]$, then $u_j^+ = \{u_j^+(k)\}$ and $u_j^- = \{u_j^-(k)\}$ for $k \in [1, T]$. Therefore, $u_j = u_j^+ - u_j^-$ for all $j \in Z^+$. Considering that $L_\infty^+ > 0$ and $\lambda \in (\frac{a2^{(p+1)}}{pL_\infty^+}, +\infty)$, we fix $\lambda > \frac{a2^{(p+1)}}{pL_\infty^+}$ and fix l such that $L_\infty^+ > l > \frac{a2^{(p+1)}}{p\lambda}$. Now, we claim $\{u_j^-\}$ is bounded. By direct computation, we have

$$\begin{aligned} \|u_j^-\|^p &= a\left|u_j^-(1) + u_j^-(T)\right|^p + \sum_{k=1}^{T-1} a\left|\Delta u_j^-(k)\right|^p \\ &= a\left|-u_j^-(T+1) + u_j^-(T)\right|^p + \sum_{k=1}^{T-1} a\left|\Delta u_j^-(k)\right|^p \\ &\le -\sum_{k=1}^{T} a\left[\phi_p\left(\Delta u_j(k)\right)\Delta u_j^-(k)\right] \\ &= -\left\langle \Phi'(u_j), u_j^-\right\rangle, \end{aligned} \tag{4.7}$$

where $u_j^-(T+1) = -u_j^-(1)$, for all $j \in Z^+$. Moreover, by definition of u_j^- and since $f(k,x) \ge 0$ for all $x < 0$, we have

$$\begin{aligned} \|u_j^-\|^p &\le -\left\langle \Phi'(u_j), u_j^-\right\rangle + \lambda \sum_{k=1}^{T} f\left(k, u_j(k)\right) u_j^-(k) \\ &= -\left\langle \Phi'(u_j), u_j^-\right\rangle + \lambda\left\langle \Psi'(u_j), u_j^-\right\rangle \\ &= -\left\langle I'_\lambda(u_j), u_j^-\right\rangle \end{aligned} \tag{4.8}$$

for all $j \in Z^+$. This, combined with the formulas

$$\lim_{j\to+\infty} I'_\lambda(u_j) = 0, \qquad \lim_{j\to+\infty} \frac{-\langle I'_\lambda(u_j), u_j^-\rangle}{\|u_j^-\|} = 0,$$

gives us

$$\lim_{j\to+\infty} \left\|u_j^-\right\| = 0.$$

Hence, our claim is proved. Therefore, there exists $Q > 0$ such that $\|u_j^-\| \le Q$ for all $j \in Z^+$. Using a similar argument to (4.2) produces at once

$$\left|u_j^-(k)\right| \le \frac{Q}{a}\left(\frac{1}{2}\right)^{1/p} = L \tag{4.9}$$

for all $k \in [1, T]$ and $j \in Z^+$.

Now, arguing by contradiction, assume that $\{u_j\}$ is unbounded, that is, $\{u_j^+\}$ is unbounded. From

$$\lim_{s\to+\infty}\frac{F(k,s)}{s^p}\geq L_\infty^+>l,$$

there exists $\delta>0$ such that $F(k,s)>ls^p$ for all $s>\delta$ and $k\in[1,T]$. Let

$$M_\delta=\max_{1\leq k\leq T}\bigl\{\bigl|F(k,u)-l|u|^p\bigr|:-L\leq u\leq\delta\bigr\}.$$

Then it is easy to check that

$$F\bigl(k,u_j(k)\bigr)\geq l\bigl|u_j(k)\bigr|^p-M_\delta\quad\text{for } k\in[1,T]\text{ and } j\in Z^+. \tag{4.10}$$

This, combined with (4.1), gives us

$$I_\lambda(u_j)=\frac{1}{p}\|u_j\|^p-\lambda\Psi(u_j)\leq\frac{a2^{(p+1)}}{p}\|u_j\|_p^p-\lambda l\|u_j\|_p^p+T\lambda M_\delta,$$

that is,

$$I_\lambda(u_j)\leq\left(\frac{a2^{(p+1)}}{p}-\lambda l\right)\|u_j\|_p^p+T\lambda M_\delta$$

for $j\in Z^+$. Since $\|u_j\|\to+\infty$ and $\frac{\bar{a}2^{(p+1)}}{p}-\lambda l<0$, one has $I_\lambda(u_j)\to-\infty$ and this is absurd. Hence I_λ satisfies the P.S. condition.

Finally, we prove that I_λ is unbounded from below. Arguing as before, we have $I_\lambda(u_n)\to-\infty$ as $\|u_n\|\to+\infty$. The rest of the proof is similar to Theorem 3.1 and is omitted. The proof is complete. □

Acknowledgements
The authors would like to thank the editor and the referees for their valuable comments and suggestions, which improved the quality of our manuscript.

Funding
This work is supported by the Natural Science Foundation of Guangdong Province, China (No. 2014A030310334 and No. 2016A030313005) and the Innovation Project of Department of Education of Guangdong Province, China (No. 2015KTSCX147).

Abbreviations
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally to this work. They both read and approved the manuscript.

Author details
[1]School of Mathematics and Computational Sciences, Wuyi University, Jiangmen, P.R. China. [2]School of Financial Mathematics and Statistics, Guangdong University of Finance, Guangzhou, P.R. China.

Publisher's Note

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Received: 5 January 2018 Accepted: 22 May 2018 Published online: 30 May 2018

References

1. Agarwal, R.P.: Difference Equations and Inequalities. Dekker, New York (2000)
2. Bai, D., Henderson, J., Zeng, Y.: Positive solutions of discrete Neumann boundary value problems with sign-changing nonlinearities. Bound. Value Probl. **2015**, Article ID 231 (2015)
3. Bai, D., Lian, H., Wang, H.: Exact multiplicity of solutions for discrete second order Neumann boundary value problems. Bound. Value Probl. **2015**, Article ID 229 (2015)
4. Bisci, G.M., Repovs, D.: Existence of solutions for *p*-Laplacian discrete equations. Appl. Math. Comput. **242**, 454–461 (2014)
5. Bonanno, G.: A critical point theorem via the Ekeland variational principle. Nonlinear Anal. **75**, 2992–3007 (2012)
6. Bonanno, G.: Relations between the mountain pass theorem and local minima. Adv. Nonlinear Anal. **1**, 205–220 (2012)
7. Bonanno, G., Candito, P., D'Aguì, G.: Variational methods on finite dimensional Banach spaces and discrete problems. Adv. Nonlinear Stud. **14**, 915–939 (2014)
8. Bonanno, G., Candito, P., D'Aguì, G.: Positive solutions for a nonlinear paremeter-depending algebraic system. Electron. J. Differ. Equ. **2015**, Article ID 17 (2015)
9. Bonanno, G., D'Aguì, G.: Two non-zero solutions for elliptic Dirichlet problems. Z. Anal. Anwend. **35**, 449–464 (2016)
10. Bonanno, G., Jebelean, P., Serban, C.: Superlinear discrete problems. Appl. Math. Lett. **52**, 162–168 (2016)
11. Cabada, A., Iannizzotto, A.: Existence of homoclinic constant sign solutions for a difference equation on the integers. Appl. Math. Comput. **224**, 216–223 (2013)
12. Cabada, A., Tersian, S.: Existence of heteroclinic solutions for a discrete *p*-Laplacian problems with a parameter. Nonlinear Anal., Real World Appl. **224**, 216–223 (2013)
13. Candito, P., D'Aguì, G.: Three solutions for a discrete nonlinear Neumann problem involving the *p*-Laplacian. Adv. Differ. Equ. **2010**, Article ID 862016 (2010)
14. Candito, P., D'Aguì, G.: Constant-sign solutions for a nonlinear Neumann problem involving the discrete *p*-Laplacian. Opusc. Math. **34**, 683–690 (2014)
15. Candito, P., Giovannelli, N.: Multiple solutions for a discrete boundary value problem involving the *p*-Laplacian. Comput. Math. Appl. **21**, 649–659 (2015)
16. D'Aguì, G., Mawhin, J., Sciammetta, A.: Positive solutions for a discrete two point nonlinear boundary value problem with *p*-Laplacian. J. Math. Anal. Appl. **447**, 383–397 (2017)
17. Deng, X., Liu, X., Zhang, Y., Shi, H.: Periodic and subharmonic solutions for a 2*n*th-order difference equation involving *p*-Laplacian. Indag. Math. **24**, 613–625 (2013)
18. Iannizzotto, A., Tersian, S.: Multiple homoclinic orbits for the discrete *p*-Laplacian via critical point theory. J. Math. Anal. Appl. **403**, 173–182 (2013)
19. Jiang, D., O'Regan, D., Agarwal, R.P.: A generalized upper and lower solution method for singular discrete boundary value problems for the one-dimensional *p*-Laplacian. J. Appl. Anal. **11**, 35–47 (2005)
20. Kelly, W., Peterson, A.: Difference Equation: An Introduction with Applications. Academic Press, San Diego (1991)
21. Kuang, J.: Existence of homoclinic solutions for higher-order periodic difference equations with *p*-Laplacian. J. Math. Anal. Appl. **417**, 904–917 (2014)
22. Kuang, J., Guo, Z.: Homoclinic solutions of a class of periodic difference equations with asymptotically linear nonlinearities. Nonlinear Anal. **89**, 208–218 (2013)
23. Pucci, P., Radulescu, V.: The impact of the mountain pass theory in nonlinear analysis: a mathematical survey. Boll. Unione Mat. Ital. **9**, 543–582 (2013)
24. Radulescu, V.D., Repovs, D.D.: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton (2015)
25. Stevič, S.: Solvable subclasses of a class of nonlinear second-order difference equations. Adv. Nonlinear Anal. **5**(2), 147–165 (2016)
26. Vasilyev, A.V., Vasilyev, V.B.: On solvability of some difference-discrete equations. Opusc. Math. **36**(4), 525–539 (2016)