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Hölder continuity of weak solution to a nonlinear problem with non-standard growth conditions

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Abstract

We study the Hölder continuity of weak solution u to an equation arising in the stationary motion of electrorheological fluids. To this end, we first obtain higher integrability of Du in our case by establishing a reverse Hölder inequality. Next, based on the result of locally higher integrability of Du and difference quotient argument, we derive a Nikolskii type inequality; then in view of the fractional Sobolev embedding theorem and a bootstrap argument we obtain the main result. Here, the analysis and the existence theory of a weak solution to our equation are similar to the weak solution which has been established in the literature with $\frac{3d}{d+2} \le p_\infty \le p(x) \le p_0 < \infty$, and in this paper we confine ourselves to considering $p(x) \in (\frac{3d}{d+2}, 2)$ and space dimension $d = 2, 3$.

Keywords: Higher integrability; Hölder continuity; Nonlinear problem; Fractional Sobolev space; Electrorheological fluids

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded Lipschitz domain. This paper deals with a nonlinear problem (1.6) which arises in the steady motions of a special incompressible non-Newtonian fluid: electrorheological fluids, one example of smart fluids that change their viscosity rapidly when an electric field is applied. In the field of mechatronics, this fluid is actively being researched and numerous research activities on this fluid have been performed in various engineering applications. Also in the mathematical community such materials are intensively investigated being non-Newtonian fluids [9, 10, 12–14, 34]. Note that one of the first mathematical investigations of non-Newtonian models was carried out by Ladyzhenskaya in 1966 [26–28]; the author considered the modified Navier–Stokes equations

$$\begin{cases} u_t - \operatorname{div} a(Du) + D\phi = -\operatorname{div}(u \otimes u) + f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u : \Omega \longrightarrow \mathbb{R}^d$, $\phi : \Omega \longrightarrow \mathbb{R}$, are the unknown velocity, pressure, respectively. $f : \Omega \longrightarrow \mathbb{R}^d$ is a given density of the bulk force. Du denotes the symmetric part of the velocity gradient ∇u, namely $Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $a : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}^{d\times d}$ depends in a nonlinear way

by Du. We note that there is abundant literature on the power-law model

$$a(Du) = \nu_0\big(\nu_1 + |Du|^2\big)^{\frac{q-2}{2}} Du + \nu_2 Du, \tag{1.2}$$

with $\nu_0 > 0, \nu_1, \nu_2 \geq 0$, here $q > 1$ is a positive constant. For example, the existence of measure-valued solutions was shown to (1.1)–(1.2) for $q > \frac{2d}{d+2}$ in [29, 36], for $q > \frac{2d}{d+2}$, the existence of a weak solution has been studied in [6, 15, 30–32]. In [46], Wolf constructed a weak solution $u \in L^q(0,T;V_q(\Omega)) \cap C_w(0,T;L^2(\Omega))$ to (1.1)–(1.2) for the power with $q > 2\frac{d+1}{d+2}$. In 2014, Bae and Jin [5] studied the local in time existence of a weak solution to (1.1)–(1.2) for $\frac{3d}{d+2} < q < 2$ when $d = 2,3$ and the global in time existence of a weak solution for $q \geq \frac{11}{5}$, when $d = 3$.

When q as a positive constant is replaced by a variable exponent $p(x)$, then the model of (1.2) can be seen as the variable exponent power-law model. According to the model proposed by Rajagopal and Růžička [38, 41], the system of electrorheological fluids reads

$$\begin{cases} \operatorname{div} E = 0, \qquad \operatorname{curl} E = 0 & \text{in } \Omega, \\ E \cdot n = E_0 \cdot n & \text{on } \partial\Omega, \\ u_t - \operatorname{div} S(Du,E) + u \cdot \nabla u + \nabla\phi = f + \chi^E E \cdot \nabla E & \text{in } Q_T, \\ \operatorname{div} u = 0, \qquad u|_{t=0} = u_0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with f, E_0, u_0 are given. $Q_T = [0,T] \times \Omega$, $E : Q_T \longrightarrow \mathbb{R}^d$ is the electric field, $u : Q_T \longrightarrow \mathbb{R}^d$ is the velocity, $\phi : Q_T \longrightarrow \mathbb{R}$ is the pressure, $f : Q_T \longrightarrow \mathbb{R}^d$ is the mechanical force and χ^E is the positive constant dielectric susceptibility. $S(D,E) : \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}_{\text{sym}}$ denotes the stress tensor, which is under non-standard growth conditions

$$\begin{aligned} S(Du,E) = {} & \alpha_{21}\Big(\big(1 + |Du|^2\big)^{\frac{p(|E|^2)-1}{2}} - 1\Big) E \otimes E \\ & + \big(\alpha_{31} + \alpha_{33}|E|^2\big)\big(1 + |Du|^2\big)^{\frac{p(|E|^2)-2}{2}} Du \\ & + \alpha_{51}\big(1 + |Du|^2\big)^{\frac{p(|E|^2)-2}{2}} \big((Du)E \otimes E + E \otimes (Du)E\big). \end{aligned} \tag{1.4}$$

Here α_{ij} are material constants such that

$$\alpha_{31} > 0, \qquad \alpha_{33} > 0, \qquad \alpha_{33} + \frac{4}{3}\alpha_{51} > 0,$$

and $p = p(|E|^2) > 1$ is continuous. We shall note that the system (1.4) is separated into the quasi-static Maxwell's equation $(1.4)_1$–$(1.4)_2$ (cf. [25]) and the equation of motion and the conservation of mass $(1.4)_3$-$(1.4)_5$, where E can be viewed as a parameter.

Note that the higher differentiability of weak solutions to (1.3)–(1.4) had been obtained in [40, 41], the first regularity result for the model of electrorheological fluids proposed in [41], and in any case the first in a point-wise sense. A further step, the Hausdorff dimension of the singular set $\Omega\backslash\Omega_0$ has been studied in [3]. Related regularity results in the stationary case can also be found in [1, 2, 10, 45] and the references therein. For the non-stationary case, one can refer [4, 41].

In this paper, we are interested in the (interior) regularity properties of weak solutions to the stationary case of (1.3)–(1.4):

$$\operatorname{div} u = 0, \qquad -\operatorname{div} S(Du,E) + u\cdot\nabla u + \nabla\phi = f + \chi^E E\cdot\nabla E. \tag{1.5}$$

The issue of regularity of solutions to (1.5) has been performed in [41] where the author proves the existence of a $W^{2,2}$ solution to (1.5). Here we analyze the system arising from (1.5) as

$$\begin{cases} -\operatorname{div} S(Du) + u\cdot\nabla u + \nabla\phi = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $f:\Omega\longrightarrow\mathbb{R}^d$, and

$$S(Du) = \bigl(\mu_0 + |Du|^2\bigr)^{\frac{p(x)-2}{2}} Du, \tag{1.7}$$

with $\mu_0 > 0$. $p(\cdot):\Omega\longrightarrow[1,\infty)$ is a given Hölder (log-Hölder) continuous function that satisfies

$$\frac{3d}{d+2} < p_\infty \le p(x) \le p_0 < 2, \qquad \bigl|p(x)-p(y)\bigr| \le \omega\bigl(|x-y|\bigr) \le c_0|x-y|^{2\theta_1}, \tag{1.8}$$

for all $x,y\in\bar\Omega$, where $c_0\ge 1$ is a constant, $\theta_1 = \frac{A_d(d+2)-3d}{2A_d}\in(0,1)$ and A_d be a constant defined in (3.29), $p_\infty := \min_{x\in\bar\Omega} p(x)$ and $p_0 := \max_{x\in\bar\Omega} p(x)$, $\omega:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is the modulus of $p(\cdot)$, which satisfies

$$\omega(6R) < 1, \qquad \omega(R)\log\frac{1}{R} \le L, \tag{1.9}$$

for all $0 < R \le 1$ and $L > 0$ is a constant. In this paper, we assume $\mu_0 = 1$, then from the definition of $S(Du)$,

$$\bigl|S(Du)\bigr| \le \bigl(1+|Du|^2\bigr)^{\frac{p(x)-1}{2}}. \tag{1.10}$$

The purpose of this paper is to study the Hölder continuity of a weak solution u to (1.6), to this end, various higher integrability results are important to overcome the lack of standard growth conditions of the system. Thus, we first improve the power of integrability of $Du\in L^{p(x)}_{\mathrm{loc}}$, $p(x)\in(1,2)$ by establishing a reverse Hölder inequality, and at this point, using the Gerhing lemma 2.4, we can deduce that $Du\in L^{p(x)(1+\delta_1)}$ for some positive constant $\delta_1 > 0$. Next, by a difference quotient argument, we proceed to show the fractional differentiability of Du. For this purpose, we construct a Nikolskii type inequality (3.38), from which, by the fractional order Sobolev embedding theorem and a standard bootstrap argument, we have $Du\in L^\eta$ with $\eta\ge d$. Note that the self-improving property to a class of elliptic system was first observed by Elcrat and Meyers in [33] (see also [20] and [43]), but their argument is based on the reverse Hölder inequality and a modification of the Gehring lemma. Finally, by the Sobolev embedding theorem, we derive the Hölder continuity of u.

In the whole paper, the key point is to suppose that $p(\cdot)$ satisfies the log-Hölder conditions (1.8)–(1.9), which implies that

$$\frac{1}{r} \sim \frac{1}{r^{p_2/p_1}} \quad \text{and} \quad \frac{1}{r} \sim \frac{1}{r^{p_2^2/p_1^2}},$$

for all $r \in (0,1]$ when $p_1 := \inf_{x\in B_{3r}} p(x)$, $p_2 := \sup_{x\in B_{3r}} p(x)$. Moreover, when $p(x)$ satisfies the log-Hölder continuous conditions, one can use the Korn inequality with variable exponent case (Lemma 2.3), which is the main tool to prove the local higher integrability of ∇u in terms of Du. We also observe that, for the sake of brevity and in order to highlight the main ideas, we confine ourselves to the considered homogeneous case of (1.6). For the non-homogeneous case ($f \in W^{-1,p'(x)}(\Omega)$), there are some technique difficulties in the proof of higher integrability of Du (Lemma 3.1), and we will investigate it in our future work. The result of this paper reads as follows.

Theorem 1.1 *Suppose $f = 0$. Let $\Omega \subset \mathbb{R}^d$ ($d = 2,3$) be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $p(\cdot) : \Omega \longrightarrow (\frac{3d}{d+2}, 2)$ be* log*-Hölder continuous where $\frac{3d}{d+2} < p_\infty \le p \le p_0 < 2$ satisfies (1.8)–(1.9), and $S(Du)$ satisfies (1.10). $(u,\phi) \in (V_{p(x)}, L_0^{p'(x)}(\Omega))$ are the solutions of (1.6)–(1.7). Then*

$$u \in C^\alpha(\Omega) \quad \textit{for some } \alpha \in (0,1).$$

The rest of paper is organized as follows. In Sect. 2, we present some notions of variable exponents spaces, the definition of a weak solution to (1.6), the property of a difference quotient with variable exponents, and formulate some lemma which will be used in later. In Sect. 3, we first prove the locally higher integrability of Du (Lemma 3.1). Next, by the known result of the difference quotient argument, the log-Hölder continuity of $p(\cdot)$, we derive the fractional differentiability of Du, then, by the fractional Sobolev space embedding theorem and a standard bootstrap argument, we obtain the higher integrability of Du (the power of integrability is bigger than that in Lemma 3.1). At last, we prove the main result of Theorem 1.1.

2 Preliminaries

2.1 Basic notions

In the present paper we shall often write p or $p(\cdot)$ instead of $p(x)$ if there is no danger of confusion and the exponent q denotes a constant. c denotes a general constant which may vary in different estimates. If the dependence needs to be explicitly stressed, some notations like $c', c_0, c_1, c(k_0)$ will be used. $A \sim B$ means there exist constants c_1 and c_2 such that $c_1 B \le A \le c_2 B$. $B_r(x_0) := \{x : \operatorname{dist}(x, x_0) < r\}$, we denote the average integral of u on B_r as $(u)_r := (u)_{B_r} = \frac{1}{|B_r|}\int_{B_r} u\,dx$. We recall in what follows some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (for more details one can refer to [8, 11, 16–18, 22–24, 37, 39, 44] and the references therein). Let $P(\Omega)$ be the set consisting of a Lebesgue measurable function $p(\cdot) : \Omega \longrightarrow [1,\infty]$, where $\Omega \subset \mathbb{R}^d$ ($d \ge 2$) is nonempty. Now, for any $p(x) \in P(\Omega)$, let us introduce the spaces which are used in this paper,

$$V_{p(x)} := \left\{u : u \in W^{1,p(x)}(\Omega), \operatorname{div} u = 0\right\},$$

$$L_0^{p(x)}(\Omega) := \left\{ u : u \in L^{p(x)}(\Omega), \int_\Omega u\,dx = 0 \right\}.$$

Next, let us introduce the embedding properties of the generalized Lebesgue space. Firstly, we know that

$$L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega), \tag{2.1}$$

if and only if

$$q(x) < p(x) \quad \text{a.e. in } \Omega.$$

Moreover, if $q \in P(\Omega)$ and $q(x) < p^*(x)$ and for all $x \in \bar{\Omega}$, then the embedding $W^{1,p(x)} \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = np(x)/(n - p(x))$ if $p(x) < n$ or $p^*(x) = +\infty$ if $p(x) = n$. In what follows we denote $L^{p'(x)}(\Omega)$ as the conjugate of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$, then for all $p(x) \in P(\Omega), u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$ we have

$$\int_\Omega \left|u(x)v(x)\right| dx \le 2\left|u(x)\right|_{L^{p(x)}} \left|v(x)\right|_{L^{p'(x)}}.$$

From the definition of variable exponent Lebesgue space above, now we introduce a basic property of $L^{p(x)}(\Omega)$.

Lemma 2.1 *Let $p(x) \in P(\Omega)$ satisfy $1 \le t_1 \le p(x) \le t_2 < \infty$, then for every $u \in L^{p(x)}(\Omega)$*

$$\min\left\{ \|u\|_{L^{p(x)}(\Omega)}^{t_1}, \|u\|_{L^{p(x)}(\Omega)}^{t_2} \right\} \le \int_\Omega |u|^{p(x)}\,dx \le \max\left\{ \|u\|_{L^{p(x)}(\Omega)}^{t_1}, \|u\|_{L^{p(x)}(\Omega)}^{t_2} \right\}. \tag{2.2}$$

A proof can be retrieved e.g. from Lemma 3.2.5 in [11]. For convenience, we may denote inequality (2.2) as

$$\|u\|_{L^{p(x)}(\Omega)}^{s_1} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|_{L^{p(x)}(\Omega)}^{s_2}, \tag{2.3}$$

where s_1, s_2 equal to t_1 or t_2.

The following conclusion from fluid dynamics ensures local bounds of ∇u in terms of Du on the scale of L^q space.

Lemma 2.2 (Korn inequality) *Let $1 < \gamma_1 \le q \le \gamma_2$ and assume that $u \in L^q(B_r(x_0), \mathbb{R}^d)$ satisfies $Du \in L^q(B_r(x_0), \mathbb{R}^{d\times d}_{\text{sym}})$. Then $\nabla u \in L^q(B_r(x_0), \mathbb{R}^{d\times d})$ and for a constant $c = c(d, \gamma_1, \gamma_2)$ we have*

$$\fint_{B_r(x_0)} |\nabla u|^q\,dx \le c \fint_{B_r(x_0)} |Du|^q\,dx + c\left(\fint_{B_r(x_0)} \left| \frac{u - (u)_{x_0,r}}{r} \right| dx \right)^q. \tag{2.4}$$

Additionally, if $u = 0$ on $\partial B_r(x_0)$, then

$$\fint_{B_r(x_0)} |\nabla u|^q\,dx \le c \fint_{B_r(x_0)} |Du|^q\,dx.$$

Its proof may be found, e.g., in [35]. In addition, we shall use the Korn inequality with variable exponent case, and we formulate it in the form we need (cf. Theorem 14.3.23 in [11]).

Lemma 2.3 *Let $B_r \subset \mathbb{R}^d$ be a bounded ball, let $p(x)$ satisfies log-Hölder conditions and $1 < p_\infty \le p(x) \le p_0 < \infty$. Then*

$$\begin{aligned} &\|\nabla u - (\nabla u)_r\|_{L^{p(x)}(B_r)} \le c\|Du - (Du)_r\|_{L^{p(x)}(B_r)}, \\ &\|\nabla u\|_{L^{p(x)}(B_r)} \le c\|Du - (Du)_r\|_{L^{p(x)}(B_r)} + \frac{c}{r}\|u - (u)_r\|_{L^{p(x)}(B_r)}, \end{aligned} \tag{2.5}$$

for all $u \in W^{1,p(x)}(B_r)$.

Next, we introduce the Gehring lemma in a version formulated by Giaqunta and Giusti (see, e.g., Chapter V, Proposition 1.1 in [20] or Theorem 6.6 in [21]).

Lemma 2.4 *Let $\Omega \subset R^d$, $0 < m < 1$, and $f \in L^1_{\mathrm{loc}}(\Omega)$, $g \in L^\sigma_{\mathrm{loc}}(\Omega)$ for some $\sigma > 1$ be two nonnegative functions such that for any ball B_ρ with $B_{3\rho} \subset\subset \Omega$*

$$\fint_{B_\rho} f\,dx \le b_1\left(\fint_{B_{3\rho}} f^m\,dx\right)^{\frac{1}{m}} + b_2\fint_{B_{3\rho}} g\,dx + k\fint_{B_{3\rho}} f\,dx,$$

where $b_1, b_2 > 1$ and $0 < k \le k_0 = k_0(m,d)$. Then there exists a constant $\gamma_0 = \gamma_0(d,m,b_1) > 1$ such that

$$f \in L^\gamma_{\mathrm{loc}} \quad \text{for all } 1 < \gamma < \min\{\gamma_0, \sigma\}.$$

In order to show the interior higher integrability of Du, we shall use the following lemma, which is a well-known result (Bogovskii theorem), and we restate in the form we need (cf. [7, 19]).

Lemma 2.5 *Let $B_R \subset \mathbb{R}^d$ and let $f \in L^q(B_R)$ with $1 < q_1 < q < q_2$ be such that $(f)_R = 0$. Then there exists $v \in W^{1,q}_0(B_R;\mathbb{R}^d)$ satisfying*

$$\operatorname{div} v = f$$

and such that

$$\int_{B_R} |\nabla v|^{q_3}\,dx \le c\int_{B_R} |f|^{q_3}\,dx,$$

for every $q_3 \in [q_1, q]$, where $c = c(d, q_1, q_2)$ is independent of $R > 0$. Moreover, if the support of f is contained in $B_r \subset B_R$, the same result holds for v.

Recalling the structure of $S = S(Du)$ in (1.7) with $\mu_0 = 1$, then we have

$$S_{ij}(\xi) = \begin{cases} (1 + |\xi|^2)^{\frac{p(x)-2}{2}}\xi_{ij}, & \xi \in R^{d\times d}_{\mathrm{sym}}, \xi \ne 0, \\ 0, & \xi = 0, \end{cases}$$

with $\frac{3d}{d+2} < p_\infty \le p(x) \le p_0 \le 2$. Define

$$S(\xi) := A(x,\xi) = \left(1+|\xi|^2\right)^{\frac{p(x)-2}{2}}\xi, \quad \xi \in R^{d\times d}_{\mathrm{sym}}.$$

Then $A : \Omega \times R^{d\times d}_{\mathrm{sym}} \longrightarrow R^{d\times d}_{\mathrm{sym}}$ satisfy

$$\begin{aligned}
&\left|A(x,\xi) - A(x_0,\xi)\right| \\
&\quad \le \omega\left(|x-x_0|\right)\left[\left(1+|\xi|^2\right)^{\frac{p(x)-1}{2}} + \left(1+|\xi|^2\right)^{\frac{p(x_0)-1}{2}}\right]\left(1+\log\left(1+|\xi|\right)\right), \qquad (2.6)\\
&A(x,\xi)\cdot\xi \ge |\xi|^{p(x)} - 1. \qquad (2.7)
\end{aligned}$$

From the definition of $S_{ij}(\xi)$, we can also obtain

$$\frac{\partial S_{ij}(\xi)}{\partial \xi_{kl}} = \left(p(x)-2\right)\left(1+|\xi|^2\right)^{\frac{p(x)-4}{2}}\xi_{kl}\xi_{ij} + \left(1+|\xi|^2\right)^{\frac{p(x)-2}{2}}\delta_{ki}\delta_{lj},$$

where δ_{ij} is Kronecker's delta, for all $\xi,\eta \in R^{d\times d}_{\mathrm{sym}}$, $|\xi|+|\eta|>0$,

$$\begin{aligned}
&\left(S_{ij}(\xi) - S_{ij}(\eta)\right)(\xi_{ij}-\eta_{ij}) \\
&\quad = \int_0^1 \frac{d}{dt} S_{ij}\left(\eta + t(\xi-\eta)\right) dt(\xi_{ij}-\eta_{ij}) \\
&\quad \ge \int_0^1 \left(p(\cdot)-1\right)\left(1+\left(\eta+t(\xi-\eta)\right)^2\right)^{\frac{p(\cdot)-2}{2}}|\xi-\eta|^2\,dt \\
&\quad \ge \int_0^1 \left(p(\cdot)-1\right)\left(1+\left|\eta+t(\xi-\eta)\right|\right)^{p(\cdot)-2}|\xi-\eta|^2\,dt \\
&\quad \ge \left(p(\cdot)-1\right)\left(1+|\xi|+|\eta|\right)^{p(\cdot)-2}|\xi-\eta|^2 \\
&\quad \ge k_0\left(1+|\xi|+|\eta|\right)^{p(\cdot)-2}|\xi-\eta|^2, \qquad (2.8)
\end{aligned}$$

where $k_0 = p_\infty - 1$, and in the third inequality we have taken into account the inequality

$$\left(1+|\xi|+|\eta|\right)^{-(2-p(\cdot))} \le \int_0^1 \left(1+|\xi+t\eta|\right)^{-(2-p(\cdot))}dt, \quad \text{a.e. in } \Omega.$$

2.2 The property of difference quotient

In the whole paper, we will employ the difference

$$\triangle_{\lambda,k}u(x) := u(x+\lambda e_k) - u(x),$$

where $e_k = (0,\dots,1,\dots,0)$ and 1 at the kth place ($k = 1,\dots,d$). Moreover, for simplicity, we may repeat, using the parameter p_1, p_2:

$$p_1 := \inf_{x\in B_{3r}} p(x), \qquad p_2 := \sup_{x\in B_{3r}} p(x).$$

Let $B_r = B_r(x_0) \subset \Omega$ be a ball such that $\bar{B}_{6r} \subset \Omega$, in what follows, we will repeatedly use the following fact.

Lemma 2.6 *Let $1 \le p(\cdot) \le q_0 < \infty$ (q_0 is a constant) and $f \in W^{1,p(\cdot)}(\Omega)$, then, for all $V \subset\subset \Omega \subset \mathbb{R}^d$, there exists a constant $c > 0$ such that*

$$\int_{B_{mr}} |\triangle_{\lambda,k} u|^{p(x)}\,dx \le c|\lambda|^{p_1} \int_{B_{(m+1)r}} \left|\frac{\partial u}{\partial x_k}\right|^{p(x)} dx, \tag{2.9}$$

for all $|\lambda| < r < 1, k = 1,\dots,d$, $m = 1,2$.

Proof We first assume $u(x)$ is smooth, then for all $x \in V, i = 1,2,\dots,d$ and $0 < |h| < \frac{1}{2}\operatorname{dist}(V,\partial\Omega)$

$$u(x + he_i) - u(x) = h\int_0^1 \frac{\partial u(x + the_i)}{\partial x_i}\,dt,$$

from above, then we have

$$\begin{aligned}
\int_V \left|D^h u(x)\right|^{p(\cdot)} dx &= \int_V \left\{\sum_{i=1}^d \left|\frac{u(x + he_i) - u(x)}{h}\right|^2\right\}^{\frac{p(\cdot)}{2}} dx \\
&\le \int_V \left\{\sum_{i=1}^d \left[\int_0^1 \left|\nabla u(x + the_i)\right| dt\right]^2\right\}^{\frac{p(\cdot)}{2}} dx \\
&\le c\sum_{i=1}^d \int_V \int_0^1 \left|\nabla u(x + the_i)\right|^{p(\cdot)} dt\,dx \\
&\le c\int_\Omega \left|\nabla u(x)\right|^{p(\cdot)} dx,
\end{aligned}$$

where c is independent of $p(\cdot)$, and in the second inequality we have taken into account the fact

$$\left(\left|u(x)\right| + \left|v(x)\right|\right)^{p(x)} \le 2^{\sup_{x\in V} p(x)-1}\left(\left|u(x)\right|^{p(x)} + \left|v(x)\right|^{p(x)}\right), \quad p(x) \ge 1. \tag{2.10}$$

Indeed, when $|u|$ or $|v|$ equal to zero, the above inequality is obvious, and without loss of generality, we may assume $|u|, |v| \neq 0$, and

$$V = V_1 \cup V_2, \qquad V_1 := \left\{x : |u| \ge |v|\right\}, \qquad V_2 := \left\{x : |v| \ge |u|\right\}.$$

It is only enough to consider $|u| \ge |v|$ in V_1, set $t := \frac{|u|}{|v|} \ge 1$, observe the function

$$f(t) = \frac{(1 + t)^{p(x)}}{1 + t^{p(x)}}, \quad t \ge 1, p(x) \ge 1,$$

then we obtain, for any fixed x, $\sup_{t\ge 1} f(t) = 2^{p(x)-1}$. Hence,

$$\frac{(1 + t)^{p(x)}}{1 + t^{p(x)}} \le 2^{p(x)-1}, \quad \text{a.e. } x \in V_1,$$

whence (2.10) in V_1, by set $t := \frac{|v|}{|u|}$, we have same conclusion in V_2, thus we obtain (2.10). From the definition of a generalized Lebesgue space, we obtain (2.9) if u is smooth, then, for all $u \in W^{1,p(\cdot)}(\Omega)$, (2.9) holds. □

2.3 Definition of the weak solutions to (1.6)

Suppose $\frac{3d}{d+2} \le p_\infty \le p(x) \le p_0 < \infty$, and $f \in W^{-1,p'(\cdot)}(\Omega)$ is given, then $(u,\phi) \in (V_{p(\cdot)}, L_0^{p(\cdot)'}(\Omega))$ is said to be a weak solution to (1.6)–(1.10), if and only if

$$\int_\Omega S(Du)\cdot D\varphi\,dx - \int_\Omega u\otimes u\cdot\nabla\varphi\,dx - \int_\Omega \phi\operatorname{div}\varphi\,dx + (f,\varphi) = 0, \tag{2.11}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, or

$$\int_\Omega S(Du)\cdot D\varphi\,dx - \int_\Omega u\otimes u\cdot\nabla\varphi\,dx + (f,\varphi) = 0, \tag{2.12}$$

for all $\varphi \in V_{p(x)}$. For more details, one can refer for instance to [11] (Chap. 14, Sect. 4, p. 472) or [41].

3 Hölder continuity of weak solutions

We note that the starting point for any comparison and freezing argument in the setting of variable $p(x)$-growth problems is a quantitative higher integrability result. Hence, in order to obtain the interior differentiability of weak solution to (1.6), we shall first show the locally higher integrability of Du. At this point, we define a global positive constant $\alpha \in (1, [(d+2)p_\infty - d]/2d)$, from which we then have the following result.

Lemma 3.1 *Suppose $f = 0$. Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $p(\cdot) : \Omega \longrightarrow (\frac{3d}{d+2}, 2)$ be* log-*Hölder continuous with $\frac{3d}{d+2} < p_\infty \le p \le p_0 < 2$ satisfying* (1.8) *and* (1.9), *and S satisfying* (1.10). *$(u,\phi) \in (V_{p(x)}, L_0^{p'(x)}(\Omega))$ is the solution of* (1.6)–(1.7). *Then there exist constants $c, \delta_1 > 0$ both depending on d, p_0, p_∞ and $r_0 \in (0,1)$ suitable small, such that if $B_{2r} \subset\subset \Omega$ for any $r \in (0, r_0)$, then*

$$\left(\fint_{B_r} |Du|^{p(x)(1+\delta_1)}\,dx\right)^{\frac{1}{1+\delta_1}} \le c\fint_{B_{2r}} |Du|^{p(x)}\,dx + c\fint_{B_{2r}} \left(|\nabla u|^{p_\infty} + |u|^{p_\infty^*} + 1\right)dx. \tag{3.1}$$

Proof Without loss of generality, we first set $r_0 = \frac{1}{2}$, and we will specify it in later. Let $\eta \in C_0^\infty(B_{2r})$ with $r \in (0, r_0)$ be a cut-off function such that

$$\begin{cases} \eta = 1 & \text{in } B_r, \\ 0 \le \eta \le 1 & \text{in } B_{2r}, \\ |\nabla\eta| \le \frac{c}{r} & \text{in } B_{2r}, \end{cases}$$

where c is a positive constant independent of r. Let

$$\varphi = \eta^2\big(u - (u)_{2r}\big) + w, \tag{3.2}$$

where the function w is defined according to Lemma 2.5 as a solution to

$$\operatorname{div} w = -\operatorname{div}\big(\eta^2\big(u - (u)_{2r}\big)\big) = -\big(u - (u)_{2r}\big)\cdot\nabla\big(\eta^2\big). \tag{3.3}$$

It is obvious that such a function w exists, since

$$\operatorname{div} u = 0, \qquad \int_{B_{2r}} \big(u - (u)_{2r}\big) \cdot \nabla\big(\eta^2\big)\, dx = 0.$$

We claim that $w \in W_0^{1,p_2}(B_{2r})$. In fact, from Lemma 2.5, we have

$$\fint_{B_{2r}} |\nabla w|^q\, dx \le c \fint_{B_{2r}} \left|\frac{u - (u)_{2r}}{r}\right|^q dx, \tag{3.4}$$

for some exponent $q > p_2$ such that the right hand side is finite.

Taking into account (2.12), we let φ in (3.2) be a test function, thus

$$\begin{aligned} 0 &= \int_{B_{2r}} \eta^2 S(Du) \cdot Du\, dx + 2\int_{B_{2r}} \eta S(Du) \cdot \big(\big(u - (u)_{2r}\big) \cdot \nabla\eta\big)\, dx \\ &\quad + \int_{B_{2r}} S(Du) \cdot Dw\, dx + \int_{B_{2r}} (u \cdot \nabla u) \cdot \big(\big(u - (u)_{2r}\big)\eta^2\big)\, dx \\ &\quad + \int_{B_{2r}} (u \cdot \nabla u) \cdot w\, dx \\ &:= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.5}$$

Now, we estimate the terms I_1–I_5. Using (2.7), we first infer that

$$I_1 \ge \int_{B_{2r}} \eta^2 |Du|^{p(x)}\, dx - cr^d. \tag{3.6}$$

Note that $2p'(x) \ge 2$ and $\eta \in [0,1]$, thus, by (1.10) and the Young inequality

$$|I_2| \le \varepsilon \int_{B_{2r}} \eta^2 |Du|^{p(x)}\, dx + c(\varepsilon) \int_{B_{2r}} \left|\frac{u - (u)_{2r}}{r}\right|^{p(x)} dx + cr^d, \tag{3.7}$$

where ε is a positive constant that will be specified later.

Likewise, taking into account (3.4), we arrive at

$$\begin{aligned} |I_3| &\le \varepsilon \int_{B_{2r}} |Du|^{p(x)}\, dx + c(\varepsilon) \int_{B_{2r}} |Dw|^{p(x)}\, dx + cr^d \\ &\le \varepsilon \int_{B_{2r}} |Du|^{p(x)}\, dx + c(\varepsilon) \int_{B_{2r}} \left|\frac{u - (u)_{2r}}{r}\right|^{p_2} dx + cr^d. \end{aligned} \tag{3.8}$$

Observe that

$$\big\|u - (u)_{2r}\big\|_{L^q(B_{2r})} \le c(q)\|u\|_{L^q(B_{2r})}$$

for any $q \ge 1$, with $c(q) \ge 1$. Thus, appealing to the Young and Hölder inequalities, we have

$$\begin{aligned} |I_4| &\le \|\nabla u\|_{L^{(\frac{1}{2}(\frac{p_\infty}{\alpha})^*)'}(B_{2r})} \big\|u - (u)_{2r}\big\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \|u\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \\ &\le c\|\nabla u\|_{L^{(\frac{1}{2}(\frac{p_\infty}{\alpha})^*)'}(B_{2r})} \|u\|^2_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \end{aligned}$$

$$\leq c\int_{B_{2r}} |\nabla u|^{\frac{dp_\infty}{(d+2)p_\infty-2d\alpha}}\,dx + c\int_{B_{2r}} |u|^{(\frac{p_\infty}{\alpha})^*}\,dx. \tag{3.9}$$

Similarly, using the Hölder and Young inequalities again, from the property of w in (3.4) and noting that $w \in W_0^{1,p_2}(B_{2r})$, the term I_5 can be estimated as

$$\begin{aligned}
|I_5| &\leq \|\nabla u\|_{L^{\frac{dp_\infty}{(d+2)p_\infty-2d\alpha}}(B_{2r})} \|u\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \|w\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \\
&\leq c\|\nabla u\|_{L^{\frac{dp_\infty}{(d+2)p_\infty-2d\alpha}}(B_{2r})} \|u\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \|\nabla w\|_{L^{\frac{p_\infty}{\alpha}}(B_{2r})} \\
&\leq c\|\nabla u\|_{L^{\frac{dp_\infty}{(d+2)p_\infty-2d\alpha}}(B_{2r})} \|u\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \|u-(u)_{2r}\|_{L^{(\frac{p_\infty}{\alpha})^*}(B_{2r})} \\
&\leq c\int_{B_{2r}} |\nabla u|^{\frac{dp_\infty}{(d+2)p_\infty-2d\alpha}}\,dx + c\int_{B_{2r}} |u|^{(\frac{p_\infty}{\alpha})^*}\,dx.
\end{aligned} \tag{3.10}$$

Inserting the estimation (3.6)–(3.10) into (3.5) and choosing $\varepsilon = \frac{1}{4}$, we conclude that

$$\begin{aligned}
\fint_{B_r} |Du|^{p(x)}\,dx &\leq c\fint_{B_{2r}} |Du|^{p(x)}\,dx + c\fint_{B_{2r}} |\nabla u|^{\frac{dp_\infty}{(d+2)p_\infty-2d\alpha}} + |u|^{(\frac{p_\infty}{\alpha})^*} + 1\,dx \\
&\quad + c\left(\fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^{p(x)} + \left|\frac{u-(u)_{2r}}{r}\right|^{p_2} dx\right).
\end{aligned} \tag{3.11}$$

Next, we shall show a reverse Hölder inequality. In view of the Sobolev–Poincaré and Korn inequalities in Lemma 2.2, the last term on the right hand side of (3.11) can be estimated as

$$\begin{aligned}
\fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^{p(x)} dx &\leq 1 + \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^{p_2} dx \\
&\leq 1 + |B_{2r}|^{-\frac{p_2}{(p_\infty/\alpha)^*}} \left(\int_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^{(\frac{p_\infty}{\alpha})^*}\right)^{\frac{p_2}{(p_\infty/\alpha)^*}} \\
&\leq 1 + |B_{2r}|^{-\frac{p_2}{(p_\infty/\alpha)^*}} \left(\int_{B_{2r}} |\nabla u|^{\frac{p_\infty}{\alpha}}\,dx\right)^{\frac{p_2\alpha}{p_\infty}} \\
&\leq 1 + |B_{2r}|^{1-\frac{p_2}{(p_\infty/\alpha)^*}} \fint_{B_{2r}} |\nabla u|^{\frac{p_\infty}{\alpha}}\,dx \\
&\leq 1 + c\left[\fint_{B_{2r}} |Du|^{\frac{p_\infty}{\alpha}}\,dx + \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^{\frac{p_\infty}{\alpha}} dx\right] \\
&\leq c\left[1 + \left(\fint_{B_{2r}} |Du|^{\frac{p_\infty}{\alpha}}\,dx\right)^{\alpha}\right] + c\fint_{B_{2r}} |\nabla u|^{\frac{p_\infty}{\alpha}}\,dx,
\end{aligned} \tag{3.12}$$

where $c = c(d, p_0, p_\infty)$ and in the fourth inequality, we have taken into account that $p_2\alpha/p_\infty > 1$ and $\int_{B_{2r}} |\nabla u|^{p_\infty/\alpha}\,dx \leq 1$ for any $r \in (0, r_0']$ with $r_0' \leq 1$ suitable small, since we have absolute continuity of the integral. At this stage, we have determined $r_0 = \min\{\frac{1}{2}, r_0'\}$.

Now, inserting (3.12) into (3.11), we conclude that

$$\fint_{B_r} |Du|^{p(x)}\,dx \leq c\fint_{B_{2r}} |Du|^{p(x)}\,dx + c\left(\fint_{B_{2r}} |Du|^{\frac{p_\infty}{\alpha}}\,dx\right)^{\alpha}$$

$$+ c\fint_{B_{2r}} |\nabla u|^{\frac{dp_\infty}{(d+2)p_\infty - 2d\alpha}} + |\nabla u|^{\frac{p_\infty}{\alpha}} + |u|^{(\frac{p_\infty}{\alpha})^*} + 1\,dx. \tag{3.13}$$

Taking into account Lemma 2.4, we set

$$g := |\nabla u|^{\frac{dp_\infty}{(d+2)p_\infty - 2d\alpha}} + |\nabla u|^{\frac{p_\infty}{\alpha}} + |u|^{(\frac{p_\infty}{\alpha})^*} + 1,$$
$$f := |Du|^{p(x)}.$$

From the definition of α, we can see that

$$\frac{d}{(d+2)p_\infty - 2d\alpha} \in (0,1), \qquad \left(\frac{p_\infty}{\alpha}\right)^* < \frac{p_\infty^*}{\alpha} < p_\infty^*.$$

Thus, we infer that $g \in L^\sigma(B_{2r})$ for some $\sigma = \sigma(d, p_\infty, \alpha) = \sigma(d, p_\infty) > 1$. At this point, there exists a constant $\delta_1 > 0$ such that $\gamma = 1 + \delta_1$ in Lemma 2.4, then the result (3.1) holds. □

Based on the interior higher integrability of Du, we now turn to a proof of the Hölder continuity of u. For the main difficult result from the difference quotient of $S(Du)$ in (3.15), for dealing with it, we need the monotonicity (2.8) and the growth condition (2.6) of $S(\cdot)$. Furthermore, we may repeatedly use the log-Hölder property of $p(x)$ and the local higher integrability of Du.

Proof of Theorem 1.1 For $i, j = 1, 2, \ldots, d$. Let $\xi \in C_0^\infty$ be a cut-off function for B_{2r}, i.e.,

$$\begin{cases} \xi = 1 & \text{in } B_r, \\ 0 \le \xi \le 1 & \text{in } B_{2r}, \\ |\frac{\partial \xi}{\partial x_i}| \le \frac{c}{r}, \qquad |\frac{\partial^2 \xi}{\partial x_i \partial x_j}| \le \frac{c}{r^2} & \text{in } B_{2r}, \end{cases}$$

where $c > 0$ is a positive constant independent of r. Define

$$\varphi = \triangle_{-\lambda,k}\big(\xi^2 \triangle_{\lambda,k} u\big), \tag{3.14}$$

where $|\lambda| < r < 1$, $k = 1, \ldots, d$. One can see that φ is an admissible test function in (2.11). Now we divide the proof into several steps.

Step 1 (Fractional differentiability of Du). To begin with, we choose φ in (3.14) as a test function, inserting it into (2.11) with $f = 0$, which implies

$$\begin{aligned} &\int_{B_{2r}} S_{ij}(Du) D_{ij}\big(\triangle_{-\lambda,k}\big(\xi^2 \triangle_{\lambda,k} u\big)\big)\,dx + \int_{B_{2r}} u_i \partial_i u_j \triangle_{-\lambda,k}\big(\xi^2 \triangle_{\lambda,k} u_j\big)\,dx \\ &\quad = \int_{B_{2r}} \phi \partial_i\big(\triangle_{-\lambda,k}\big(\xi^2 \triangle_{\lambda,k} u_i\big)\big)\,dx. \end{aligned}$$

Observe that

$$\begin{aligned} &\int_{B_{2r}} S_{ij}(Du) D_{ij}\big(\triangle_{-\lambda,k}\big(\xi^2 \triangle_{\lambda,k} u\big)\big)\,dx \\ &\quad = \int_{B_{2r}} \big[\triangle_{\lambda,k} S_{ij}(Du)\big] \xi^2 \triangle_{\lambda,k} D_{ij} u\,dx \end{aligned}$$

$$+\int_{B_{2r}} S_{ij}(Du)\triangle_{-\lambda,k}\left(\xi\left(\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_j+\frac{\partial\xi}{\partial x_j}\triangle_{\lambda,k}u_i\right)\right)dx$$

and

$$\begin{aligned}
&\triangle_{\lambda,k}S(Du)\\
&\quad=\left(1+\left|Du(x+\lambda e_k)\right|^2\right)^{\frac{p(x+\lambda e_k)-2}{2}}Du(x+\lambda e_k)\\
&\qquad-\left(1+\left|Du(x)\right|^2\right)^{\frac{p(x+\lambda e_k)-2}{2}}Du(x)\\
&\qquad+\left[\left(1+\left|Du(x)\right|^2\right)^{\frac{p(x+\lambda e_k)-2}{2}}Du(x)-\left(1+\left|Du(x)\right|^2\right)^{\frac{p(x)-2}{2}}Du(x)\right].
\end{aligned}\tag{3.15}$$

Then from (2.8) we arrive at

$$\begin{aligned}
&k_0\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(x+\lambda e_k)-2}\cdot|\triangle_{\lambda,k}Du|^2\xi^2\,dx\\
&\quad\le-\int_{B_{2r}} S_{ij}(Du)\triangle_{-\lambda,k}\left(\xi\left(\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_j+\frac{\partial\xi}{\partial x_j}\triangle_{\lambda,k}u_i\right)\right)dx\\
&\qquad-\int_{B_{2r}} u_i\partial_i u_j\left(\triangle_{-\lambda,k}\left(\xi^2\triangle_{\lambda,k}u_j\right)\right)dx+\int_{B_{2r}}\phi\partial_i\left(\triangle_{-\lambda,k}\left(\xi^2\triangle_{\lambda,k}u_i\right)\right)dx\\
&\qquad-\int_{B_{2r}}\left[\left(1+\left|Du(x)\right|^2\right)^{\frac{p(x+\lambda e_k)-2}{2}}D_{ij}u(x)-\left(1+\left|Du(x)\right|^2\right)^{\frac{p(x)-2}{2}}D_{ij}u(x)\right]\\
&\qquad\times\triangle_{\lambda,k}D_{ij}u\,dx\\
&\quad=:H_1+H_2+H_3+H_4.
\end{aligned}\tag{3.16}$$

Since $p(x)$ is Hölder continuous, we can choose $0<r<1$ suitable small such that

$$p_2\le p_1(1+\delta_1)\le p(x)(1+\delta_1).$$

By Lemma 3.1, we can see that

$$\begin{aligned}
&k_0\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(x+\lambda e_k)-2}\cdot|\triangle_{\lambda,k}Du|^2\xi^2\,dx\\
&\quad\le k_0\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p_2-2}\cdot|\triangle_{\lambda,k}Du|^2\xi^2\,dx\\
&\quad\le ck_0\int_{B_{3r}}\left(1+\left|Du(x)\right|\right)^{p_2}dx\le c.
\end{aligned}$$

From the previous inequality, one can see that, for suitable small $r\in(0,1]$ and $|\lambda|<r$,

$$\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(x+\lambda e_k)-2}\cdot|\triangle_{\lambda,k}Du|^2\xi^2\,dx<1,\tag{3.17}$$

this conclusion will be heavily used in the following estimate. Furthermore, taking into account the Korn inequality (2.4) and (3.1), we are in a position to obtain

$$\int_{B_{2r}}|\nabla u|^{p(x+\lambda e_k)}\,dx<\int_{B_{3r}}\left(|\nabla u|^{p_2}+1\right)dx\le c\tag{3.18}$$

for all $|\lambda| < r$, where c is a determined positive constant independent of λ, and from now on $r \in (0,1]$ is a fixed constant. For simplicity of notation, in the following paper, we may denote $p(x+\lambda e_k)$ as $p(\bar{x})$. Now, we turn to an estimate of H_1, H_3–H_4.

Estimation of H_1. By Hölder inequality with variable exponent, from (1.10), (2.3) and (2.9), we find that

$$\begin{aligned}
H_1 &\le 2\sum_{i,j=1}^{d}\|S_{ij}(Du)\|_{L^{p'(\bar{x})}(B_{2r})} \\
&\quad\times\left\|\triangle_{-\lambda,k}\left(\xi\left(\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_j+\frac{\partial\xi}{\partial x_j}\triangle_{\lambda,k}u_i\right)\right)\right\|_{L^{p(\bar{x})}(B_{2r})} \\
&\le c|\lambda|^{p_1/p_2}\left(\int_{B_{2r}}\left(1+|Du|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}} \\
&\quad\times\sum_{i,j=1}^{d}\left\|\frac{\partial}{\partial x_k}\left(\xi\left(\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_j+\frac{\partial\xi}{\partial x_j}\triangle_{\lambda,k}u_i\right)\right)\right\|_{L^{p(\bar{x})}(B_{2r})} \\
&\le c\frac{|\lambda|^{p_1/p_2}}{r^{p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}} \\
&\quad\times\left[\frac{|\lambda|^{p_1/p_2}}{r^{p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{s_1}}+\|\xi\nabla(\triangle_{\lambda,k}u)\|_{L^{p(\bar{x})}(B_{2r})}\right],
\end{aligned} \tag{3.19}$$

for all $0<|\lambda|<r<1$, and s_1 equal to p_1 or p_2, q_1 equal to p_1' or p_2', and in the last inequality we have used the fact $|Du| \le |\nabla u|$. Note that

$$\xi\nabla(\triangle_{\lambda,k}u) = \nabla(\xi\triangle_{\lambda,k}u) - \nabla\xi\triangle_{\lambda,k}u \quad\text{and}\quad \left(\nabla(\xi\triangle_{\lambda,k}u)\right)_{2r} = 0 = \left(D(\xi\triangle_{\lambda,k}u)\right)_{2r}.$$

Thus, by the Korn inequality (2.5), we infer that

$$\begin{aligned}
&\|\xi\nabla(\triangle_{\lambda,k}u)\|_{L^{p(\bar{x})}(B_{2r})} \\
&\quad\le \|\nabla(\xi\triangle_{\lambda,k}u)\|_{L^{p(\bar{x})}(B_{2r})}+\|\nabla\xi\triangle_{\lambda,k}u\|_{L^{p(\bar{x})}(B_{2r})} \\
&\quad\le \|D(\xi\triangle_{\lambda,k}u)\|_{L^{p(\bar{x})}(B_{2r})}+\|\nabla\xi\triangle_{\lambda,k}u\|_{L^{p(\bar{x})}(B_{2r})} \\
&\quad\le c\left(\int_{B_{2r}}\left|\xi D(\triangle_{\lambda,k}u)\right|^{p(\bar{x})}dx\right)^{\frac{1}{s_1}}+c\frac{|\lambda|^{p_1/p_2}}{r^{p_2/p_1}}\left(\int_{B_{2r}}|\nabla u|^{p(\bar{x})}dx\right)^{\frac{1}{s_1}},
\end{aligned} \tag{3.20}$$

where in the last inequality, for simplicity, we just denote the exponent as $1/s_1$, and it cannot add any confusion when the exponent is replaced by one of another character.

Now, inserting (3.20) into (3.19), we obtain

$$\begin{aligned}
H_1 &\le \frac{c|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}+\frac{1}{s_1}} \\
&\quad+\frac{c|\lambda|^{p_1/p_2}}{r^{p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}}\cdot\left(\int_{B_{2r}}\left|\xi D(\triangle_{\lambda,k}u)\right|^{p(\bar{x})}dx\right)^{\frac{1}{s_1}} \\
&:= A+B.
\end{aligned} \tag{3.21}$$

Taking into account (3.18), we can see that the term A in the previous inequality is bounded from above for fixed r suitable small. On the other hand, observe that, for any suitable function f, g and $q \le s < 2$, by the Hölder inequality,

$$\begin{aligned}\int |f|^s\,dx &= \int \left(|g|^{\frac{s(q-2)}{2}}|f|^s\right)|g|^{\frac{s(2-q)}{2}}\,dx \\ &\le \left\| |g|^{\frac{s(q-2)}{2}}|f|^s \right\|_{L^{\frac{2}{s}}} \left\| |g|^{\frac{s(2-q)}{2}} \right\|_{L^{\frac{2}{2-s}}}.\end{aligned}$$

Thus, in order to estimate the term B in (3.21), take

$$g = \left(1 + \left|Du(x+\lambda e_k)\right| + \left|Du(x)\right|\right), \qquad f = \xi D(\triangle_{\lambda,k}u),$$

$s = q = p(\bar{x})$ in previous inequality. If $\frac{t_1}{s_1} \ge 1$, by the Hölder inequality, the Young inequality with $k_0/4$ and (2.3), we obtain

$$\begin{aligned}B &\le \frac{c|\lambda|^{p_1/p_2}}{r^{p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}} \\ &\quad\times\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx\right)^{\frac{t_1}{2s_1}} \\ &\quad\times\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}dx\right)^{\frac{2-t_2}{2s_1}} \\ &\le \frac{k_0}{4}\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx\right)^{\frac{t_1}{s_1}} \\ &\quad+c(k_0)\frac{|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2}{q_1}} \\ &\quad\times\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}dx\right)^{\frac{2-t_2}{s_1}} \\ &\le \frac{k_0}{4}\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx \\ &\quad+c(k_0)\frac{|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2}{q_1}} \\ &\quad\times\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}dx\right)^{\frac{2-t_2}{s_1}},\end{aligned} \tag{3.22}$$

where t_1, t_2 are equal to p_1 or p_2, and in the last inequality we have used the fact $\frac{t_1}{s_1} \ge 1$ and (3.17).

Likewise, if $0 < \frac{t_1}{s_1} < 1$, that is, $t_1 = p_1, s_1 = p_2$, by the Hölder inequality, the Young inequality with $k_0/4$ and (2.3), we arrive at

$$\begin{aligned}B &\le \frac{c|\lambda|^{p_1/p_2}}{r^{p_2/p_1}}\left(\int_{B_{2r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}} \\ &\quad\times\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx\right)^{\frac{p_1}{2p_2}}\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{2p_2}} \\
&\le \frac{k_0}{4}\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx \\
&\quad + c(k_0)\left(\int_{B_{2r}} \left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2p_2}{q_1(2p_2-p_1)}} \\
&\quad \times \frac{|\lambda|^{2p_1/(2p_2-p_1)}}{r^{2p_2^2/p_1(2p_2-p_1)}}\left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{2p_2-p_1}} \\
&\le \frac{k_0}{4}\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx \\
&\quad + c(k_0)\left(\int_{B_{2r}} \left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2p_2}{q_1(2p_2-p_1)}} \\
&\quad \times \frac{|\lambda|^{2p_1/(2p_2-p_1)}}{r^{2p_2^2/p_1^2}}\left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{2p_2-p_1}}.
\end{aligned}
\tag{3.23}
$$

Combining (3.22) with (3.23), we obtain

$$
\begin{aligned}
B &\le \frac{k_0}{4}\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx \\
&\quad + c(k_0)\frac{|\lambda|^{2p_1/(2p_2-p_1)}}{r^{2p_2^2/p_1^2}}\left(\int_{B_{2r}} \left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2p_2}{q_1(2p_2-p_1)}} \\
&\quad \times \left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{2p_2-p_1}} \\
&\quad + c(k_0)\frac{|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\left(\int_{B_{2r}} \left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2}{q_1}} \\
&\quad \times \left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{s_1}}.
\end{aligned}
\tag{3.24}
$$

Inserting (3.24) into (3.21), we finally obtain

$$
\begin{aligned}
H_1 &\le \frac{c|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\left[\left(\int_{B_{2r}} \left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{q_1}+\frac{1}{s_1}}\right. \\
&\quad + \left(\int_{B_{2r}} \left(1+|\nabla u|^{p\bar{x})}\right)dx\right)^{\frac{2}{q_1}} \\
&\quad \left.\times \left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{s_1}}\right] \\
&\quad + c\frac{|\lambda|^{2p_1/(2p_2-p_1)}}{r^{2p_2^2/p_1^2}}\left(\int_{B_{2r}} \left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{2p_2}{q_1(2p_2-p_1)}} \\
&\quad \times \left(\int_{B_{2r}} \left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}\,dx\right)^{\frac{2-t_2}{2p_2-p_1}}
\end{aligned}
$$

$$+\frac{k_0}{4}\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx, \tag{3.25}$$

for all $0<|\lambda|<r<1$.

Estimation of H_3. To begin with, we note that

$$\frac{\partial}{\partial x_i}\left(\triangle_{-\lambda,k}\left(\xi^2\triangle_{\lambda,k}u_i\right)\right)=2\left(\triangle_{-\lambda,k}\left(\xi\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_i\right)\right),$$

then from (2.9), (2.3), similar to (3.24), we can see that

$$\begin{aligned}
H_3&=2\int_{B_{2r}}\phi\left(\triangle_{-\lambda,k}\left(\xi\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_i\right)\right)dx\\
&\le c|\lambda|^{p_1/p_2}\|\phi\|_{L^{p'(\bar{x})}(B_{2r})}\left(\int_{B_{2r}}\left|\nabla\left(\xi\frac{\partial\xi}{\partial x_i}\triangle_{\lambda,k}u_i\right)\right|^{p(\bar{x})}dx\right)^{\frac{1}{t_3}}\\
&\le c|\lambda|^{p_1/p_2}\|\phi\|_{L^{p'(x)}(B_{3r})}\\
&\quad\times\left[\frac{|\lambda|^{p_1/p_2}}{r^{2p_2/p_1}}\left(\int_{B_{2r}}|\nabla u|^{p(\bar{x})}dx\right)^{\frac{1}{t_3}}+\frac{1}{r^{p_2/p_1}}\left(\int_{B_{2r}}\left|\xi D(\triangle_{\lambda,k}u)\right|^{p(\bar{x})}dx\right)^{\frac{1}{s_1}}\right]\\
&\le c\frac{|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\|\phi\|_{L^{p'(x)}(B_{3r})}\left(\int_{B_{3r}}\left(1+|\nabla u|^{p(\bar{x})}\right)dx\right)^{\frac{1}{t_3}}\\
&\quad+c\frac{|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\|\phi\|^2_{L^{p'(x)}(B_{3r})}\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}dx\right)^{\frac{2-t_2}{s_1}}\\
&\quad+c(k_0)\frac{|\lambda|^{2p_2/(2p_2-p_1)}}{r^{2p_2^2/p_1^2}}\|\phi\|^{2p_2/(2p_2-p_1)}_{L^{p'(x)}(B_{3r})}\\
&\quad\times\left(\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})}dx\right)^{\frac{2-t_2}{2p_2-p_1}}\\
&\quad+\frac{k_0}{4}\int_{B_{2r}}\left(1+\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)^{p(\bar{x})-2}\cdot\left|D(\triangle_{\lambda,k}u)\right|^2\xi^2\,dx,
\end{aligned}\tag{3.26}$$

where t_3 is equal to p_1 or p_2, and we replace t_3 by s_1 in the second inequality if there is no danger of any possible confusion, since from (3.19) and (3.21), we infer that $(\int_{B_{2r}}|\xi D(\triangle_{\lambda,k}u)|^p\,dx)^{\frac{1}{s_1}}$ is biggest for all cases of the value of t_3.

Estimation of H_4. Now, since $0<r<1$ is suitably small, we have

$$p_2\left(1+\frac{\delta_1}{4}\right)\le p_1(1+\delta_1)\le p(x)(1+\delta_1).$$

From (2.6) and Lemma 3.1, we can find that

$$\begin{aligned}
H_4&\le c\omega(|\lambda|)\int_{B_{2r}}\left(1+\left|Du(x)\right|^2\right)^{\frac{p_2-1}{2}}\\
&\quad\times\left(\log\left(1+\left|Du(x)\right|\right)+1\right)\left(\left|Du(x+\lambda e_k)\right|+\left|Du(x)\right|\right)dx\\
&\le c\omega(|\lambda|)\int_{B_{2r}}\left(1+\left|Du(x)\right|\right)^{p_2}\left(\log^{\frac{p_2}{p_2-1}}\left(1+\left|Du(x)\right|\right)+1\right)\\
&\quad+\left|Du(x+\lambda e_k)\right|^{p_2}dx
\end{aligned}$$

$$
\begin{aligned}
&\le c|\lambda|^{2\theta_1}\int_{B_{3r}}\bigl(1+\bigl|Du(x)\bigr|\bigr)^{p_2(1+\frac{\delta_1}{4})}\,dx\\
&\le c|\lambda|^{2\theta_1}\int_{B_{3r}}\bigl(1+\bigl|Du(x)\bigr|\bigr)^{p(x)(1+\delta_1)}\,dx\le c\frac{\lambda^{2\theta_1}}{r^{\delta_1}d},
\end{aligned}
\tag{3.27}
$$

for all $0<|\lambda|\le r$.

Inserting (3.25)–(3.27) into (3.16), we finally obtain

$$
\begin{aligned}
&\frac{k_0}{2}\int_{B_{2r}}\bigl(1+\bigl|Du(x+\lambda e_k)\bigr|+\bigl|Du(x)\bigr|\bigr)^{p(\bar{x})-2}\cdot\bigl|D(\triangle_{\lambda,k}u)\bigr|^2\xi^2\,dx\\
&\quad\le H_2+c\frac{|\lambda|^{2p_1/p_2}}{r^{2p_2/p_1}}\biggl[\biggl(\int_{B_{3r}}\bigl(1+|\nabla u|^{p(x)}\bigr)\,dx\biggr)^{\frac{1}{q_1}+\frac{1}{s_1}}\\
&\qquad+\|\phi\|_{L^{p'(x)}(B_{3r})}\biggl(\int_{B_{3r}}\bigl(1+|\nabla u|^{p(x)}\bigr)\,dx\biggr)^{\frac{1}{t_3}}\\
&\qquad+\biggl(\|\phi\|^2_{L^{p'(x)}(B_{3r})}+\biggl(\int_{B_{3r}}\bigl(1+|\nabla u|^{p(x)}\bigr)\,dx\biggr)^{\frac{2}{q_1}}\biggr)\\
&\qquad\times\biggl(\int_{B_{2r}}\bigl(1+\bigl|Du(x+\lambda e_k)\bigr|+\bigl|Du(x)\bigr|\bigr)^{p(\bar{x})}\,dx\biggr)^{\frac{2-t_2}{s_1}}\biggr]\\
&\qquad+\frac{|\lambda|^{2p_2/(2p_2-p_1)}}{r^{2p_2^2/p_1^2}}\biggl(\int_{B_{2r}}\bigl(1+\bigl|Du(x+\lambda e_k)\bigr|+\bigl|Du(x)\bigr|\bigr)^{p(\bar{x})}\,dx\biggr)^{\frac{2-t_2}{2p_2-p_1}}\\
&\qquad\times\biggl(\int_{B_{3r}}\bigl(1+|\nabla u|^{p(x)}\bigr)\,dx\biggr)^{\frac{2p_2}{q_1(2p_2-p_1)}}\\
&\qquad+\frac{|\lambda|^{2p_2/(2p_2-p_1)}}{r^{2p_2^2/p_1^2}}\biggl(\int_{B_{2r}}\bigl(1+\bigl|Du(x+\lambda e_k)\bigr|+\bigl|Du(x)\bigr|\bigr)^{p(\bar{x})}\,dx\biggr)^{\frac{2-t_2}{2p_2-p_1}}\\
&\qquad\times\|\phi\|^{2p_2/(2p_2-p_1)}_{L^{p'(x)}(B_{3r})}+c\frac{\lambda^{2\theta_1}}{r^{\delta_1}d}.
\end{aligned}
\tag{3.28}
$$

Estimation of H_2. To estimate H_2, we first note that

$$
\begin{aligned}
H_2&=\int_{B_{2r}}(\triangle_{\lambda,k}u_i)\frac{\partial u_j(x+\lambda e_k)}{\partial x_i}\xi^2\triangle_{\lambda,k}u_j\,dx\\
&\quad+\int_{B_{2r}}u_i\biggl(\triangle_{\lambda,k}\frac{\partial u_j}{\partial x_i}\biggr)\xi^2\triangle_{\lambda,k}u_j\,dx\\
&:=C+D,
\end{aligned}
$$

where $i,j=1,\dots,d$. From now on, we assume for the moment that we have proved

$$
u\in W^{1,s}_{\mathrm{loc}}(\Omega),
\tag{3.29}
$$

with $s\in[p_\infty,A_d]$, $A_d\le d$, and

$$
A_d:=\begin{cases}d, & d=2,\\ \frac{3p_0d}{(d+2)p_0-2\bar{p}_\infty}, & d=3,\end{cases}
$$

here we denote $3d/(d+2)$ by $\bar{p}_\infty$, we will see that the assumption of (3.29) is valid later. From the Hölder inequality, we have

$$C \le \int_{B_{2r}} |\triangle_{\lambda,k} u|^2 \left|\nabla u(x+\lambda e_k)\right| dx \le c\|\triangle_{\lambda,k} u\|^2_{L^{2s'}(B_{2r})} \|u\|_{W^{1,s}(B_{3r})}. \tag{3.30}$$

We let $s^* = \frac{ds}{d-s}$, since $s < 2s' < s^*$, by interpolation, it follows that

$$\|\triangle_{\lambda,k} u\|^2_{L^{2s'}(B_{2r})} \le \|\triangle_{\lambda,k} u\|^{2(1-\theta)}_{L^{s^*}(B_{2r})} \cdot \|\triangle_{\lambda,k} u\|^{2\theta}_{L^s(B_{2r})} \le c|\lambda|^{2\theta} \|u\|^2_{W^{1,s}(B_{3r})}, \tag{3.31}$$

with $\theta = \frac{s(d+2)-3d}{2s}$.

Combining (3.30) with (3.31), we arrive at

$$C \le c|\lambda|^{2\theta} \|u\|^3_{W^{1,s}(B_{3r})}. \tag{3.32}$$

For the term D, by integration by parts, we get

$$D = -\int_{B_{2r}} u_i (\triangle_{\lambda,k} u_j)^2 \xi \frac{\partial \xi}{\partial x_i} dx.$$

Similar to the estimation of the term C, we obtain

$$D \le \frac{c}{r} \|\triangle_{\lambda,k} u\|^2_{L^{2s'}(B_{2r})} \|u\|_{W^{1,s}(B_{3r})} \le \frac{c}{r} |\lambda|^{2\theta} \|u\|^3_{W^{1,s}(B_{3r})}. \tag{3.33}$$

Together (3.32) with (3.33), we finally obtain

$$H_2 \le c\left(1+\frac{1}{r}\right)\lambda^{2\theta} \|u\|^3_{W^{1,s}(B_{3r})} \le \frac{c\lambda^{2\theta}}{r^2} \|u\|^3_{W^{1,s}(B_{3r})}. \tag{3.34}$$

Since $s \in [p_\infty, A_d]$, we can see that

$$\begin{cases} \frac{p_1}{p_2} \ge \frac{p_\infty}{p_0} \ge \theta, \\ \frac{2p_1}{2p_2-p_1} \ge \frac{2p_\infty}{2p_0-p_\infty} \ge \theta, \end{cases} \tag{3.35}$$

and from (1.8), (1.9), we have

$$r^{-(p_2-p_1)} = 2^{p_2-p_1} e^{(p_2-p_1)\log\frac{1}{2r}} \le 2^{p_2-p_1} e^{\omega(2r)\log\frac{1}{2r}} \le c(L), \tag{3.36}$$

for all $0 < r < 1$. Moreover, since δ_1 is independent of r, we can choose $\delta \in (0, \delta_1]$ to be a constant, replacing δ_1 by δ in (3.27), such that $\delta d \le 2$. Now, taking into account (3.35), (3.36), inserting (3.34) into (3.28), we finally obtain

$$\begin{aligned} &\int_{B_{2r}} \left(1 + \left|Du(x+\lambda e_k)\right| + \left|Du(x)\right|\right)^{p(\bar{x})-2} \cdot \left|\triangle_{\lambda,k} D(u)\right|^2 \xi^2 \, dx \\ &\quad \le \frac{c|\lambda|^{2p_1/p_2}}{r^{2(1+(p_2-p_1)/p_1)}} + \frac{c|\lambda|^{2p_1/(2p_2-p_1)}}{r^{2(1+(p_2-p_1)(p_2+p_1)/p_1^2)}} + \frac{c}{r^2}|\lambda|^{2\theta} + c\frac{\lambda^{2\theta_1}}{r^{\delta d}} \\ &\quad \le \frac{c}{r^2}|\lambda|^{2\theta}, \end{aligned} \tag{3.37}$$

for all $0 < |\lambda| < r < 1$ with c depends on $\|u\|_{W^{p(x)}_{\mathrm{loc}}(\Omega)}$, $\|u\|_{W^{1,s}_{\mathrm{loc}}(\Omega)}$, L and $\|\phi\|_{L^{p'(x)}_{\mathrm{loc}}(\Omega)}$, and in the last inequality we have used the fact $\theta_1 \geq \theta$.

In what follows, we set

$$\gamma := \frac{2s}{s+2-\bar{p}_\infty} \in \left[\frac{2p_\infty}{2p_\infty+2-\bar{p}_\infty}, \frac{2A_d}{2A_d+2-\bar{p}_\infty}\right].$$

Observe that

$$\frac{2s(\gamma-2)}{2\gamma} = \bar{p}_\infty - 2 \quad \text{for all } s \in [p_\infty, A_d].$$

Taking into account (3.19), with the aid of the Hölder inequality, we infer that

$$\begin{aligned}
&\int_{B_r} \left|\triangle_{\lambda,k} D(u)\right|^\gamma dx \\
&\quad \leq \left(\int_{B_{2r}} \left(1 + \left|Du(x+\lambda e_k)\right| + \left|Du(x)\right|\right)^{\bar{p}_\infty - 2} \left|\triangle_{\lambda,k} D(u)\right|^2 \xi^2 \, dx\right)^{\frac{\gamma}{2}} \\
&\qquad \times \left(\int_{B_{2r}} \left(1 + \left|Du(x+\lambda e_k)\right| + \left|Du(x)\right|\right)^{s} dx\right)^{\frac{2-\gamma}{2}} \\
&\quad \leq \left(\int_{B_{2r}} \left(1 + \left|Du(x+\lambda e_k)\right| + \left|Du(x)\right|\right)^{p(\bar{x}) - 2} \left|\triangle_{\lambda,k} D(u)\right|^2 \xi^2 \, dx\right)^{\frac{\gamma}{2}} \\
&\qquad \times \left(\int_{B_{2r}} \left(1 + \left|Du(x+\lambda e_k)\right| + \left|Du(x)\right|\right)^{s} dx\right)^{\frac{2-\gamma}{2}} \leq \frac{c|\lambda|^{\gamma\theta}}{r^2},
\end{aligned} \tag{3.38}$$

where in the first inequality, we have taken into account $\frac{\gamma}{2-\gamma}(2-\bar{p}_\infty) = s$.

Appealing to (3.38), the equivalence of the Nikolskii fractional space and the Sobolev space [42], we have

$$Du \in W^{t,\gamma}_{\mathrm{loc}}(\Omega), \quad \text{for all } t \in [0,\theta],$$

then the fractional order Sobolev embedding theorem implies that

$$Du \in L^{\frac{d\gamma}{d-t\gamma}} \quad \text{for all } t \in [0,\theta]. \tag{3.39}$$

Step 2 (Higher integrability of Du). We set

$$T(s) := \frac{d\gamma}{d-\theta\gamma} = \frac{4sd}{10d - 2\bar{p}_\infty d - 4s}.$$

Then, from the definition of $\bar{p}_\infty$, by a direct calculation, we find that

$$T(s) - s \geq \sigma := \frac{6d + 2\bar{p}_\infty d + 4p_\infty}{10d - 2\bar{p}_\infty d - 4p_\infty} > 0. \tag{3.40}$$

Taking into account (3.39), (3.40), we obtain

$$Du \in L^\eta(B_{\frac{r}{2}}), \quad \text{for all } \eta \in \left[1, T(s)\right]. \tag{3.41}$$

We use that $u \in V_{p(x)}$ is any weak solution to (1.6) and $p(x) \geq p_\infty$. Thus, we claim that $s = p_\infty$ such that (3.29) holds. We set

$$s_0 := p_\infty, \qquad s_1 := s_0 + \frac{\sigma}{2}.$$

In virtue of (3.40), we see that

$$T(s_0) > s_1 > s_0 = p_\infty.$$

Taking into account (3.41),

$$Du \in L^{s_1}(B_{\frac{r}{2}}).$$

By the Korn inequality, we have

$$u \in W^{1,s_1}_{\mathrm{loc}}(\Omega), \tag{3.42}$$

from the above, it follows that (3.29) holds with $s = s_1$. Now, if $s_1 > A_d$, we have proved the higher integrability of Du, and we can derive the Hölder continuity of u by the Sobolev imbedding theorem, since $Du \in L^\eta(B_{\frac{r}{2}})$ for all $\eta \in [1, T(A_d)]$, and $T(A_d) > d$. If $s_1 \leq A_d$, we continue the process above, such that there is a $s_i > A_d$ $(i > 1)$. Without loss of generality, we assume that $s_1 \leq A_d$, then from (3.29), (3.39)–(3.42),

$$Du \in L^{\eta_1}(B_{\frac{r}{2}}), \quad \eta_1 \in \bigl[1, T(s_1)\bigr].$$

Note that $T(s_1) > T(s_0)$, then we can increase the power of integrability of Du by a standard bootstrap argument. We set

$$\tilde{s} := \sup\bigl\{s \in [p_\infty, A_d] : u \in W^{1,s}_{\mathrm{loc}}(\Omega)\bigr\}. \tag{3.43}$$

We may assume that $s = A_d$, otherwise, $s < A_d$. Define

$$\bar{s} := \tilde{s} - \frac{\sigma}{4},$$

by the definition of $\tilde{s}$, we can obtain $u \in W^{1,\bar{s}}_{\mathrm{loc}}(\Omega)$, since $\tilde{s} > p_\infty + \sigma$. From (3.41), it follows that

$$Du \in L^\eta(B_{\frac{r}{2}}), \quad \text{for all } \eta \in \bigl[1, T(\bar{s})\bigr].$$

Then, choose $\eta := T(\bar{s}) - \frac{\sigma}{4}$, and taking into account (3.40), we find that

$$\eta \geq \bar{s} + \sigma - \frac{\sigma}{4} \geq \tilde{s} + \frac{\sigma}{2},$$

it contradicts (3.43).

By the conclusion above, from now on we have

$$u \in W^{1,s}_{\mathrm{loc}}(\Omega), \quad \text{for all } s \in [p_\infty, A_d],$$

then, from (3.41), we have

$$u \in W^{1,T(s)}_{\mathrm{loc}}(\Omega), \quad \text{for all } s \in [p_\infty, A_d].$$

Note that $T(A_d) > T(s)$, thus

$$u \in W^{1,\eta}_{\mathrm{loc}}(\Omega), \quad \text{for all } \eta \in \bigl[p_\infty, T(A_d)\bigr], \tag{3.44}$$

with $T(A_d) > d$. Now, making use of the Sobolev embedding theorem, from (3.44), $u \in C^\alpha(\Omega)$, for some $\alpha \in (0,1)$. □

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