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Existence and multiplicity of solutions for a p-Kirchhoff equation on $\mathbb{R}^N$

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Abstract

In this paper, we consider the following p-Kirchhoff equation:

$$\bigl[M\bigl(\|u\|^p\bigr)\bigr]^{p-1}\bigl(-\Delta_p u + V(x)|u|^{p-2}u\bigr) = f(x,u), \quad x \in \mathbb{R}^N, \tag{P}$$

where $f(x,u) = \lambda g(x)|u|^{q-2}u + h(x)|u|^{r-2}u$, $1 < q < p < r < p^*$ ($p^* = \frac{Np}{N-p}$ if $N \geq p$, $p^* = \infty$ if $N \leq p$). Using variational methods, we prove that, under proper assumptions, there exist $\lambda_0, \lambda_1 > 0$ such that problem (P) has a solution for all $\lambda \in [0, \lambda_0)$ and has a sequence of solutions for all $\lambda \in [0, \lambda_1)$.

MSC: 35B38; 35J20; 35J62

Keywords: p-Kirchhoff equation; Variational methods; Existence and multiplicity of solutions

1 Introduction and main results

In this paper, we consider the following p-Kirchhoff equation:

$$\bigl[M\bigl(\|u\|^p\bigr)\bigr]^{p-1}\bigl(-\Delta_p u + V(x)|u|^{p-2}u\bigr) = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where M, V are continuous functions, $f(x,u) = \lambda g(x)|u|^{q-2}u + h(x)|u|^{r-2}u$ $(1 < q < p < r < p^*)$ is concave and convex, and

$$\|u\|^p = \int_{\mathbb{R}^N} \bigl(|Du|^p + V(x)|u|^p\bigr)\,dx \quad (1 < p < N).$$

Since the pioneering work of Lions [1], much attention has been paid to the existence of nontrivial solutions, multiplicity of solutions, ground state solutions, sign-changing solutions, and concentration of solutions for problem (1.1). For example, for the following Kirchhoff equation:

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

Li and Ye [2] and Guo [3] showed the existence of a ground state solution for problem (1.2) with $N = 3$, where the potential $V(x) \in C(\mathbb{R}^3)$ and it satisfies $V(x) \leq \liminf_{|y|\to+\infty} V(y) \triangleq$

$V_\infty < +\infty$. Sun and Wu [4] investigated the existence and non-existence of nontrivial solutions with the following assumption: $V(x) \geq 0$ and there exists $c > 0$ such that meas$\{x \in \mathbb{R}^N : V(x) < c\}$ is nonempty and has finite measure. Wu [5] proved that problem (1.2) has a nontrivial solution and a sequence of high energy solutions where $V(x)$ is continuous and satisfies $\inf V(x) \geq a_1 > 0$ and for each $M > 0$, meas$\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$. Nie and Wu [6] treated (1.2) where the potential is a radial symmetric function. Chen et al. [7] considered equation (1.2) when $f(x,u) = \lambda a(x)|u|^{q-2}u + b(x)|u|^{r-2}u$ $(1 < q < p = 2 < r < 2^*)$.

Moreover, for p-Kirchhoff-type problem of the following form:

$$-\left[a + \lambda M\left(\|u\|^p\right)\right]\left[-\Delta_p u + b|u|^{p-2}u\right] = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

Cheng and Dai [8] proved the existence and non-existence of positive solutions, where $M(t)$ satisfies

(M) There exists $\sigma \in (0,1)$ such that $\hat{M}(t) \geq \sigma[M(t)]t$, here $\hat{M}(t) = \int_0^t M(s)\,ds$.

Furthermore, the authors in [9] dealt with problem (1.3) for the special case $M(t) = t$ and $p = 2$. Recently, Chen and Zhu [10] considered problem (1.3) for $M(t) = t^\tau$ and $f(u) = |u|^{m-2}u + \mu|u|^{q-2}u$. Similar consideration can be found in [11–13].

However, p-Kirchhoff problem in the following form:

$$-\left[M\left(\|u\|^p\right)\right]^{p-1}\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

or p-Kirchhoff problem like (1.1) seems to be considered by few researchers as far as we know. Alves et al. [14] and Corrêa and Figueiredo [15] established the existence of a positive solution for problem (1.4) by the mountain pass lemma, where M is assumed to satisfy the following conditions:

(H_1) $M(t) \geq m_0$ for all $t \geq 0$.

(H_2) $\hat{M}(t) \geq [M(t)]^{p-1}t$ for all $t \geq 0$, where $\hat{M}(t) = \int_0^t [M(s)]^{p-1}\,ds$.

In [16], Liu established the existence of infinitely many solutions to a Kirchhoff-type equation like (1.1). They treated the problem with M satisfying (H_1) and

(H_3) $M(t) \leq m_1$ for all $t > 0$.

Very recently, Figueiredo and Nascimento [17] and Santos Junior [18] considered solutions of problem (1.1) by minimization argument and minimax method, respectively, where $p = 2$ and M satisfies (H_1) and

(H_4) The function $t \mapsto M(t)$ is increasing and the function $t \mapsto \frac{M(t)}{t}$ is decreasing.

Subsequently, Li et al. [19] investigated the existence, multiplicity, and asymptotic behavior of solutions for problem (1.4), where M could be zero at zero, i.e., the problem is degenerate.

Note that $M(t) = a + bt$ does not satisfy (H_2) for $p = 2$ and (H_3) for all $1 < p < N$. Moreover, $M(t) = a + bt^k$ fails to satisfy (H_2), (H_3) for all $k > 0$, and (H_4) for all $k > 1$. In this paper, we will assume proper conditions on M, which cover the typical case $M(t) = a + bt^k$ and the degenerate case. Furthermore, our assumption on the potential V is totally different from all the previous works which were concerned with Kirchhoff-type problems to the best of our knowledge. The assumption on V is related to the functions g, h in the nonlinearity f. The potential V is not necessarily radial and can be unbounded or decaying to zero as $|x| \to +\infty$ according to different functions g and h. See assumptions (V) and (M_1)–(M_5) below.

Before stating our main results, we introduce some function spaces and then present two embedding theorems, which is important to investigating our problem. For any $s \in (1,+\infty)$ and any continuous function $K(x):\mathbb{R}^N \to \mathbb{R}, K(x) \geq 0, \not\equiv 0$, we define the weighted Lebesgue space $L^s(\mathbb{R}^N, K)$ equipped with the norm

$$\|u\|_{L^s(\mathbb{R}^N,K)} = \left(\int_{\mathbb{R}^N} K(x)|u|^s\,dx\right)^{1/s}. \tag{1.5}$$

Throughout the article we assume $V(x)$ satisfies

(V) $V(x) \in C(\mathbb{R}^N)$, $V(x) \geq 0$, and $\{x \in \mathbb{R}^N : V(x) = 0\} \subset B_{R_0}$ for some $R_0 > 0$, where $B_{R_0} = \{x||x| \leq R_0, x \in \mathbb{R}^N\}$.

The natural functional space to study problem (1.1) is X with respect to the norm

$$\|u\|^p = \int_{\mathbb{R}^N} \left(|Du|^p + V(x)|u|^p\right) dx. \tag{1.6}$$

The following theorem is due to Lyberopoulos [20]. Denote $B_R = \{x|x \in \mathbb{R}^N, |x| \leq R\}$ and $B_R^C = \mathbb{R}^N \backslash B_R$.

Theorem 1.1 *Let* $p < r < p^*$, $V(x)$ *satisfies* (V), $h(x) \in C(\mathbb{R}^N)$, *and* $h(x) \geq 0, \not\equiv 0$ *such that*

$$\mathcal{M} := \lim_{R\to+\infty} m(R) < +\infty, \tag{1.7}$$

where

$$m(R) := \sup_{x\in B_R^C} \frac{(h(x))^{p^*-p}}{(V(x))^{p^*-r}}.$$

Then the embedding $X \hookrightarrow L^r(\mathbb{R}^N, h)$ *is continuous. Furthermore, if* $\mathcal{M} = 0$, *then the embedding is compact.*

Theorem 1.2 *Let* $1 < q < p$, $V(x)$ *satisfies* (V), $g(x) \in C(\mathbb{R}^N)$, $g(x) \geq 0, \not\equiv 0$ *such that*

$$\mathcal{L} := \lim_{R\to+\infty} l(R) < +\infty, \tag{1.8}$$

where

$$l(R) := \int_{B_R^C} g^{\frac{p}{p-q}} V^{-\frac{q}{p-q}}\,dx.$$

Then the embedding $X \hookrightarrow L^q(\mathbb{R}^N, g)$ *is continuous. Furthermore, if* $\mathcal{L} = 0$, *then the embedding is compact.*

Proof This theorem can be seen as a corollary of Theorem 2.3 in [21]. Here we give a detailed proof for the readers convenience. Let $\varphi_R \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \leq \varphi_R \leq 1$, $\varphi_R(x) = 0$ for $|x| < R$, $\varphi_R(x) = 1$ for $|x| > R + 1$, and $|D\varphi_R(x)| \leq C$. For any fixed $R > R_0$, we write $u = \varphi_R u + (1 - \varphi_R)u$. Then it follows from Hölder's inequality that

$$\|\varphi_R u\|^q_{L^q(\mathbb{R}^N,g)} \leq \int_{B_R^C} g|u|^q\,dx \leq \left(\int_{B_R^C} V|u|^p\,dx\right)^{\frac{q}{p}} \left(\int_{B_R^C} g^{\frac{p}{p-q}} V^{-\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{p}}$$

$$\leq \big(l(R)\big)^{\frac{p-q}{p}} \left(\int_{B_R^C} \big(|Du|^p + V|u|^p\big)\,dx\right)^{\frac{q}{p}}. \tag{1.9}$$

Furthermore, by the Sobolev embedding theorem, we have

$$\begin{aligned} \big\|(1-\varphi_R)u\big\|^q_{L^q(\mathbb{R}^N,g)} &\leq \int_{B_{R+1}} g|u|^q\,dx \leq C\int_{B_{R+1}} |u|^q\,dx \\ &\leq C\biggl(\int_{B_{R+1}} |Du|^p\,dx\biggr)^{q/p} \\ &\leq C\biggl(\int_{B_{R+1}} \big(|Du|^p + V(x)|u|^p\big)\,dx\biggr)^{q/p}. \end{aligned} \tag{1.10}$$

Combining (1.9) with (1.10), we obtain the continuity of the embedding $X \hookrightarrow L^q(\mathbb{R}^N, g)$.

In the following, we prove the embedding $X \hookrightarrow L^q(\mathbb{R}^N, g)$ is compact. Let $\mathcal{L} = 0$ and suppose that $u_n \rightharpoonup 0$ weakly in X. Then $\|u_n\|_X$ is bounded. Hence it follows from (1.9) that for any $\varepsilon > 0$, there exists $R > 0$ sufficiently large such that

$$\|\varphi_R u_n\|_{L^q(\mathbb{R}^N,g)} \leq \frac{\epsilon}{2}.$$

Moreover, by the Rellich–Kondrachov theorem, $\|(1-\varphi_R)u_n\|_{L^q(\mathbb{R}^N,g)} \to 0$, and so there exists $n(\epsilon) \in \mathbb{N}$ such that, for all $n \geq n(\epsilon)$,

$$\big\|(1-\varphi_R)u_n\big\|_{L^q(\mathbb{R}^N,g)} \leq \frac{\epsilon}{2}.$$

Hence, for any $\epsilon > 0$, there exist R and n sufficiently large such that

$$\|u\|_{L^q(\mathbb{R}^N,g)} \leq \|\varphi_R u_n\|_{L^q(\mathbb{R}^N,g)} + \big\|(1-\varphi_R)u_n\big\|_{L^q(\mathbb{R}^N,g)} \leq \epsilon,$$

which implies the embedding $X \hookrightarrow L^q(\mathbb{R}^N, g)$ is compact. □

In the rest of the paper, we assume

(A) The function V satisfies (V) and the functions M, g, h are continuous and nonnegative such that $\mathcal{M} = \mathcal{L} = 0$, where $\mathcal{M}$ and $\mathcal{L}$ are defined by (1.7) and (1.8), respectively.

By Theorems 1.1 and 1.2, if $\mathcal{M} = \mathcal{L} = 0$, then the embedding $X \hookrightarrow L^q(\mathbb{R}^N, g)$ and $X \hookrightarrow L^r(\mathbb{R}^N, h)$ is compact for $1 < q < p < r < p^*$. Let S_q and S_r be the best embedding constants, then

$$\int_{\mathbb{R}^N} g|u|^q\,dx \leq S_q^{-q/p}\|u\|^q, \qquad \int_{\mathbb{R}^N} h|u|^r\,dx \leq S_r^{-r/p}\|u\|^r. \tag{1.11}$$

Since X is a reflexive and separable Banach space, it is well known that there exist $e_j \in X$ and $e_j^* \in X^*$ $(j = 1, 2, \ldots)$ such that

(1) $\langle e_i, e_j^*\rangle = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

(2) $X = \overline{\operatorname{span}\{e_1, e_2, \ldots\}}$, $X^* = \overline{\operatorname{span}\{e_1^*, e_2^*, \ldots\}}$.

Set

$$X_i = \operatorname{span}\{e_i\}, \qquad Y_k = \bigoplus_{i=1}^{k} X_i, \qquad Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}. \tag{1.12}$$

Motivated by [8, 19], we make the following assumptions on M:

(M_1) There exists $\sigma > 0$ such that

$$\hat{M}(t) \geq \sigma\big[M(t)\big]^{p-1}t$$

holds for all $t \geq 0$, where $\hat{M}(t) = \int_0^t [M(s)]^{p-1}\,ds$.

(M_2) $M(t) \geq m_0 > 0$ for all $t \geq 0$.

(M_3) $M(t)$ is nonnegative and increasing for all $t \geq 0$.

(M_4) There exists $\rho > 0$ such that

$$\frac{\sigma}{p}\big[M(\rho^p)\big]^{p-1} > \frac{1}{r}S_r^{-r/p}\rho^{r-p},$$

where S_r is the best embedding constant of $X \hookrightarrow L^r(\mathbb{R}^n, h)$.

(M_5) There exists $\gamma_1 > 0$ such that

$$\frac{\sigma}{p}\big[M(\gamma_1^p)\big]^{p-1}\gamma_1^p \geq \frac{\beta_1^r\gamma_1^r}{4r},$$

where

$$\beta_1 = \sup_{u\in Z_1,\|u\|=1}\left(\int_{\mathbb{R}^N} h|u|^r\,dx\right)^{1/r}.$$

The main results of our paper read as follows.

Theorem 1.3 *Assume* (A), (M_1) *and* (M_2) *or* (M_3), (M_4). *Suppose also* $p < \sigma r$ *and* $1 < q < p < r < p^*$. *Then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has a solution for all* $\lambda \in [0, \lambda_0)$.

Theorem 1.4 *Assume* (A), (M_1) *and* (M_2) *or* (M_3), (M_4). *Suppose also* $p < \sigma r$ *and* $1 < q < p < r < p^*$. *Then there exists* $\lambda_1 > 0$ *such that problem* (1.1) *has a sequence* $\{u_n\}$ *of solutions in* X *with* $J(u_n) \to \infty$ *as* $n \to \infty$ *for all* $\lambda \in [0, \lambda_1)$.

Remark 1.5 Set $M(t) = a + bt^k$ $(a, b, k > 0)$. Then we can easily deduce that M satisfies (M_1) for all $p > 1$ and $0 < \sigma \leq \frac{1}{(p-1)k+1}$.

Remark 1.6 Let $M(t) = a + b\ln(1 + t)$ $(a, b > 0, t \geq 0)$. Assume $p > 1, b(p-1) < a$, then by direct calculation, one has

$$\hat{M}(t) = \int_0^t \big[M(t)\big]^{p-1}\,dt \geq t\big[M(t)\big]^{p-1}\left(1 - \frac{b(p-1)}{a}\right).$$

Consequently, M satisfies (M_1) for $0 < \sigma \leq 1 - \frac{b(p-1)}{a}$.

Remark 1.7 Clearly, assumptions (M_1), (M_3), (M_4) or (M_1), (M_3), (M_5) cover the degenerate case.

2 Proofs of the main results

The associated energy functional to equation (1.1) is

$$J(u) = \frac{1}{p}\hat{M}\big(\|u\|^p\big) - \frac{\lambda}{q}\int_{\mathbb{R}^N} g|u|^q\,dx - \frac{1}{r}\int_{\mathbb{R}^N} h|u|^r\,dx. \tag{2.1}$$

For any $v \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\big\langle J'(u), v\big\rangle &= \big[M\big(\|u\|^p\big)\big]^{p-1}\int_{\mathbb{R}^N}\big(|\nabla u|^{p-2}\nabla u\cdot\nabla v + V|u|^{p-2}uv\big)\,dx \\ &\quad - \lambda\int_{\mathbb{R}^N} g|u|^{q-2}uv\,dx - \int_{\mathbb{R}^N} h|u|^{r-2}uv\,dx. \end{aligned}\tag{2.2}$$

We say that $\{u_n\}$ is a $(PS)_c$ sequence for the functional J if

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X^*, \tag{2.3}$$

where X^* denotes the dual space of X. If every $(PS)_c$ sequence of J has a strong convergent subsequence, then we say that J satisfies the (PS) condition.

The proof of Theorem 1.3 mainly relies on the following mountain pass lemma in [22] (see also [23]).

Lemma 2.1 *Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ with $J(0) = 0$. Suppose*
(H_1) *there are $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for $\|u\|_E = \rho$;*
(H_2) *there is $e \in E$, $\|e\|_E > \rho$ such that $J(e) < 0$. Define*

$$\Gamma = \big\{\gamma \in C^1\big([0,1],E\big) \,|\, \gamma(0) = 0, \gamma(1) = e\big\}.$$

Then

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J\big(\gamma(t)\big) \geq \alpha$$

is finite and $J(\cdot)$ possesses a $(PS)_c$ sequence at level c. Furthermore, if J satisfies the (PS) condition, then c is a critical value of J.

In the following, we shall verify J satisfies all conditions of the mountain pass lemma.

Lemma 2.2 *Assume* (A), (M_1) *and* (M_2) *or* (M_3). *Suppose also $p < \sigma r$. Then any $(PS)_c$ sequence of J is bounded.*

Proof Let $\{u_n\}$ be any $(PS)_c$ sequence of J and satisfy (2.3).

By (M_1) and (A), we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq J(u_n) - \frac{1}{r}\big\langle J'(u_n), u_n\big\rangle \\ &= \frac{1}{p}\hat{M}\big(\|u_n\|^p\big) - \frac{1}{r}\big[M\big(\|u_n\|^p\big)\big]^{p-1}\|u_n\|^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_{\mathbb{R}^N} g|u_n|^q\,dx \\ &\geq \left(\frac{\sigma}{p} - \frac{1}{r}\right)\big[M\big(\|u_n\|^p\big)\big]^{p-1}\|u_n\|^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)S_q^{-q/p}\|u_n\|^q. \end{aligned}\tag{2.4}$$

Case 1. If (M_2) holds. Then we deduce from (2.4) that

$$c+1+\|u_n\| \geq \left(\frac{\sigma}{p}-\frac{1}{r}\right)m_0^{p-1}\|u_n\|^p - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)S_q^{-q/p}\|u_n\|^q. \tag{2.5}$$

Hence $\{u_n\}$ is bounded.

Case 2. If (M_3) holds. Let $\tau_0 > 0$ be fixed. If $\|u_n\|^p \geq \tau_0$, then

$$c+1+\|u_n\| \geq \left(\frac{\sigma}{p}-\frac{1}{r}\right)\left[M(\tau_0)\right]^{p-1}\|u_n\|^p - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)S_q^{-q/p}\|u_n\|^q, \tag{2.6}$$

which implies $\{u_n\}$ is bounded. □

Lemma 2.3 *Assume* (A), (M_1) *and* (M_2) *or* (M_4). *Then there are* $\rho, \alpha > 0$ *such that* $J(u) \geq \alpha$ *for* $\|u\| = \rho$.

Proof *Case* 1. (M_2) is satisfied. It follows from (1.11), (2.1), and (M_1)–(M_2) that

$$\begin{aligned} J(u) &\geq \frac{\sigma}{p}m_0^{p-1}\|u\|^p - \frac{\lambda}{q}S_q^{-q/p}\|u\|^q - \frac{1}{r}S_r^{-r/p}\|u\|^r \\ &= \|u\|^q\left(\frac{\sigma}{p}m_0^{p-1}\|u\|^{p-q} - \frac{\lambda}{q}S_q^{-q/p} - \frac{1}{r}S_r^{-r/p}\|u\|^{r-q}\right). \end{aligned} \tag{2.7}$$

Denote $\phi(t) = At^{p-q} - B\lambda - Ct^{r-q}$ with

$$A = \sigma m_0^{p-1}/p, \qquad B = S_q^{-q/p}/q, \qquad C = S_r^{-r/p}/r. \tag{2.8}$$

Obviously, $\phi(t)$ attains its maximum

$$\phi(t_0) = \frac{r-p}{r-q}At_0^{p-q} - B\lambda$$

at

$$t = t_0 = \left(\frac{A(p-q)}{C(r-q)}\right)^{1/(r-p)}.$$

Let $\lambda_0 = \frac{A(r-p)}{B(r-q)}t_0^{p-q}$, $\rho = t_0$, and $\alpha = t_0^q\phi(t_0)$. Then $J(u) \geq \alpha > 0$ for $\|u\| = \rho$ and $\lambda \in [0, \lambda_0)$.

Case 2. (M_4) is fulfilled. Let $\|u\| = \rho$. Then, by (1.11), (2.1), and (M_1), there hold

$$\begin{aligned} J(u) &\geq \frac{\sigma}{p}\left[M\left(\|u\|^p\right)\right]^{p-1}\|u\|^p - \frac{\lambda}{q}S_q^{-q/p}\|u\|^q - \frac{1}{r}S_r^{-r/p}\|u\|^r \\ &= \rho^q\left(A(\rho)\rho^{p-q} - B\lambda - C\rho^{r-q}\right), \end{aligned} \tag{2.9}$$

where $A(\rho) = \frac{\sigma}{p}[M(\rho^p)]^{p-1}$ and B, C is defined by (2.8). In view of (M_4), $J(u) \geq \alpha > 0$ for all $0 < \lambda < \lambda_0 = \frac{1}{B}[A(\rho)\rho^{p-q} - C\rho^{r-q}]$. □

Lemma 2.4 *Assume* (A), (M_1) *and* $p < \sigma r$. *Then there is* $e \in X$ *with* $\|e\| > \rho$ *such that* $J(e) < 0$.

Proof By integrating (M_1), we obtain

$$\hat{M}(t) \le \hat{M}(t_1)\left(\frac{t}{t_1}\right)^{1/\sigma} \quad \text{for all } t \ge t_1 > 0. \tag{2.10}$$

Hence, for $\|tu\|^p \ge t_1$,

$$J(tu) \le \frac{1}{p}\hat{M}(t_1)\left(\frac{\|u\|^p}{t_1}\right)^{1/\sigma} t^{\frac{p}{\sigma}} - t^q\frac{\lambda}{q}\int_{\mathbb{R}^N} g|u|^q\,dx - t^r\frac{1}{r}\int_{\mathbb{R}^N} h|u|^r\,dx. \tag{2.11}$$

Consequently, $J(tu) < 0$ if $t \ge R$ for some $R > 0$ sufficiently large. □

Lemma 2.5 *Assume* (A), (M_1) *and* (M_2) *or* (M_3). *Then any* $(PS)_c$ *sequence of* J *has a strong convergent subsequence.*

Proof Let $\{u_n\}$ be any $(PS)_c$ sequence of J and satisfy (2.3). By Lemma 2.2, $\{u_n\}$ is bounded. Passing to a subsequence if necessary, we have

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X,\\ &u_n \to u \quad \text{in } L^q(\mathbb{R}^N, g) \text{ and in } L^r(\mathbb{R}^N, h),\\ &u_n \to u \quad \text{almost everywhere in } \mathbb{R}^N. \end{aligned}$$

Denote $P_n = \langle J'(u_n), u_n - u\rangle$ and

$$Q_n = \left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u\nabla(u_n - u) + V|u|^{p-2}u(u_n - u)\right)dx.$$

We can easily obtain that

$$\begin{aligned} &\lim_{n\to\infty} P_n = 0, \qquad \lim_{n\to\infty} Q_n = 0,\\ &\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x)|u_n|^{q-2}u_n(u_n - u)\,dx = 0,\\ &\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|u_n|^{r-2}u_n(u_n - u)\,dx = 0. \end{aligned}$$

Since

$$\begin{aligned} P_n - Q_n = {}&\left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)\,dx\\ &+ \left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_{\mathbb{R}^N} V\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\,dx\\ &- \lambda\int_{\mathbb{R}^N} g(x)|u_n|^{q-2}u_n(u_n - u)\,dx - \int_{\mathbb{R}^N} h(x)|u_n|^{r-2}u_n(u_n - u)\,dx, \end{aligned}$$

we can deduce that

$$\lim_{n\to\infty}\Bigg\{\left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)\,dx$$

$$+\left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_{\mathbb{R}^N}V\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\,dx\Bigg\}=0. \tag{2.12}$$

Case 1. (M_2) holds. Using the standard inequality in $\mathbb{R}^N$ given by

$$\left\langle |x|^{p-2}x-|y|^{p-2}y,x-y\right\rangle \geq C_p|x-y|^p \quad \text{if } p\geq 2 \tag{2.13}$$

or

$$\left\langle |x|^{p-2}x-|y|^{p-2}y,x-y\right\rangle \geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}} \quad \text{if } 2>p>1, \tag{2.14}$$

we obtain from (2.12) that $\|u_n-u\|\to 0$ as $n\to\infty$.

Case 2. If (M_3) holds, then due to the degenerate nature of (1.1), two situations must be considered: either $\inf_n\|u_n\|>0$ or $\inf_n\|u_n\|=0$.

Case 2-1: $\inf_n\|u_n\|>0$. Then we can deduce from (2.12)–(2.14) that $\|u_n-u\|\to 0$ as Case 1.

Case 2-2: $\inf_n\|u_n\|=0$. If 0 is an accumulation point for the sequence $\{\|u_n\|\}$, then there is a subsequence of $\{u_n\}$ (not relabelled) such that $u_n\to 0$. Hence $0=J(0)=\lim_{n\to\infty}J(u_n)=c$. By Lemma 2.3, $c>0$. This is impossible. Consequently, 0 is an isolated point of $\{\|u_n\|\}$. Therefore, there is a subsequence of $\{u_n\}$ (not relabelled) such that $\inf_n\|u_n\|>0$, and we can proceed as before.

This completes the proof. □

Proof of Theorem 1.3 The conclusion follows by Lemmas 2.2–2.5 immediately. □

To get multiplicity result of problem (1.1), we need the following fountain theorem.

Lemma 2.6 (Fountain theorem [24]) *Let X be a Banach space with the norm $\|\cdot\|$, and let X_i be a sequence of subspace of X with* $\dim X_i<\infty$ *for each $i\in\mathbb{N}$. Further, set*

$$X=\overline{\bigoplus_{i=1}^{\infty}X_i},\qquad Y_k=\bigoplus_{i=1}^{k}X_i,\qquad Z_k=\overline{\bigoplus_{i=k}^{\infty}X_i}.$$

Consider an even functional $\Phi\in C^1(X,\mathbb{R})$. Assume, for each $k\in\mathbb{N}$, there exist $\rho_k>\gamma_k>0$ such that

(Φ_1) $a_k:=\max_{u\in Y_k,\|u\|=\rho_k}\Phi(u)\leq 0$;

(Φ_2) $b_k:=\inf_{u\in Z_k,\|u\|=\gamma_k}\Phi(u)\to+\infty,k\to+\infty$;

(Φ_3) Φ *satisfies the* $(PS)_c$ *condition for every* $c>0$.

Then Φ has an unbounded sequence of critical values.

Proof of Theorem 1.4 Obviously the functional J is even. It remains to verify that J satisfies (Φ_1)–(Φ_3) in Lemma 2.6.

It follows from (2.10) that

$$\hat{M}(t)\leq C_1t^{1/\sigma}+C_2$$

for positive constants C_1, C_2 and for all $t \geq 0$. Hence

$$J(u) \leq \frac{1}{p}\left(C_1\|u\|^{\frac{p}{\sigma}} + C_2\right) - \frac{\lambda}{q}\int_{\mathbb{R}^N} g|u|^q\,dx - \frac{1}{r}\int_{\mathbb{R}^N} h|u|^r\,dx. \tag{2.15}$$

Since all norms are equivalent on the finite dimensional space Y_k, we have, for all $u \in Y_k$,

$$J(u) \leq \frac{1}{p}\left(C_1\|u\|^{\frac{p}{\sigma}} + C_2\right) - \lambda C_3\|u\|^q - C_4\|u\|^r, \tag{2.16}$$

where C_3, C_4 are positive constants. Therefore $a_k := \max_{u\in Y_k,\|u\|=\rho_k} J(u) < 0$ for $\|u\| = \rho_k$ sufficiently large. This gives (Φ_1).

Denote $\beta_k = \sup_{u\in Z_k,\|u\|=1}(\int_{\mathbb{R}^N} h|u|^r\,dx)^{1/r}$. Since $Z_{k+1} \subset Z_k$, we deduce that $0 \leq \beta_{k+1} \leq \beta_k$. Hence $\beta_k \to \beta_0 \geq 0$ as $k \to +\infty$. By the definition of β_k, there exists $u_k \in Z_k$ with $\|u_k\| = 1$ such that

$$-\frac{1}{k} \leq \beta_k - \left(\int_{\mathbb{R}^N} h|u_k|^r\,dx\right)^{1/r} \leq 0$$

for all $k \geq 1$. Therefore there exists a subsequence of $\{u_k\}$ (not relabelled) such that $u_k \rightharpoonup u$ in X and $\langle u, e_j^*\rangle = \lim_{k\to\infty}\langle u_k, e_j^*\rangle = 0$ for all $j \geq 1$. Consequently, $u = 0$. This implies $u_k \rightharpoonup 0$ in X and so $u_k \to 0$ in $L^r(\mathbb{R}^N, h)$. Thus $\beta_0 = 0$. The proof of (Φ_2) is divided into the following two cases.

Case 1: (M_2) holds. For any $u \in Z_k$, there holds

$$J(u) \geq \frac{\sigma}{p}m_0^{p-1}\|u\|^p - \frac{\lambda}{q}S_q^{-q/p}\|u\|^q - \frac{1}{r}\beta_k^r\|u\|^r. \tag{2.17}$$

Set

$$\gamma_k = \left(\frac{\sigma m_0^{p-1} r}{4p\beta_k^r}\right)^{\frac{1}{r-p}}, \qquad \lambda_1 = \frac{\sigma q m_0^{p-1}}{2p}\gamma_1^{p-q}S_q^{q/p}.$$

Then

$$J(u) \geq \frac{\sigma}{4p}m_0^{p-1}\gamma_k^p \tag{2.18}$$

for all $\lambda \in (0, \lambda_1)$ and $\|u\| = \gamma_k$. Hence (Φ_2) is fulfilled.

Case 2: (M_3), (M_5) hold. For $\|u\| = \rho$, we have

$$J(u) \geq \frac{\sigma}{p}\left[M\left(\rho^p\right)\right]^{p-1}\rho^p - \frac{\lambda}{q}S_q^{-q/p}\rho^q - \frac{1}{r}S_r^{-r/p}\rho^r. \tag{2.19}$$

Set

$$\widetilde{\gamma}_k = \left(\frac{\sigma[M(\gamma_1^p)]^{p-1} r}{4p\beta_k^r}\right)^{\frac{1}{r-p}}, \qquad \widetilde{\lambda}_1 = \frac{\sigma q[M(\gamma_1^p)]^{p-1}}{2p}\gamma_1^{p-q}S_q^{q/p}.$$

Then by (M_5)

$$J(u) \geq \frac{\sigma}{4p}\left[M\left(\widetilde{\gamma}_1^p\right)\right]^{p-1}\gamma_k^p \tag{2.20}$$

for all $\lambda \in (0, \widetilde{\lambda}_1)$ and $\|u\| = \widetilde{\gamma}_k$. Hence (Φ_2) is fulfilled.

By Lemma 2.5, we obtain (Φ_3). Consequently, the conclusion follows by the fountain theorem. □

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No.

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