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Regularity to the spherically symmetric compressible Navier–Stokes equations with density-dependent viscosity

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Abstract

This paper is concerned with the dynamics for the compressible Navier–Stokes equations with density-dependent viscosity in bounded annular domains in R^3. In the paper, we shall analyze the spherical symmetric model and establish the regularity in H^2 and H^4 under certain assumptions imposed on the initial data.

Keywords: Navier–Stokes equations; Density-dependent viscosity; Regularity; Spherical symmetry

1 Introduction

It is well known that the compressible isentropic Navier–Stokes equations which describe the motion of compressible fluids can be written in Eulerian coordinates as

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \tag{1.1}$$

$$(\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \operatorname{div}\big(\mu(\rho) D(\mathbf{U})\big) - \nabla\big(\lambda(\rho)\operatorname{div}\mathbf{U}\big) + \nabla P(\rho) = 0, \tag{1.2}$$

where $\rho(\mathbf{x}, t)$, $\mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma$ $(\gamma > 1)$ stand for the fluid density, velocity and pressure, respectively, and the strain tensor is given by

$$D(\mathbf{U}) = \frac{\nabla \mathbf{U} + (\nabla \mathbf{U})^T}{2}. \tag{1.3}$$

The Lamé viscosity coefficients $\mu(\rho)$, $\lambda(\rho)$ satisfy the natural restrictions

$$\mu(\rho) > 0, \qquad \mu(\rho) + N\lambda(\rho) \geq 0. \tag{1.4}$$

For simplicity of the presentation, we consider only the viscosity terms $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ and $D(\mathbf{U}) = \nabla \mathbf{U}$. Then (1.1)–(1.2) become

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \tag{1.5}$$

$$(\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P(\rho) - \operatorname{div}(\rho \nabla \mathbf{U}) = 0. \tag{1.6}$$

We are concerned with the spherically symmetric solutions of system (1.5)–(1.6) in bounded annular domains $G = \{\mathbf{x} \in R^3, 0 < a < |\mathbf{x}| < b < +\infty\}$. To this end, we denote

$$|\mathbf{x}| = r, \qquad \rho(\mathbf{x}, t) = \rho(r, t), \qquad \mathbf{U}(\mathbf{x}, t) = u(r, t)\frac{\mathbf{x}}{r}, \tag{1.7}$$

which leads to the following system of equations for $r > 0$:

$$\rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \tag{1.8}$$

$$(\rho u)_t + \left(\rho u^2 + \rho^\gamma\right)_r + \frac{2\rho u^2}{r} - (\rho u)_r - \rho\left(\frac{2u}{r}\right)_r = 0. \tag{1.9}$$

We shall consider problem (1.8)–(1.9) in the region G subject to the initial data

$$(\rho, u)(r, 0) = (\rho_0, u_0)(r), \quad r \in [a, b], \tag{1.10}$$

and the boundary condition

$$u(a, t) = u(b, t) = 0, \quad t \in [0, T]. \tag{1.11}$$

First we find it convenient to transfer problems (1.8)–(1.11) into that in Lagrangian coordinates and draw the desired results. We introduce the following coordinate transformation:

$$x(r, t) = \int_a^r \rho(s, \tau)s^2\, ds, \quad t = \tau, \tag{1.12}$$

then the boundaries $r = a$ and $r = b$ become

$$x = 0, \qquad x = \int_a^b \rho(s, \tau)s^2\, ds = \int_a^b \rho_0(s)s^2\, ds, \tag{1.13}$$

where $\int_a^b \rho_0(s)s^2\, ds$ is the total initial mass and, without loss of generality, we can normalize it to 1. So in terms of Lagrangian coordinates, the domain G becomes $\Omega = (0, 1)$. The relations between Lagrangian and Eulerian coordinates are satisfied by

$$\frac{\partial x}{\partial r} = \rho r^2, \qquad \frac{\partial x}{\partial t} = \rho u r^2. \tag{1.14}$$

The initial boundary value problem (1.8)–(1.11) are changed to

$$\rho_t + \rho^2\left(r^2 u\right)_x = 0, \tag{1.15}$$

$$\frac{u_t}{r^2} + \left(\rho^\gamma\right)_x = \left(\rho^2\left(r^2 u\right)_x\right)_x - \frac{2\rho_x u}{r}, \tag{1.16}$$

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in [0, 1], \tag{1.17}$$

$$u(0, t) = u(1, t) = 0, \quad t > 0. \tag{1.18}$$

Much progress was achieved recently on the compressible Navier–Stokes equations with density-dependent viscosity coefficient. Firstly let us recall some well-known results as regards the one-dimensional compressible isentropic Navier–Stokes equations with the flow density being connected with the infinite vacuum, [19, 24, 25] for the local well-posedness and the global existence of weak solutions to an initial boundary value problem with the viscous gas being connected vacuum states with jump discontinuities, [9, 10] for the global behavior with the initial density being piecewise smooth; [28–31, 34] for the local existence, the global existence, the asymptotic behavior and the uniqueness of weak solutions with a viscous gas being connected vacuum states with continuous density.

In spatial multi-dimension, there is a huge literature as regards the global existence, the regularity and the asymptotic behavior of a solution to system (1.1)–(1.2) with constant viscosity, we refer the reader to [2, 3, 11–14, 16, 20–23, 33] and the references therein. For the 3-D flow of a compressible fluid with cylindrical symmetry, the global existence and the large-time behavior of generalized solutions have been proved in [1, 4, 5, 7, 15, 23, 26, 27, 32] for the isentropic and the nonisentropic cases. The corresponding study of the regularity of a solution for any given initial datum has been carried out in [17]. For the 3-D flow of compressible fluid with spherical symmetry, there are some interesting results, [18] for the global well-posedness of classical solutions with large oscillations and vacuum; [33] for the global existence and uniqueness of the weak solution without a solid core; [14] for the structure of the solution; [21] for the global existence of the exterior problem and the initial boundary value problem. Besides, we would like to refer to [6, 8] as regards the existence and regularity of solutions for micropolar fluid with spherical symmetry in the three-dimensional case.

In the paper, we shall analyze the spherical symmetric model and focus on the initial boundary problem of an isentropic compressible fluid. We show the regularity in H^2 and H^4 under certain assumptions imposed on the initial data.

The notation in this paper will be as follows:

L^p, $1 \le p \le +\infty$, $W^{m,p}$, $m \in N$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on $[0,1]$. To denote various constants, we use C_i ($i = 1,2,4$) to denote the generic positive constant depending only on the H^i norm of initial datum (ρ_0, u_0), $\min_{x\in[0,1]} \rho_0(x)$ and variable t, respectively. In addition, $\|\cdot\|$ denotes the norm in the space L^2.

The basic assumption of this paper is the following:

$$\inf_{[0,1]} \rho_0 > \underline{\rho}, \tag{1.19}$$

for some constant $\underline{\rho} > 0$.

Theorem 1.1 *Let $\gamma > 1$. Assume that the initial data $(\rho_0, u_0) \in H^2(\Omega) \times H^2(\Omega)$ and* (1.19) *hold, then there exists a unique generalized global solution $(\rho(t), u(t)) \in (H^2(\Omega))^2$ to the problem* (1.15)–(1.18) *verifying that, for any $T > 0$,*

$$\rho \in L^\infty\big([0,T], H^2(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big), \tag{1.20}$$

$$u \in L^\infty\big([0,T], H^2(\Omega)\big) \cap L^2\big([0,T], H^3(\Omega)\big), \tag{1.21}$$

$$u_t \in L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big). \tag{1.22}$$

Theorem 1.2 *Let $\gamma > 1$. Assume that the initial data satisfies* (1.19) *and* $(\rho_0, u_0) \in H^4(\Omega) \times H^4(\Omega)$, *then there exists a unique generalized global solution* $(\rho(t), u(t)) \in (H^4(\Omega))^2$ *to the problem* (1.15)–(1.18) *verifying that, for any* $T > 0$,

$$\rho \in L^\infty\big([0,T], H^4(\Omega)\big) \cap L^2\big([0,T], H^4(\Omega)\big), \tag{1.23}$$

$$u \in L^\infty\big([0,T], H^4(\Omega)\big) \cap L^2\big([0,T], H^4(\Omega)\big), \tag{1.24}$$

$$u_t \in L^\infty\big([0,T], H^2(\Omega)\big) \cap L^2\big([0,T], H^3(\Omega)\big), \tag{1.25}$$

$$u_{tt} \in L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big). \tag{1.26}$$

Corollary 1.1 *Under assumptions of Theorem* 1.2, (1.23)–(1.24) *imply* $(\rho(t), u(t))$ *is the classical solution verifying that, for any* $t > 0$,

$$\|\rho(t)\|_{C^{3+1/2}} + \|u(t)\|_{C^{3+1/2}} \le C_4. \tag{1.27}$$

2 Proof of Theorem 1.1

This section is devoted to deriving the estimates of the solutions to prove Theorem 1.1 which will be presented in a sequence of lemmas. We begin with the following lemma.

Lemma 2.1 (Theorem 2.2 in [21]) *Under the assumptions in Theorem* 1.1, *then there exist positive constants* $\rho_* > 0$ *and* $\rho^* > 0$ *so that the unique global solution* $(\rho(t), u(t))$ *to problem* (1.15)–(1.18) *exists and satisfies, for any* $T > 0$,

$$0 < \rho_* \le \rho(x,t) \le \rho^*, \tag{2.1}$$

$$\int_0^1 \big(u^2 + (\rho - \bar{\rho})^2 + u_x^2 + u_t^2 + \rho_x^2\big)(x,t)\,dx + \int_0^t \int_0^1 \big(\rho_x^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2\big)(x,s)\,dx\,ds \le C_1, \quad t \in [0,T], \tag{2.2}$$

where $\bar{\rho} = \frac{1}{b-a}\int_a^b \rho(s,t)s^2\,ds$.

Lemma 2.2 *Under the conditions in Theorem* 1.1, *for any* $T > 0$,

$$\|\rho_{xx}(t)\|^2 + \int_0^t \|\rho_{xx}(s)\|^2\,ds \le C_2, \quad t \in [0,T], \tag{2.3}$$

$$\|u_{xx}(t)\|^2 + \int_0^t \int_0^1 u_{xxx}^2(x,s)\,dx\,ds \le C_2, \quad t \in [0,T]. \tag{2.4}$$

Proof We infer from (1.16) that

$$\frac{u_t}{r^2} = -\gamma\rho^{\gamma-1}\rho_x + 2\rho\rho_x\big(r^2u\big)_x + \rho^2\big(r^2u_{xx} + 4rr_xu_x + 2r_x^2u + 2rr_{xx}u\big) - \frac{2\rho_x u}{r}. \tag{2.5}$$

Multiplying (2.5) by u_{xx} in $L^2(\Omega)$, we deduce

$$\int_0^1 \rho^2r^2u_{xx}^2\,dx = \int_0^1 u_{xx}\bigg(\frac{u_t}{r^2} + \gamma\rho^{\gamma-1}\rho_x - 2\rho\rho_x\big(r^2u\big)_x - \rho^2\big(4rr_xu_x + 2r_x^2u + 2rr_{xx}u\big) + \frac{2\rho_x u}{r}\bigg)\,dx. \tag{2.6}$$

Using Young's inequality, Sobolev's embedding theorem, and Lemma 2.1, we deduce from (2.6) that

$$\begin{aligned}\int_0^1 u_{xx}^2\,dx &\le C_1\int_0^1\left(u_t^2+\rho_x^2+\rho_x^2u_x^2\right)dx+\frac{1}{4}\int_0^1 u_{xx}^2\,dx\\ &\le C_2+C_1\|u_x\|_{L^\infty}^2\int_0^1\rho_x^2+\frac{1}{4}\int_0^1 u_{xx}^2\,dx\\ &\le C_2+\frac{1}{2}\int_0^1 u_{xx}^2\,dx,\end{aligned}$$

whence

$$\int_0^1 u_{xx}^2\,dx\le C_2. \tag{2.7}$$

Differentiating (1.16) with respect to x, and exploiting (1.15), we have

$$\left(\frac{u_t}{r^2}\right)_x=\left(-\rho^\gamma+\rho^2\left(r^2u\right)_x\right)_{xx}-\left(\frac{2\rho_x u}{r}\right)_x, \tag{2.8}$$

which gives

$$\rho_{txx}+\gamma\rho^{\gamma-1}\rho_{xx}=E_0(x,t) \tag{2.9}$$

with

$$E_0(x,t)=\frac{2\rho_x u r_x}{r^2}-\frac{2\rho_{xx}u+2\rho_x u_x}{r}-\gamma(\gamma-1)\rho^{\gamma-2}\rho_x^2+\frac{2u_t r_x}{r^3}-\frac{u_{tx}}{r^2}.$$

Multiplying (2.9) by ρ_{xx}, integrating the resultant over $[0,1]$, we deduce

$$\begin{aligned}&\frac{d}{dt}\left\|\rho_{xx}(t)\right\|^2+\int_0^1\gamma\rho^{\gamma-1}\rho_{xx}^2\,dx\\ &\quad=\int_0^1\left(\frac{2\rho_x u r_x}{r^2}-\frac{2\rho_{xx}u+2\rho_x u_x}{r}-\gamma(\gamma-1)\rho^{\gamma-2}\rho_x^2+\frac{2u_t r_x}{r^3}-\frac{u_{tx}}{r^2}\right)\rho_{xx}\,dx\\ &\quad\le C_1\|\rho_{xx}\|\left(\|u_{tx}\|+\|u_t r_x\|+\left\|\rho_x^2\right\|+\|\rho_x u_x\|+\|\rho_x u r_x\|\right)-2\int_0^1\frac{u}{r}\rho_{xx}^2\,dx.\end{aligned}$$

Using the Young inequality and the interpolation inequality to the above inequality, then we get

$$\begin{aligned}\frac{d}{dt}\left\|\rho_{xx}(t)\right\|^2&\le C_1\left(\|u_{tx}\|^2+\|u_t\|^2+\|\rho_x\|^2+\|u_x\|^2+\|u\|^2\right)\\ &\quad+C_1\|\rho_{xx}\|^2\left(1+\|u\|_{L^\infty}^2\right).\end{aligned} \tag{2.10}$$

Integrating (2.10) with respect to t, using initial condition $\rho_0\in H^2$ and Lemma 2.1, we have

$$\left\|\rho_{xx}(t)\right\|^2\le C_2+C_1\int_0^t\left\|\rho_{xx}(s)\right\|^2\,ds,\quad \forall t\in[0,T].$$

Then using the Gronwall inequality to the above inequality, we can get (2.3).

Differentiating (1.16) with respect to x, we can obtain

$$u_{xxx} = -\frac{1}{\rho^2 r^2}\Big[(4ru\rho_x^2 r_x + 2r^2\rho_x^2 u_x + 2\rho\rho_{xx}(2rr_x u + r^2 u_x) + 4\rho\rho_x(2r_x^2 u + 4rr_x u_x + 2rr_{xx}u + r^2 u_{xx}) + \rho^2(6r_x r_{xx}u + 6r_x^2 u_x + 6rr_{xx}u + r^2 u_{xx}) + \rho^2(6r_x r_{xx}u + 6r_x^2 u_x + 6rr_{xx}u_x + 6rr_x u_{xx} + 2rr_{xxx}u) + \frac{2\rho_{xx}u + 2\rho_x u_x}{r} - \frac{2\rho_x u r_x}{r^2} + \frac{u_{tx}}{r^2} - \frac{u_t r_x}{r^3} + \gamma\rho^{\gamma-1}\rho_{xx} + \gamma(\gamma-1)\rho^{\gamma-2}\rho_x^2\Big]. \tag{2.11}$$

Integrating (2.11) over x and t, applying the embedding theorem, Lemma 2.1, (2.7) and (2.3), we conclude, for any $t \in [0, T]$,

$$\begin{aligned}\int_0^t\int_0^1 u_{xxx}^2\,dx\,ds &\le C_1\int_0^t\int_0^1 (u_{tx}^2 + \rho_x^4 + \rho_{xx}^2 + \rho_x^4 u_x^2 + \rho_x^2 u_x^2 + \rho_{xx}^2 u_x^2 + u_t^2)(x,s)\,dx\,ds \\ &\le C_1 + C_1\int_0^t \|u_x\|_{L^\infty}^2 \int_0^1 (\rho_{xx}^2 + \rho_x^4)\,dx\,ds + C_1\int_0^t \|\rho_x\|_{L^\infty}^2\|u_{xx}\|^2\,ds \\ &\le C_2,\end{aligned}$$

which, along with (2.7), gives (2.4). The proof is complete. □

Proof of Theorem 1.1 Clearly, (1.20)–(1.22) follow from Lemmas 2.1–2.2. This completes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

In this section, we shall complete the proof of Theorem 1.2. To this end, we assume that in this section that all assumptions in Theorem 1.2 hold. We begin with the following lemma.

Lemma 3.1 *The following estimate holds for any* $T > 0$:

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{ttx}(s)\|^2\,ds \le C_4 + C_2\int_0^t \|u_{txx}(s)\|^2\,ds, \quad t \in [0, T]. \tag{3.1}$$

Proof We easily infer from (1.16) and (2.1)–(2.4) that

$$\|u_t(t)\| \le C_1(\|\rho_x(t)\| + \|u_x(t)\| + \|u_{xx}(t)\|). \tag{3.2}$$

Differentiating (1.16) with respect to x and using Lemmas 2.1–2.2, we have

$$\|u_{tx}(t)\| \le C_2(\|\rho_x(t)\| + \|\rho_{xx}(t)\| + \|u_x(t)\| + \|u_{xx}(t)\| + \|u_{xxx}(t)\|), \tag{3.3}$$

or

$$\|u_{xxx}(t)\| \le C_2(\|\rho_x(t)\| + \|\rho_{xx}(t)\| + \|u_x(t)\| + \|u_{xx}(t)\| + \|u_{tx}(t)\|). \tag{3.4}$$

Differentiating (1.16) with respect to x twice and using the Cauchy–Schwarz inequality, we have

$$\|u_{txx}(t)\| \le C_2\big(\|\rho_x(t)\|_{H^2} + \|u_x(t)\|_{H^3}\big), \tag{3.5}$$

$$\|u_{xxxx}(t)\| \le C_2\big(\|\rho_x(t)\|_{H^2} + \|u_x(t)\|_{H^2} + \|u_{txx}(t)\|\big). \tag{3.6}$$

After differentiating (1.16) with respect to t, using (3.2)–(3.3) and (3.5), we can get

$$\begin{aligned}\|u_{tt}(t)\| &\le C_2\big(\|u_{tx}(t)\| + \|u_{txx}(t)\| + \|u_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\| + \|u_{xx}(t)\|\big) &(3.7)\\ &\le C_2\big(\|\rho_x(t)\|_{H^2} + \|u_x(t)\|_{H^3}\big). &(3.8)\end{aligned}$$

Now differentiating (1.16) with respect to t twice, multiplying the resulting equation by $(\frac{u}{r^2})_{tt}$ in $L^2(\Omega)$, and using integration by parts and (1.18), we conclude

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\Big(\frac{u}{r^2}\Big)_{tt}^2 dx = &-\int_0^1\big(-\rho^\gamma + \rho^2(r^2u)_x\big)_{tt}\Big(\frac{u}{r^2}\Big)_{ttx} dx\\ &-\int_0^1\Big(\frac{2\rho_x u}{r} - \frac{u^2}{r^3}\Big)_{tt}\Big(\frac{u}{r^2}\Big)_{tt} dx =: A_1 + A_2. \end{aligned}\tag{3.9}$$

We use (2.1)–(2.4), (3.2)–(3.8), (1.16) and the interpolation inequality to deduce that, for any small $\varepsilon \in (0,1)$,

$$\begin{aligned}A_1 &= -\int_0^1\big(-\rho^\gamma + \rho^2(r^2u)_x\big)_{tt}\Big(\frac{u}{r^2}\Big)_{ttx} dx\\ &\le -\int_0^1 \rho^2 u_{ttx}^2\,dx + C_2\big(\|\rho_{tt}u_x\| + \|\rho_t^2 u_x\| + \|\rho_t u_t\| + \|\rho_t u_x\| + \|\rho_t u_{xt}\| + \|u_x\| + \|u_t\|\\ &\quad + \|u_x u_t\| + \|u_{tx}\| + \|u_{tt}\|\big)\big(\|u_{ttx}\| + \|u_{tt}\| + \|u_x u_t\| + \|u_{tx}\| + \|u_t\| + \|u_x\| + \|u^3\|\big)\\ &\le -\int_0^1 \rho^2 u_{ttx}^2\,dx + \varepsilon\|u_{ttx}(t)\|^2 + C_2\big(\|\rho_{tt}(t)\|_{H^1}^2 + \|\rho_t(t)\|^2\\ &\quad + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|u_x(t)\|_{H^1}^2 + \|\rho_x(t)\|_{H^1}^2\big)\\ &\le -C_2^{-1}\|u_{ttx}(t)\|^2 + C_2\big(\|u_x(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_t(t)\|_{H^2}^2\big), \end{aligned}\tag{3.10}$$

$$\begin{aligned}A_2 &= -\int_0^1\Big(\frac{2\rho_x u}{r} - \frac{u^2}{r^3}\Big)_{tt}\Big(\frac{u}{r^2}\Big)_{tt} dx\\ &\le C_2\big(\|\rho_{ttx}\| + \|\rho_{tx}u_t\| + \|\rho_x u_{tt}\| + \|\rho_{tx}u^2\| + \|u_t^2\| + \|u_{tt}\| + \|u_t\| + \|\rho_{xt}u\|\\ &\quad + \|\rho_x u_t\| + \|\rho_x u^3\| + \|u_t\| + \|u^3\|\big)\big(\|u_{tt}\| + \|u_t\| + \|u_t r_t\| + \|u r_{tt}\| + \|u r_t^2\|\big)\\ &\le C_2\big(\|u_t(t)\|^2 + \|u_{tt}(t)\|^2 + \|u_{txx}(t)\|^2 + \|u_{tx}(t)\|^2 + \|\rho_x(t)\|^2 + \|u_x(t)\|^2\big). \end{aligned}\tag{3.11}$$

Integrating (3.9) with respect to t, and using (2.1)–(2.4), initial condition (1.16) and (3.10)–(3.11), then we obtain (3.1). □

Lemma 3.2 *For any $T > 0$ and $\varepsilon \in (0,1)$, the following estimate holds*:

$$\|u_{tx}(t)\|^2 + \int_0^t\|u_{txx}(s)\|^2 \le C_4 + C_2^{-1}\varepsilon^2\int_0^t\|u_{ttx}(s)\|^2\,ds, \quad t \in [0,T]. \tag{3.12}$$

Proof Differentiating (1.16) with respect to t and x, then multiplying the resultant by $(\frac{u}{r^2})_{tx}$ in $L^2[0,1]$, and integrating by parts, we know that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{u}{r}\right)_{tx}^2 dx\\
&= \left(-\rho^\gamma+\rho^2\left(r^2u\right)_x\right)_{tx}\left(\frac{u}{r^2}\right)_{tx}\Big|_0^1 - \int_0^1\left(-\rho^\gamma+\rho^2\left(r^2u\right)_x\right)_{tx}\left(\frac{u}{r^2}\right)_{txx} dx\\
&\quad+\int_0^1\left(\frac{u^2}{r^3}-\frac{2\rho_x u}{r}\right)_{tx}\left(\frac{u}{r^2}\right)_{tx} dx\\
&= B_0(x,t)+B_1(t)+B_2(t),
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
B_0(x,t) &= \left(-\rho^\gamma+\rho^2\left(r^2u\right)_x\right)_{tx}\left(\frac{u}{r^2}\right)_{tx}\Big|_0^1,\\
B_1(t) &= -\int_0^1\left(-\rho^\gamma+\rho^2\left(r^2u\right)_x\right)_{tx}\left(\frac{u}{r^2}\right)_{txx} dx,\\
B_2(t) &= \int_0^1\left(\frac{u^2}{r^3}-\frac{2\rho_x u}{r}\right)_{tx}\left(\frac{u}{r^2}\right)_{tx} dx.
\end{aligned}$$

Now using Young's inequality several times, and employing (2.1)–(2.4) and the interpolation inequality, after some calculation, we have, for any $\varepsilon\in(0,1)$,

$$\begin{aligned}
B_0(x,t) &\le C_2\big(\|\rho_{tx}\|_{L^\infty}+\|\rho_t\rho_x\|_{L^\infty}+\|\rho_x\rho_t u_x\|_{L^\infty}+\|\rho_{tx}u_x\|_{L^\infty}+\|\rho_t u_{xx}\|_{L^\infty}\\
&\qquad+\|u_{txx}\|_{L^\infty}+\|u_{tx}\|_{L^\infty}\big)\|u_{tx}\|_{L^\infty}\\
&\le C_2\big(\|u_x\|_{H^2}+\|\rho_x\|_{H^1}+\|u_{tx}\|^{\frac12}\|u_{txx}\|^{\frac12}+\|u_{txx}\|^{\frac12}\|u_{txxx}\|^{\frac12}\big)\|u_{tx}\|^{\frac12}\|u_{txx}\|^{\frac12}\\
&\le C_2^{-1}\varepsilon^2\big(\|u_{txx}\|^2+\|u_{txxx}\|^2\big)+C_2\varepsilon^{-6}\big(\|u_{tx}\|^2+\|u_x\|_{H^2}+\|\rho_x\|_{H^1}\big),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
B_1(t) &\le -\int_0^1\rho^2u_{txx}^2\,dx+\varepsilon\int_0^1\rho^2u_{txx}^2\,dx+C_2\big(\|\rho_{tx}(t)\|_{H^1}^2+\|\rho_x(t)\|_{H^1}^2\\
&\qquad+\|u_x(t)\|_{H^1}^2+\|u_t(t)\|^2+\|u_{tx}(t)\|^2\big),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
B_2(t) &\le C_2\big(\|u_xu_t\|+\|u_{tx}\|+\|u_x\|+\|u_t\|+\|u^3\|+\|\rho_{txx}\|+\|\rho_{tx}u_x\|+\|\rho_{xx}u_t\|\\
&\qquad+\|\rho_xu_{tx}\|+\|\rho_{xx}\|+\|\rho_xu_x\|+\|\rho_{tx}\|+\|\rho_xu_t\|+\|\rho_x\|\big)\big(\|u_{tx}\|+\|u_x\|\big)\\
&\le C_2\big(\|u_x\|_{H^1}^2+\|u_t\|_{H^1}^2+\|\rho_x\|_{H^1}^2+\|\rho_{tx}\|^2+\|\rho_{txx}\|^2\big)\\
&\le C_2\big(\|u_x\|_{H^2}^2+\|u_t\|_{H^1}^2+\|\rho_x\|_{H^1}^2\big).
\end{aligned} \tag{3.16}$$

Differentiating (1.16) with respect to x and t, and using Lemmas 2.1–2.2 and (3.7)–(3.8), we conclude

$$\|u_{txxx}(t)\|\le C_2\big(\|u_x(t)\|_{H^2}+\|\rho_x(t)\|_{H^1}+\|u_{tx}(t)\|_{H^1}+\|u_{ttx}(t)\|\big). \tag{3.17}$$

We integrate (3.13) with respect to t, use (3.3), (3.14)–(3.16) and Lemmas 2.1–2.2 to obtain (3.12). The proof is complete. □

Lemma 3.3 *The following estimates hold for any $T > 0$:*

$$\|u_{tt}(t)\|^2 + \|u_{tx}(t)\|^2 + \int_0^t (\|u_{txx}\|^2 + \|u_{ttx}\|^2)(s) \le C_4, \quad t \in [0,T], \tag{3.18}$$

$$\|\rho_{xxx}(t)\|^2 + \|u_{xxx}(t)\|^2 + \int_0^t (\|\rho_{xxx}\|^2 + \|u_{xxxx}\|^2)(s)\,ds \le C_4, \quad t \in [0,T]. \tag{3.19}$$

Proof We insert (3.1) into (3.12) and pick ε small enough to get (3.18).

Differentiating (2.9) with respect to x, we have

$$\rho_{txxx} + \gamma\rho^{\gamma-1}\rho_{xxx} = E_1(x,t), \tag{3.20}$$

where

$$E_1(x,t) = E_{0x}(x,t) - \gamma(\gamma-1)\rho^{\gamma-2}\rho_x\rho_{xx}.$$

Taking into account estimate (3.18), from (2.1)–(2.4) we can get

$$\begin{aligned}\|E_1(t)\| &\le C_2(\|E_{0x}\| + \|\rho_x\rho_{xx}\|)\\ &\le C_2(\|\rho_x u_x\| + \|\rho_{xx}\| + \|\rho_{xx}u_x\| + \|\rho_{xxx}\| + \|\rho_x u_{xx}\| + \|\rho_{xx}\rho_x\|\\ &\quad + \|u_{tx}\| + \|u_t\rho_x\| + \|u_{txx}\|)\\ &\le C_2(\|\rho_x(t)\|_{H^2} + \|u_x(t)\|_{H^2} + \|u_t(t)\|_{H^2}),\end{aligned}$$

which, along with Lemmas 2.1–2.2 and (3.18), implies

$$\int_0^t \|E_1(s)\|^2\,ds \le C_4 + C_2\int_0^t \|\rho_{xxx}(s)\|^2\,ds. \tag{3.21}$$

After multiplying (3.20) by ρ_{xxx} in $L^2[0,1]$, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\rho_{xxx}\|^2 + \gamma\int_0^1 \rho^{\gamma-1}\rho_{xxx}^2\,dx \le C_1\|E_1(t)\|^2, \tag{3.22}$$

which implies

$$\frac{d}{dt}\|\rho_{xxx}\|^2 \le C_1\|E_1(t)\|^2. \tag{3.23}$$

Integrating (3.23) with respect to t, using (3.21), we conclude

$$\|\rho_{xxx}\|^2 \le C_4 + C_2\int_0^t \|\rho_{xxx}(s)\|^2\,ds,$$

which, after applying the Gronwall inequality and (3.22), yields

$$\|\rho_{xxx}(t)\|^2 + \int_0^t \|\rho_{xxx}(s)\|^2\,ds \le C_4, \quad \forall t \in [0,T]. \tag{3.24}$$

By (3.4), (3.6), (3.18) and Lemmas 2.1–2.2, we conclude

$$\left\|u_{xxx}(t)\right\|^2 + \int_0^t \left\|u_{xxxx}(s)\right\|^2 ds \le C_4, \quad \forall t \in [0,T],$$

which, along with (3.24), gives (3.19). The proof is complete. □

Lemma 3.4 *The following estimates hold for any $T > 0$:*

$$\left\|\rho_{xxxx}(t)\right\|^2 + \int_0^t \left\|\rho_{xxxx}(s)\right\|^2 ds \le C_4, \quad t \in [0,T], \tag{3.25}$$

$$\left\|u_{txx}(t)\right\|^2 + \left\|u_{xxxx}(t)\right\|^2 + \int_0^t \left\|u_{xxxxx}(s)\right\|^2 ds \le C_4, \quad t \in [0,T]. \tag{3.26}$$

Proof Differentiating (1.16) with respect to t, and using Lemmas 2.1–2.2 and Lemma 3.3, we can get

$$\begin{aligned}\left\|u_{txx}(t)\right\| &\le C_1\left(\left\|u_t(t)\right\|_{H^1} + \left\|\rho_x(t)\right\| + \left\|u_x(t)\right\|_{H^1} + \left\|u_{tt}(t)\right\|\right)\\ &\le C_4, \quad \forall t \in [0,T].\end{aligned} \tag{3.27}$$

Differentiating (3.20) with respect to x, we have

$$\rho_{txxxx} + \gamma\rho^{\gamma-1}\rho_{xxxx} = E_2(x,t), \tag{3.28}$$

where

$$E_2(x,t) = E_{1x}(x,t) - \gamma(\gamma-1)\rho_x\rho_{xxx} \tag{3.29}$$

and

$$E_{1x}(x,t) = E_{0xx}(x,t) - \gamma(\gamma-1)\left(\rho^{\gamma-2}\rho_x\rho_{xx}\right)_x.$$

An easy calculation with the interpolation inequality, (3.2)–(3.8) and Lemmas 2.1–2.2 and Lemma 3.3 gives

$$\begin{aligned}\left\|E_{0x}(t)\right\| &\le C_2\left(\|\rho_x u_x\| + \|\rho_{xxx}u_x\| + \|\rho_{xx}u_{xx}\| + \left\|\rho_x^3\right\| + \|u_{tx}r_x\| + \|u_t r_{xx}\| + +\|u_{txx}\|\right)\\ &\le C_2\left(\|\rho_x\|_{H^3} + \|u_x\|_{H^2} + \|u_{tx}\| + \|u_{txx}\|\right),\end{aligned} \tag{3.30}$$

$$\begin{aligned}\left\|E_{0xx}(t)\right\| &\le C_2\left(\|\rho_{xxxx}\| + \|\rho_{xx}u_x\| + \|\rho_x u_{xx}\| + \|\rho_{xxx}u_x\| + \|\rho_{xx}u_{xx}\|\right.\\ &\quad\left. + \|u_{txx}\| + \|u_{tx}u_x\| + \|u_{txxx}\|\right)\\ &\le C_2\left(\|\rho_x\|_{H^3} + \|u_x\|_{H^2} + \|u_{tx}\|_{H^2}\right),\end{aligned} \tag{3.31}$$

$$\begin{aligned}\left\|E_{1x}(t)\right\| &\le C_2\left(\left\|E_{0xx}(t)\right\| + \left\|\rho_x^2\rho_{xx}\right\| + \left\|\rho_{xx}^2\right\| + \|\rho_x\rho_{xxx}\|\right)\\ &\le C_2\left(\|\rho_x\|_{H^3} + \|u_x\|_{H^2} + \|u_{tx}\|_{H^2}\|\right).\end{aligned} \tag{3.32}$$

Taking into account estimate (3.30)–(3.32), from (3.29), we obtain

$$\left\|E_2(t)\right\| \le C_4\left(\left\|u_x(t)\right\|_{H^3} + \left\|\rho_x(t)\right\|_{H^3} + \left\|u_{tx}(t)\right\|_{H^2}\right). \tag{3.33}$$

Inserting (3.17) into (3.33), and integrating (3.33) with respect to t, using Lemmas 2.1–2.2 and Lemma 3.3, we have

$$\int_0^t \|E_2(s)\|^2\,ds \le C_4 + C_2 \int_0^t \|\rho_{xxxx}(s)\|^2\,ds, \quad \forall t \in [0,T]. \tag{3.34}$$

Multiplying (3.28) by ρ_{xxxx} in $L^2[0,1]$, we can get

$$\frac{1}{2}\frac{d}{dt}\|\rho_{xxxx}\|^2 + \gamma \int_0^1 \rho^{\gamma-1}\rho_{xxxx}^2\,dx \le C_2\|E_2(t)\|^2, \tag{3.35}$$

which implies

$$\frac{d}{dt}\|\rho_{xxxx}\|^2 \le C_2\|E_2(t)\|^2. \tag{3.36}$$

Integrating (3.36) with respect to t, using (3.34), we conclude

$$\|\rho_{xxxx}\|^2 \le C_4 + C_2 \int_0^t \|\rho_{xxxx}(s)\|^2\,ds, \quad t \in [0,T] \tag{3.37}$$

which, after using Gronwall's inequality, yields

$$\|\rho_{xxxx}(t)\|^2 \le C_4, \quad t \in [0,T]. \tag{3.38}$$

Thus, we can obtain (3.25) by virtue of (3.37)–(3.38).

By (3.6), (3.17), (3.25), Lemmas 2.1–2.2 and Lemma 3.3, we have

$$\|u_{xxxx}(t)\|^2 + \int_0^t \|u_{txxx}(s)\|^2\,ds \le C_4, \quad t \in [0,T]. \tag{3.39}$$

On the other hand, we differentiate (1.16) with respect to x three times, use Lemmas 2.1–2.2 and (3.25), (3.27) to conclude, for any $t \in [0,T]$,

$$\|u_{xxxxx}(t)\| \le C_4\big(\|u_{txxx}(t)\| + \|u_x(t)\|_{H^3} + \|\rho_x(t)\|_{H^3}\big). \tag{3.40}$$

Thus we deduce from (3.38)–(3.40) that

$$\int_0^t \|u_{xxxxx}(s)\|^2\,ds \le C_4, \quad \forall t \in [0,T]$$

which, along with (3.39), gives (3.26). This completes the proof of the lemma. □

Proof of Theorem 1.2 Applying Lemmas 2.1–2.2 and Lemmas 3.1–3.4, we readily get estimates (1.23)–(1.26). This completes the proof of Theorem 1.2. □

4 Conclusions

In this paper, we have established the regularity of global solutions for the spherically symmetric compressible fluid with density-dependent viscosity in H_2 and H_4. The biggest difference from other papers is that our domain has spherical symmetry and the viscosity coefficients are density dependent.

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