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New homoclinic solutions for a class of second-order Hamiltonian systems with a mixed condition

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Abstract

In this paper, we introduce a new mixed condition to obtain a new compact embedding theorem. Under this theorem, we study the existence and multiplicity of nontrivial homoclinic solutions for a class of second-order Hamiltonian systems with variable separated type nonlinear terms.

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1 Introduction and main results

In this paper, we consider the following second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u + \nabla W(t, u) = 0, \tag{1}$$

where $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a C^1-map and $\nabla W(t, x)$ denotes the gradient with respect to the x variable. $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a matrix-valued function. A solution $u(t)$ of problem (1) is nontrivial homoclinic (to 0) if $u \not\equiv 0$, $u(t) \to 0$ as $t \to \pm\infty$.

In the last two decades, many mathematicians have successfully used variational methods to obtain the existence and multiplicity of homoclinic orbits for problem (1) such as [1, 3–18, 20–37]. Since this problem is considered in the whole space, one of the main difficulties is the lack of compactness of embedding. In 1990, Rabinowitz [20] considered the periodic case. In his paper, $L(t)$ and $W(t, u)$ were assumed to be periodic in t and the author obtained the results by the mountain pass theorem. Without the periodic condition, Rabinowitz and Tanaka [21] introduced the following coercive condition on $L(t)$.

(L') $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric and positively definite matrix for all $t \in \mathbb{R}$, and there exists a continuous function $l : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\left(L(t)x, x\right) \geq l(t)|x|^2 \quad \text{with } l(t) \to \infty \text{ as } |t| \to \infty.$$

Using (L'), many mathematicians studied problem (1) with superquadratic, subquadratic, or asymptotic quadratic nonlinearities. In this paper, we mainly consider the superquadratic growth case. First, we recall some main results.

In 1991, Rabinowitz and Tanaka [21] used the so-called Ambrosetti–Rabinowitz ((AR)
for short) condition to obtain the existence of homoclinic solutions for problem (1).

(AR) There exists a constant $\theta > 2$ such that

$$0 < \theta W(t,x) \leq \big(\nabla W(t,x),x\big)$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$.

Condition (AR) is useful in guaranteeing the geometrical structure of the corresponding
functional and the boundedness of Palais–Smale ((PS) for short) sequence. After then,
there have been many mathematicians trying to weaken this condition. In 2009, Ding and
Lee [6] introduced the following generalized superquadratic condition.

(H_1) There exist $\epsilon \in (0,1)$ and $\kappa_1, b_0 > 0$ such that

$$\widetilde{W}(t,x) \geq b_0 \frac{(\nabla W(t,x),x)}{|x|^{2-\epsilon}} \quad \text{for all } t \in \mathbb{R} \text{ and } |x| \geq \kappa_1,$$

where

$$\widetilde{W}(t,x) = \big(\nabla W(t,x),x\big) - 2W(t,x).$$

In a recent paper, (H_1) was weakened by Wu et al. [28] with the following condition.

(H_2) $\frac{\widetilde{W}(t,x)}{W(t,x)}|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in $t \in \mathbb{R}$.

In 2011, Yang and Han [31] also made a contribution to the multiplicity of homoclinic
solutions for problem (1) under the following monotonous condition.

(H_3) $s^{-1}(\nabla W(t,sx),x)$ is an increasing function of $s \in (0,1]$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

In 2013, Lv and Tang [15] showed that the following condition is weaker than (H_3).

(H_4) There exists $\mu \geq 1$ such that

$$\mu \widetilde{W}(t,x) \geq \widetilde{W}(t,sx)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $s \in [0,1]$.

Conditions (H_1)–(H_4) can be seen as the generalizations or supplements of (AR). How-
ever, each result in the above papers implies that

(H_5) There exists $\kappa_2 > 0$ such that $\sup_{t\in\mathbb{R},|x|=\kappa_2} W(t,x) < +\infty$.

Obviously, (H_5) holds when $W(t,x)$ is periodic in t. If there is no periodic assumption, (H_5)
is an important requirement in many papers [1, 4–13, 16–18, 20, 21, 24, 27, 33, 35]. Re-
cently, Yuan and Zhang [35] considered the following variable separation nonlinear term:

$$W(t,x) = a(t)G(x), \tag{2}$$

where $a : \mathbb{R} \to \mathbb{R}^+$ is continuous, $G \in C^1(\mathbb{R}^N, \mathbb{R})$. In [35], the authors considered the fol-
lowing conditions:

(A) $a(t) \to 0$ as $|t| \to \infty$;

(L'') $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric and positively definite matrix for all $t \in \mathbb{R}$, and there
exists a constant $M_0 > 0$ such that

$$\big(L(t)x,x\big) \geq M_0|x|^2 \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Obviously, (L'') is weaker than (L'). In a recent paper, Wu et al. [28] considered a class of fractional Hamiltonian systems with variable separated type nonlinear terms. They introduced the following condition.

(L''') $\frac{a(t)}{l(t)} \to 0$ as $|t| \to \infty$, where $l(t) \in C(\mathbb{R}, \mathbb{R})$ satisfies

$$l(t) \equiv \inf_{|x|=1} (L(t)x, x).$$

Motivated by the above papers, in this paper, we introduce the following condition.

(L_1) There exists a constant $r_0 > 0$ such that

$$\lim_{|s| \to +\infty} \mathrm{meas}\left(\left\{ t \in B_{r_0}(s) : \frac{l(t)}{a(t)} \not\geq M \right\} \right) = 0, \quad \forall M > 0,$$

where $B_{r_0}(s) = \{t \in \mathbb{R} : s - r_0 \leq t \leq s + r_0\}$, $l(t) \in C(\mathbb{R}, \mathbb{R})$ and $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure.

In [35], the nonlinear term $G(x)$ is assumed to satisfy the following classical AR-type condition:

(AR') There exists a constant $\nu > 2$ such that

$$0 < \nu G(x) \leq (\nabla G(x), x)$$

for every $x \in \mathbb{R}^N \setminus \{0\}$.

From the (AR') condition, we can deduce that

(SQ) $\widetilde{G}(x) \geq 0$ for any $x \in \mathbb{R}^N$, where $\widetilde{G}(x) = (\nabla G(x), x) - 2G(x)$.

In [28], the authors considered the following weakened superquadratic growth condition.

(H_6) There exist $\nu > 2$ and $b_2, \kappa_3 > 0$ such that

$$(\nabla G(x), x) - \nu G(x) \geq -b_2|x|^2 \quad \text{for all } |x| \geq \kappa_3.$$

(H_6) is a generalization of condition (AR'), which is only a local condition. Instead of (H_6), the authors in [28] also considered the following superquadratic condition.

(H_7) $\frac{\widetilde{G}(x)}{G(x)}|x|^{2-r_0} \to +\infty$ as $|x| \to \infty$, where r_0 is the growth exponent of the subquadratic term at infinity with respect to x.

In this paper, we consider $G(x)$ satisfying the following monotonous condition.

(W_1) There exists a constant $\theta \geq 1$ such that $\theta \widetilde{G}(x) \geq \widetilde{G}(sx)$ for all $x \in \mathbb{R}^N$ and $s \in [0, 1]$.

Now we state our main results.

Theorem 1.1 *Suppose* (2), (L''), (L_1), (W_1) *and the following conditions hold*:

   (L_2) *There exists $A > 0$ such that $a(t) \leq Al(t)$ for all $t \in \mathbb{R}$;*

   (W_2) $a(t) \in C(\mathbb{R}, \mathbb{R}^+)$;

   (W_3) $G(0) = 0$ *and* $\nabla G(x) = o(|x|)$ *as* $|x| \to 0$;

   (W_4) $G(x)/|x|^2 \to +\infty$ *as* $|x| \to \infty$.

   (W_5) *There are constants $\zeta > 2$ and $d_1 > 0$ such that*

$$|G(x)| \leq d_1(|x|^2 + |x|^\zeta) \quad \text{for all } x \in \mathbb{R}^N.$$

Then problem (1) *possesses at least one nontrivial homoclinic solution.*

If $G(t)$ is even in t, we can obtain the following multiplicity result.

Theorem 1.2 *Suppose* (2), (L''), (L_1), (L_2), (W_1)–(W_5) *and* $G(-x) = G(x)$ *for all* $x \in \mathbb{R}^N$. *Then problem* (1) *possesses infinitely many solutions.*

By a condition similar to (H_2), we obtain the following.

Theorem 1.3 *Suppose* (2), (L''), (L_1), (L_2), (W_2)–(W_4), (SQ) *and the following condition holds*:

(W_6) $\frac{\widetilde{G}(x)}{G(x)}|x|^2 \to +\infty$ *as* $|x| \to \infty$.
Then problem (1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.4 *Suppose* (2), (L''), (L_1), (L_2), (W_2)–(W_4), (W_6), (SQ) *and* $G(-x) = G(x)$ *for all* $x \in \mathbb{R}^N$. *Then problem* (1) *possesses infinitely many solutions.*

Remark 1 It can be seen that $a(t)$ and $l(t)$ are positive in our theorems and the results in [28]. Then we can deduce that (L_1) is weaker than (L'''). The reader can see [2] for more details.

Remark 2 Condition (H_5) does not hold in our theorems since $a(t)$ can go to infinity as $|t| \to \infty$.

Remark 3 Condition (W_1) implies (SQ).

2 Preliminaries
Set

$$E := \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} \left(|\dot{u}(t)|^2 + \left(L(t)u(t), u(t) \right) \right) dt < \infty \right\}$$

with the inner product

$$(u, v)_E = \int_{\mathbb{R}} \left(\left(\dot{u}(t), \dot{v}(t) \right) + \left(L(t)u(t), u(t) \right) \right) dt.$$

Let $L_a^2(\mathbb{R}, \mathbb{R}^N)$ be the weighted space of measurable functions $u : \mathbb{R} \to \mathbb{R}^N$ under the norm

$$\|u\|_{L_a^2} = \left(\int_{\mathbb{R}} a(t) |u(t)|^2 \, dt \right)^{1/2}.$$

Throughout this paper, $C > 0$ denotes a universal positive constant. Note that under condition (L_1)

$$E \subset H^1(\mathbb{R}, \mathbb{R}^N) \subset L^p(\mathbb{R}, \mathbb{R}^N)$$

for any $p \in [2, +\infty]$ with the embedding being continuous, which implies that there is a positive constant $C_\infty > 0$ such that the following inequality holds:

$$\max\{\|u\|_2, \|u\|_\infty\} \leq C_\infty \|u\|. \tag{3}$$

First, we show a compact embedding theorem.

Lemma 2.1 *Under conditions (L_1)–(L_2) and (L''), the embedding $E \hookrightarrow L_a^2(\mathbb{R}, \mathbb{R}^N)$ is continuous and compact.*

Proof Suppose that $\{u_n\}$ is a bounded sequence in E. Then there exists $M_0 > 0$ such that $\|u_n\| \le M_0$. Hence there exists a weak convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in E. Assuming $v_n = u_n - u_0$, we obtain that $\{v_n\}$ is a bounded sequence in E and $v_n \rightharpoonup 0$ in E. Next, we show that $v_n \to 0$ in $L_a^2(\mathbb{R}, \mathbb{R}^N)$. It follows from the Sobolev compact embedding theorem that $v_n \to 0$ in $L^1(B_R(0), \mathbb{R}^N)$ for any $R > 0$. Choose $\{s_i\} \subset \mathbb{R}$ such that $\mathbb{R} \subset \bigcup_{i=1}^{\infty} B_{r_0}(s_i)$ and each $t \in \mathbb{R}$ is contained by two such intervals at most. Set

$$A(M, R) = \left\{ t \in B_R^c(0) : \frac{l(t)}{a(t)} \not\ge M \right\}$$

and

$$B(M, R) = \left\{ t \in B_R^c(0) : \frac{l(t)}{a(t)} \ge M \right\}.$$

On the one hand, we have

$$\int_{B(M,R)} a(t) |v_n(t)|^2 \, dt \le \frac{1}{M} \int_{B(M,R)} l(t) |v_n(t)|^2 \, dt \le \frac{\|v_n\|^2}{M} \le \frac{2M_0}{M}. \tag{4}$$

On the other hand, let $\varepsilon_R = \sup_{s_i}(\operatorname{meas}(A(M, R) \bigcap B_{r_0}(s_i)))$, we obtain

$$
\begin{aligned}
\int_{A(M,R)} a(t) |v_n(t)|^2 \, dt &\le \sum_{i=1}^{\infty} \int_{A(M,R) \bigcap B_{r_0}(s_i)} a(t) |v_n(t)|^2 \, dt \\
&\le \varepsilon_R^{\frac{1}{2}} \sum_{i=1}^{\infty} \left(\int_{A(M,R) \bigcap B_{r_0}(s_i)} a^2(t) |v_n(t)|^4 \, dt \right)^{\frac{1}{2}} \\
&\le A \varepsilon_R^{\frac{1}{2}} \sum_{i=1}^{\infty} \left(\int_{A(M,R) \bigcap B_{r_0}(s_i)} l^2(t) |v_n(t)|^4 \, dt \right)^{\frac{1}{2}} \\
&\le AC \varepsilon_R^{\frac{1}{2}} \sum_{i=1}^{\infty} \int_{B_{r_0}(s_i)} \left(|\dot{u}(t)|^2 + \left(L(t)u(t), u(t) \right) \right) dt \\
&\le 2AC \varepsilon_R^{\frac{1}{2}} \|u\|^2 \tag{5}
\end{aligned}
$$

for some $C > 0$. It follows from (L_2) that $\int_{A(M,R)} a(t) |v_n(t)|^2 \, dt \to 0$ as $R \to \infty$, which implies that

$$
\begin{aligned}
\int_{B_R^c(0)} a(t) |v_n(t)|^2 \, dt &= \int_{A(M,R)} a(t) |v_n(t)|^2 \, dt + \int_{B(M,R)} a(t) |v_n(t)|^2 \, dt \\
&\le \frac{2M_0}{M} + 2d_0 C \varepsilon_R^{\frac{1}{2}} \|u\|^2 \\
&\to 0 \quad \text{as } \min\{M, R\} \to \infty.
\end{aligned}
$$

Then we can deduce that $v_n \to 0$ in $L_a^2(\mathbb{R}, \mathbb{R}^N)$. Hence the embedding $E \hookrightarrow L_a^2(\mathbb{R}, \mathbb{R}^N)$ is continuous and compact. $\qquad\square$

From Lemma 2.1, we can easily deduce that there is a positive constant $K > 0$ such that the following inequality holds:

$$\|u\|_{L_a^2} \le K\|u\|. \tag{6}$$

Lemma 2.2 *Suppose that conditions (L''), (2), (L_1), and (W_4) hold, then we have $\nabla G(u_k) \to \nabla G(u)$ in $L_a^2(\mathbb{R}, \mathbb{R}^N)$ if $u_k \rightharpoonup u$ in E.*

Proof The proof of this lemma is similar to that of Lemma 2.3 in [28]. $\qquad\square$

The corresponding functional of (1) is defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W\big(t, u(t)\big)\, dt. \tag{7}$$

Lemma 2.3 *Suppose (L''), (2), (L_1), and (W_4) hold, then $I \in C^1(E, \mathbb{R})$ is w.l.s.c. and*

$$\langle I'(u), v \rangle = \int_{\mathbb{R}} \big(\big(\dot{u}(t), \dot{v}(t) \big) + \big(L(t)u(t), v(t) \big) \big)\, dt - \int_{\mathbb{R}} \big(\nabla W\big(t, u(t)\big), v(t) \big)\, dt \quad \forall u, v \in E.$$

Proof The proof is similar to Lemma 2.4 in [28]. $\qquad\square$

From Lemma 2.3, we obtain

$$\langle I'(u), u \rangle = \|u\|^2 - \int_{\mathbb{R}} \big(\nabla W\big(t, u(t)\big), u(t) \big)\, dt. \tag{8}$$

The following theorem is useful in proving the multiplicity of homoclinic solutions for problem (1).

Lemma 2.4 (see [19], Theorem 9.12) *Let $\mathcal{B}$ be an infinite dimensional real Banach space, and let $I \in C^1(\mathcal{B}, \mathbb{R})$ be even, satisfy the (PS) condition and $I(0) = 0$. If $\mathcal{B} = V \oplus X$, where V is finite dimensional and I satisfies the following*:
  (A_1) *There are constants $\varrho, \alpha > 0$ such that $I|_{\partial B_\varrho \cap X} \ge \alpha$; and*
  (A_2) *For each finite dimensional subspace $\widetilde{E} \subset \mathcal{B}$, there is $r = r(\widetilde{E})$ such that $I \le 0$ on $\widetilde{E} \setminus B_{r(\widetilde{E})}$.*
  Then I has an unbounded sequence of critical values.

3 Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into several lemmas.

Lemma 3.1 *Suppose that (SQ) and (W_3) hold, then $G(x) \ge 0$ for all $x \in \mathbb{R}$.*

Proof The proof of this lemma is similar to that of Lemma 7 in [15]. $\qquad\square$

Definition 3.1 A sequence $\{u_n\} \subset X$, where X is a real Banach space, is called a Cerami ((C) for short) sequence of functional I if $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$.

Definition 3.2 A functional I is said to satisfy the (C) condition if any (C) sequence of I possesses a convergent subsequence.

Lemma 3.2 *Suppose that the conditions of Theorem* 1.1 *hold, then I satisfies the* (C) *condition.*

Proof Assume that $\{u_n\} \subset E$ is a sequence such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$. Then, for some $M_1 > 0$, it follows

$$\left|I(u_n)\right| \leq M_1, \qquad \left\|I'(u_n)\right\|\left(1 + \|u_n\|\right) \leq M_1. \tag{9}$$

Next we show that $\{u_n\}$ is bounded in E. Assuming $\|u_n\| \to +\infty$ as $n \to \infty$, set $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$, which implies that there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that $w_n \rightharpoonup w_0$ in E. By (7) and (9), we get

$$\left|\int_{\mathbb{R}} \frac{W(t, u_n(t))}{\|u_n\|^2} \, dt - \frac{1}{2}\right| = \left|-\frac{I(u_n)}{\|u_n\|^2}\right| \leq \frac{M_1}{\|u_n\|^2}, \tag{10}$$

which implies that

$$\left|\int_{\mathbb{R}} \frac{W(t, u_n(t))}{\|u_n\|^2} \, dt\right| \leq 1 \tag{11}$$

for n large enough. The following discussion is divided into two cases.

Case 1: $w_0 \not\equiv 0$. Let $\Omega = \{t \in \mathbb{R} \mid |w_0(t)| > 0\}$. We can see that $\mathrm{meas}(\Omega) > 0$. Then there exists $\chi > 0$ such that $\mathrm{meas}(\Lambda) > 0$, where $\Lambda = \Omega \bigcap B_\chi(0)$. Since $\|u_n\| \to +\infty$ as $n \to \infty$ and $|u_n(t)| = |w_n(t)| \cdot \|u_n\|$, we have $|u_n(t)| \to +\infty$ as $n \to \infty$ for a.e. $t \in \Lambda$. Let $a_1 = \inf_{t \in B_\chi(0)} a(t) > 0$. By (2), (W_2), (W_4), Lemma 3.1, and Fatou's lemma, we can obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} \, dt \geq a_1 \lim_{n \to \infty} \int_{\Lambda} \frac{G(u_n(t))}{|u_n(t)|^2} \left|w_n(t)\right|^2 dt$$

$$\geq a_1 \lim_{n \to \infty} \int_{\Lambda} \frac{G(u_n(t))}{|u_n(t)|^2} \left|w_n(t)\right|^2 dt$$

$$\geq a_1 \lim_{n \to \infty} \int_{\Lambda} \frac{G(u_n(t))}{|u_n(t)|^2} \left|w_n(t)\right|^2 dt$$

$$= +\infty,$$

which contradicts (11).

Case 2: $w_0 \equiv 0$. Set a sequence $\{T_n\} \subset [0, 1]$ such that $I(T_n u_n) = \max_{T \in [0,1]} I(Tu_n)$. By Lemma 3.1, (W_2), (W_5), and Lemma 2.1, we obtain

$$0 \leq \int_{\mathbb{R}} a(t)G\left(4\sqrt{\theta M_1} w_n(t)\right) dt$$

$$\leq d_1 \left(16\theta M_1 \int_{\mathbb{R}} a(t)\left|w_n(t)\right|^2 dt + (4\sqrt{\theta M_1})^\zeta \int_{\mathbb{R}} a(t)\left|w_n(t)\right|^\zeta dt\right)$$

$$= d_1\big(16\theta M_1 + (4\sqrt{\theta M_1})^\zeta \|w_n\|_\infty^{\zeta-2}\big) \int_\mathbb{R} a(t)\big|w_n(t)\big|^2 dt$$

$$= d_1\big(16\theta M_1 + (4\sqrt{\theta M_1})^\zeta C_\infty^{\zeta-2}\big) \int_\mathbb{R} a(t)\big|w_n(t)\big|^2 dt \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$\int_\mathbb{R} a(t)G\big(4\sqrt{\theta M_1}w_n(t)\big) dt \to 0 \quad \text{as } n \to \infty. \tag{12}$$

By the definition of T_n and (12), for n large enough, we have

$$I(T_n u_n) \geq I\left(\frac{4\sqrt{\theta M_1}}{\|u_n\|}u_n\right)$$

$$= I(4\sqrt{\theta M_1}w_n)$$

$$= \frac{1}{2}\|4\sqrt{\theta M_1}w_n\|^2 - \int_\mathbb{R} a(t)G\big(4\sqrt{\theta M_1}w_n(t)\big) dt$$

$$= 8\theta M_1 - \int_\mathbb{R} a(t)G\big(4\sqrt{\theta M_1}w_n(t)\big) dt$$

$$\geq 4\theta M_1.$$

Then we obtain

$$\|T_n u_n\|^2 - \int_\mathbb{R} a(t)\big(\nabla G\big(T_n u_n(t)\big), T_n u_n(t)\big) dt$$

$$= \big\langle I'\big(T_n u_n(t)\big), T_n u_n(t)\big\rangle = T_n \frac{dI(Tu_n(t))}{dT}\bigg|_{T=T_n} = 0. \tag{13}$$

Hence, it follows from (13) and (W_1) that

$$\int_\mathbb{R} a(t)\left(\frac{1}{2}\big(\nabla G\big(u_n(t)\big), u_n(t)\big) - G\big(u_n(t)\big)\right) dt$$

$$= \frac{1}{2}\int_\mathbb{R} a(t)\widetilde{G}\big(u_n(t)\big) dt$$

$$\geq \frac{1}{2\theta}\int_\mathbb{R} a(t)\widetilde{G}\big(T_n u_n(t)\big) dt$$

$$= \frac{1}{\theta}\int_\mathbb{R}\left(\frac{1}{2}a(t)\big(\nabla G\big(T_n u_n(t)\big), T_n u_n(t)\big) - a(t)G\big(T_n u_n(t)\big)\right) dt$$

$$= \frac{1}{\theta}\left(\frac{1}{2}\|T_n u_n\|^2 - \int_\mathbb{R} a(t)G\big(T_n u_n(t)\big) dt\right)$$

$$= \frac{1}{\theta}I(T_n u_n),$$

which implies that

$$\int_\mathbb{R} a(t)\left(\frac{1}{2}\big(\nabla G\big(u_n(t)\big), u_n(t)\big) - G\big(u_n(t)\big)\right) dt \geq 4M_1 \quad \text{for } n \text{ large enough.} \tag{14}$$

However, we can deduce from (9) that

$$\left| \int_{\mathbb{R}} a(t) \left(\frac{1}{2} \left(\nabla G(u_n(t)), u_n(t) \right) - G(u_n(t)) \right) dt \right| = \left| 2I(u_n) - \langle I'(u_n), u_n \rangle \right| \leq 3M_1$$

for all $n \in \mathbb{N}$, which contradicts (14). Hence $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E, which yields

$$\langle I'(u_n) - I'(u), u_n - u \rangle \to 0 \quad \text{as } n \to \infty$$

and it follows from Lemma 2.1, Hölder's inequality, (6), and Lemma 2.2 that

$$\left| \int_{\mathbb{R}} a(t) \left(\nabla G(u_n(t)) - \nabla G(u(t)), u_n(t) - u(t) \right) dt \right| \leq \left\| \nabla G(u_n) - \nabla G(u) \right\|_{L_a^2} \|u_n - u\|_{L_a^2}$$

$$\to 0 \quad \text{as } n \to \infty.$$

Hence, we conclude that

$$\|u_n - u\|^2$$

$$= \langle I'(u_n) - I'(u), u_n - u \rangle + \int_{\mathbb{R}} a(t) \left(\nabla G(u_n(t)) - \nabla G(u(t)), u_n(t) - u(t) \right) dt$$

$$\to 0 \quad \text{as } n \to \infty.$$

We finish the proof. □

Subsequently, we need the following lemmas to prove Theorem 1.1.

Lemma 3.3 *Suppose that* (2), (L''), (L_1), (L_2), (W_1), *and* (W_3) *hold, then there exist constants* $\varrho, \alpha > 0$ *such that* $I|_{\partial B_\varrho(0)} \geq \alpha$.

Proof By (W_3), for any $\varepsilon > 0$, there is $\sigma > 0$ such that

$$\left| G(x) \right| \leq \varepsilon |x|^2 \tag{15}$$

for all $|x| \leq \sigma$. For $\varepsilon_1 = \frac{1}{4A}$, there exists $\delta_1 > 0$ such that (15) holds. Set

$$\varrho = C_\infty^{-1} \delta_1, \qquad \alpha = \frac{1}{4} \varrho^2.$$

By (7), (15), (2), (L_2), and (W_1), for any $\|u\| = \varrho$, we obtain

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} W(t, u(t)) \, dt$$

$$= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} a(t) G(u(t)) \, dt$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{1}{4A} \int_{\mathbb{R}} a(t) |u(t)|^2 \, dt$$

$$\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}} l(t)|u(t)|^2 \, dt$$

$$\geq \frac{1}{4}\|u\|^2. \tag{16}$$

By the definition of ϱ and α, (16) implies $I|_{\partial B_\varrho(0)} \geq \alpha$. □

Lemma 3.4 *Suppose that* (2), (L''), (W_2), *and* (W_4) *hold, then there exists* $\bar{e} \in E$ *such that* $\|\bar{e}\| > \varrho$ *and* $I(\bar{e}) \leq 0$, *where* ϱ *is defined in Lemma* 3.3.

Proof Set $e_0 \in C_0^\infty(-1,1)$ with $\|e_0\| = 1$. Let $a_2 = \min_{t \in B_1(0)} a(t)$ and $a_3 = \max_{t \in B_1(0)} a(t)$. For $\beta > \frac{1}{2a_2 \int_{-1}^{1} |e_0(t)|^2 \, dt}$, it follows from (W_4) that there exists $\xi > 0$ such that

$$G(x) \geq \beta |x|^2$$

for all $|x| > \xi$. By Lemma 3.1, we have

$$G(x) \geq \beta\left(|x|^2 - \xi^2\right) \tag{17}$$

for all $x \in \mathbb{R}^N$. By (2), (7), and (17), for every $\eta \in \mathbb{R} \setminus \{0\}$, we have

$$I(\eta e_0) = \frac{\eta^2}{2}\|e_0\|^2 - \int_{-1}^{1} W\left(t, \eta e_0(t)\right) dt$$

$$= \frac{\eta^2}{2}\|e_0\|^2 - \int_{-1}^{1} a(t)G\left(\eta e_0(t)\right) dt$$

$$\leq \frac{\eta^2}{2} - a_2\beta\eta^2 \int_{-1}^{1} |e_0(t)|^2 \, dt + 2a_3\beta\xi^2$$

$$= \left(\frac{1}{2} - a_2\beta \int_{-1}^{1} |e_0(t)|^2 \, dt\right)\eta^2 + 2a_3\beta\xi^2,$$

which implies that

$$I(\eta e_0) \to -\infty \quad \text{as } \eta \to +\infty.$$

Then there exists $\eta_0 \in \mathbb{R} \setminus \{0\}$ such that $\|\eta_0 e_0\| > \varrho$ and $I(\eta_0 e_0) < 0$. Letting $\bar{e}(t) = \eta_0 e_0(t)$, we finish the proof. □

Proof of Theorem 1.1 It is known that a deformation lemma can be proved when the usual (PS) condition is replaced with condition (C). From the above proofs and the mountain pass theorem (see [19], Theorem 2.2), I possesses a critical value $c \geq \alpha$ and a critical point u_0 such that $I(u_0) = c$, which means problem (1) has at least one nontrivial homoclinic solution. □

4 Proof of Theorem 1.3

Lemma 4.1 *Suppose that the conditions of Theorem* 1.2 *hold, then* I *satisfies the* (C) *condition.*

Proof Assuming that $\{u_n\} \subset E$ is a (C) sequence of I, there exists a constant $M_2 > 0$ such that

$$\left|I(u_n)\right| \leq M_2, \qquad \left\|I'(u_n)\right\|\left(1 + \|u_n\|\right) \leq M_2. \tag{18}$$

Now we show the boundedness of $\{u_n\}$. Arguing in an indirect way, we assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$, which implies that there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that $w_n \rightharpoonup w_0$ in E. Similar to the proof of Lemma 3.2, we have

$$\int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^2} \, dt \to \frac{1}{2} \quad \text{as } n \to \infty. \tag{19}$$

Case 1: $w_0 \not\equiv 0$. The proof is similar to the proof of *Case* 1 in Lemma 3.2.

Case 2: $w_0 \equiv 0$. Let $\varepsilon = 1$, then there exists $\sigma_0 > 0$ such that (15) holds for all $|x| \leq \sigma_0$. By (W_6), we obtain that for any $B > 0$, there exists $r_\infty > 0$ such that, for all $|x| \geq r_\infty$, we have

$$\frac{\widetilde{G}(x)}{G(x)} |x|^2 \geq B. \tag{20}$$

It follows from (7), (8), (20), (W_1), and (SQ) that

$$
\begin{aligned}
0 \leq \int_{\mathbb{R}} & \frac{W(t, u_n)}{\|u_n\|^2} \, dt \\
\leq & \int_{\{t \in \mathbb{R}||u_n|>r_\infty\}} \frac{W(t, u_n)}{|u_n(t)|^2} |w_n(t)|^2 \, dt + \int_{\{t \in \mathbb{R}||u_n|\leq\sigma_0\}} \frac{W(t, u_n)}{\|u_n\|^2} \, dt \\
& + \int_{\{t \in \mathbb{R}|\sigma_0\leq|u_n|\leq r_\infty\}} \frac{W(t, u_n)}{\|u_n\|^2} \, dt \\
\leq & \|w_n\|_{L^\infty}^2 \int_{\{t \in \mathbb{R}||u_n|>r_\infty\}} \frac{a(t)G(u_n)}{|u_n(t)|^2} \, dt + \int_{\{t \in \mathbb{R}||u_n|\leq\sigma_0\}} a(t)|w_n(t)|^2 \, dt \\
& + \int_{\{t \in \mathbb{R}|\sigma_0\leq|u_n|\leq r_\infty\}} \frac{a(t)G(u_n)|u_n(t)|^2}{\sigma_0^2 \|u_n\|^2} \, dt \\
\leq & \frac{\|w_n\|_{L^\infty}^2}{B} \int_{\{t \in \mathbb{R}||u_n|>r_\infty\}} a(t)\left((\nabla G(u_n), u_n) - 2G(u_n)\right) dt + \|w_n\|_{L_a^2}^2 \\
& + \frac{\max_{\sigma_0\leq|x|\leq\rho_\infty} |G(x)|}{\sigma_0^2} \int_{\{t \in \mathbb{R}|\sigma_0\leq|u_n|\leq r_\infty\}} a(t)|w_n(t)|^2 \, dt \\
\leq & \frac{\|w_n\|_{L^\infty}^2}{B} \int_{\{t \in \mathbb{R}||u_n|>r_\infty\}} a(t)\left((\nabla G(u_n), u_n) - 2G(u_n)\right) dt \\
& + \left(1 + \frac{\max_{\sigma_0\leq|x|\leq\rho_\infty} |G(x)|}{\sigma_0^2}\right) \|w_n\|_{L_a^2}^2 \\
\leq & \frac{\|w_n\|_{L^\infty}^2}{B} \left(2I(u_n) - \langle I'(u_n), u_n\rangle\right) + \left(1 + \frac{\max_{\sigma_0\leq|x|\leq\rho_\infty} |G(x)|}{\sigma_0^2}\right) \|w_n\|_{L_a^2}^2 \\
\leq & \frac{3M_2 C_\infty^2}{B} + \left(1 + \frac{\max_{\sigma_0\leq|x|\leq\rho_\infty} |G(x)|}{\sigma_0^2}\right) \|w_n\|_{L_a^2}^2.
\end{aligned}
$$

By the arbitrariness of B and Lemma 3.1, we have

$$\int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^2} \, dt < \frac{1}{4} \tag{21}$$

for n large enough, which contradicts (19). Hence $\|u_n\|$ is bounded in E. Similar to the proof of Lemma 3.2, we see that I satisfies the (C) condition. □

The following proof is similar to that of Theorem 1.1.

5 Proof of Theorems 1.2 and 1.4

As we know, Lemma 2.4 still holds when we replace the (PS) condition with the (C) condition. From Lemmas 3.2 and 4.1, we can see that I satisfies the (C) condition under the conditions of Theorems 1.2 and 1.4, respectively. Next, we only show that I satisfies the geometric conditions of Lemma 2.4. Obviously, Lemma 3.3 holds under the conditions of Theorems 1.2 and 1.4, respectively. If we take $V = \{0\}$ and $X = E$, we can see there are constants $\varrho_1, \alpha_1 > 0$ such that $I|_{\partial B_{\varrho_1} \cap X} \geq \alpha_1$ by Lemma 3.2, which means (A_1) is fulfilled. Subsequently, we show that I satisfies (A_2).

Lemma 5.1 *Suppose that* (2), (L''), (L_1), (L_2), (W_3), *and* (W_4) *hold, then I satisfies* (A_2).

Proof Let $\widetilde{E} \subset E$ be a finite dimensional subspace. For any $u \in \widetilde{E} \setminus \{0\}$ and $\vartheta > 0$, set

$$\Gamma_\vartheta(u) = \left\{ t \in \mathbb{R} : \left|u(t)\right| \geq \vartheta \|u\| \right\}.$$

Similar to Lemma 6.2 in [29], there exists $\vartheta_0 > 0$ such that

$$\mathrm{meas}\left(\Gamma_{\vartheta_0}(u)\right) \geq \vartheta_0 \tag{22}$$

for all $u \in \widetilde{E} \setminus \{0\}$. It is easy to see that there exists $\rho > 0$ such that $\mathrm{meas}\, \Upsilon_{\vartheta_0}(u) > \frac{1}{2}\vartheta_0$ for any $u \in \widetilde{E} \setminus \{0\}$, where $\Upsilon_{\vartheta_0}(u) = \Gamma_{\vartheta_0}(u) \bigcap B_\rho(0)$. Set $a_4 = \inf_{|t| \leq \rho} a(t) > 0$. By (W_4), there exists $\gamma > 0$ such that

$$G\left(u(t)\right) \geq \frac{1}{a_4 \vartheta_0^3} \left|u(t)\right|^2 \geq \frac{1}{a_4 \vartheta_0} \|u\|^2 \tag{23}$$

for all $u \in \widetilde{E}$ and $t \in \Upsilon_{\vartheta_0}(u)$ with $\|u\| \geq \gamma$. Then, for any $u \in \widetilde{E} \setminus B_\gamma$, it follows from (7), (W_1), (W_3), and (23) that

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} a(t)G\left(u(t)\right) dt \\
&= \frac{1}{2}\|u\|^2 - \int_{\Upsilon_{\vartheta_0}(u)} a(t)G\left(u(t)\right) dt - \int_{\mathbb{R} \setminus \Upsilon_{\vartheta_0}(u)} a(t)G\left(u(t)\right) dt \\
&\leq \frac{1}{2}\|u\|^2 - a_4 \int_{\Upsilon_{\vartheta_0}(u)} G\left(u(t)\right) dt \\
&\leq \frac{1}{2}\|u\|^2 - \frac{1}{\vartheta_0} \mathrm{meas}\left(\Upsilon_{\vartheta_0}(u)\right)\|u\|^2
\end{aligned}
$$

$$\leq \frac{1}{2}\|u\|^2 - \|u\|^2$$

$$= -\frac{1}{2}\|u\|^2.$$

Then there exists $r > \gamma$ such that $I|_{\widetilde{E}\setminus B_r} \leq 0$. □

Since I is even in u, I possesses an unbounded sequence of critical values by Lemma 2.4, which proves Theorems 1.2 and 1.4.

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