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Existence of solutions for several higher-order Hadamard-type fractional differential equations with integral boundary conditions on infinite interval

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Abstract

In this paper, we investigate the existence of solutions for several higher-order integral boundary value problems of Hadamard-type fractional differential equations on an infinite interval by using the monotone iterative technique and Mawhin's continuation theorem. The results enrich and extend some known conclusions of Hadamard-type fractional boundary value problems. Moreover, we give two concrete examples to illustrate the theoretical results.

MSC: 34A08; 34B15

Keywords: Hadamard fractional derivative; Fractional differential equation; Integral boundary value problem; Infinite interval

1 Introduction

In recent years, the study of fractional differential equations (FDEs for short) has been an interesting and popular field of research as it plays an important role in many areas such as control theory, electrical circuits, biology, physics, diffusion processes, finance, etc. (see [1–8]). For example, the simplified financial model can be described by FDEs as the forms:

$$\begin{cases} {}_0D_t^{q_1} x(t) = z(t) + (y(t) - a)x(t), & 0 < q_1 \le 1, \\ {}_0D_t^{q_2} y(t) = 1 - by(t) - x^2(t), & 0 < q_2 \le 1, \\ {}_0D_t^{q_3} z(t) = -x(t) - cz(t), & 0 < q_3 \le 1, \end{cases}$$

where ${}_0D_t^{(\cdot)}$ is the Caputo fractional derivative of fractional order, a, b, c are three nonnegative constants denoting the saving amount, cost per investment, and the elasticity of demand of commercial market, respectively, the state variables $x(t)$, $y(t)$, $z(t)$ represent the interest rate, investment demand, and the price index, respectively (see [2]).

As is well known, one of the interesting and important features of discussing FDEs is focused on the research of the existence solutions for nonlinear fractional initial value problems and fractional boundary value problems (BVPs for short). Some recent work can be found in [9–33] and the references therein. It is worth mentioning that the study of the Hadamard-type fractional BVPs has attracted many scholars' attention over the past four years. Hadamard-type fractional calculus was introduced by Hadamard in 1892 (see [34]). The definition of this kind of fractional derivative contains logarithmic function of arbitrary exponent in the kernel of the integral, which is different from the fractional derivatives of Riemann–Liouville and Caputo types. "Hadamard's construction is invariant in relation to dilation and is well suited to the problems containing half axes" (see [23]). Moreover, some classical methods and theories, such as fixed point theorems, coincidence degree theory, and monotone iterative technique, have been widely used to investigate Hadamard-type fractional BVPs (see [16–33]).

In [16], Benchohra, Bouriah, and Nieto investigated the following Hadamard-type FDE with periodic condition:

$$\begin{cases} {}^{H}D^{\alpha}y(t) = f(t, y(t), {}^{H}D^{\alpha}y(t)), & 0 < \alpha \leq 1, t \in [1, T], \\ y(1) = y(T), \end{cases}$$

where $T > 1$, ${}^{H}D^{\alpha}$ is the Hadamard-type fractional derivative of order α. The authors obtained the existence of solutions by means of coincidence degree theory.

In [17], Ahmad and Ntouyas discussed the following coupled Hadamard-type FDEs with Hadamard-type integral boundary conditions:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), & 1 < \alpha \leq 2, 1 < t < e, \\ D^{\beta}v(t) = g(t, u(t), v(t)), & 1 < \beta \leq 2, 1 < t < e, \\ u(1) = 0, \qquad u(e) = \frac{1}{\Gamma(\gamma)} \int_{1}^{\sigma_1} (\ln \frac{\sigma_1}{s})^{\gamma-1} \frac{u(s)}{s}\, ds, \\ v(1) = 0, \qquad v(e) = \frac{1}{\Gamma(\gamma)} \int_{1}^{\sigma_2} (\ln \frac{\sigma_2}{s})^{\gamma-1} \frac{v(s)}{s}\, ds, \end{cases}$$

where $\gamma > 0$, $1 < \sigma_1 < e$, $1 < \sigma_2 < e$, $D^{(\cdot)}$ is the Hadamard-type fractional derivative of fractional order. By using Leray–Schauder's alternative and Banach's contraction principle, the authors obtained the existence and uniqueness of solutions, respectively.

In [18], Pei, Wang, and Sun considered the following Hadamard-type fractional integro-differential equations on infinite domain:

$$\begin{cases} {}^{H}D^{\alpha}u(t) + f(t, u(t), {}^{H}I^{r}u(t), {}^{H}D^{\alpha-1}u(t)) = 0, & 1 < \alpha < 2, t \in (1, +\infty), \\ u(1) = 0, \qquad {}^{H}D^{\alpha-1}u(\infty) = \sum_{i=1}^{m} \lambda_i {}^{H}I^{\beta_i}u(\eta), \end{cases}$$

where $\eta \in (1, \infty)$, $r, \beta_i, \lambda_i \geq 0$ $(i = 1, 2, \ldots, m)$ are given constants, ${}^{H}D^{\alpha}$ is the Hadamard-type fractional derivative of order α, and ${}^{H}I^{(\cdot)}$ is the Hadamard-type fractional integral. By employing the monotone iterative technique, the existence result on positive solutions was obtained.

Motivated especially by the aforementioned work, we are concerned in this paper with the existence of solutions for two types of Hadamard-type fractional integral BVP on an infinite interval. First, by applying the monotone iterative method, we investigate the following FDE with conjugate type integral conditions on an infinite interval:

$$\begin{cases} {}^{H}D_{1+}^{\alpha}x(t) + a(t)f(t, x(t)) = 0, & n-1 < \alpha \leq n, t \in (1, +\infty), \\ x^{(m)}(1) = 0, \qquad {}^{H}D_{1+}^{\alpha-1}x(+\infty) = \int_{1}^{+\infty} g(t)x(t)\frac{dt}{t}, & m = 0, 1, \ldots, n-2, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}$, $n \geq 3$, ${}^{H}D^{\alpha}_{1^+}$ is the Hadamard-type fractional derivative of order α, $g(t) \geq 0$ satisfies $\Gamma(\alpha) - \int_1^{+\infty} g(t)(\ln t)^{\alpha-1}\frac{dt}{t} := \kappa > 0$. We assume that the following conditions hold:

(A_1) $f \in C([1,+\infty) \times [0,+\infty), [0,+\infty))$, $f(t,0) \not\equiv 0$ on any subinterval of $[1,+\infty)$ and $f(t,(1+(\ln t)^{\alpha-1})x)$ is bounded on $[1,+\infty)$ when x is bounded;

(A_2) $a(t) : [1,+\infty) \to [0,+\infty)$ is not identically zero on any subinterval of $[1,+\infty)$ and

$$0 < \int_1^{+\infty} a(t)\frac{dt}{t} < +\infty.$$

Second, we also study the existence of solutions for the following Hadamard-type FDE with integral boundary condition on an infinite interval at resonance by means of Mawhin's continuation theorem:

$$\begin{cases} {}^{H}D^{\alpha}_{1^+}x(t) + a(t)f(t,x(t),{}^{H}D^{\alpha-2}_{1^+}x(t),{}^{H}D^{\alpha-1}_{1^+}x(t)) = 0, & t \in (1,+\infty), \\ x(1) = x'(1) = 0, \qquad {}^{H}D^{\alpha-1}_{1^+}x(+\infty) = \int_1^{+\infty} g(t){}^{H}D^{\alpha-1}_{1^+}x(t)\frac{dt}{t}, \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$, ${}^{H}D^{\alpha}_{1^+}$ is the Hadamard-type fractional derivative of order α, $g(t) \geq 0$, $(1/a(t)) > 0$ on $[1,+\infty)$, $f : [1,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a-Carathéodory condition, that is, f satisfies the following three conditions:

- For each $(u,v,w) \in \mathbb{R}^3$, the mapping $t \mapsto f(t,u,v,w)$ is Lebesgue measurable;
- For a.e. $t \in [1,+\infty)$, the mapping $(u,v,w) \mapsto f(t,u,v,w)$ is continuous on $\mathbb{R}^3$;
- For each $l > 0$, there exists a function $\varphi_l : [1,+\infty) \to [0,+\infty)$ satisfying $\int_1^{+\infty} a(t)\varphi_l(t)\frac{dt}{t} < +\infty$ such that

$$\bigl|f(t,u,v,w)\bigr| \leq \varphi_l(t), \quad \text{a.e. } t \in [1,+\infty), \qquad \max\left\{\frac{|u|}{1+(\ln t)^{\alpha-1}}, \frac{|v|}{1+\ln t}, |w|\right\} \leq l.$$

And we also assume that the following condition holds:

(H_1) $\int_1^{+\infty} g(t)\frac{dt}{t} = 1$, $\int_1^{+\infty} a(t)\frac{dt}{t} < +\infty$.

In general, a boundary value problem is called resonance if the corresponding homogeneous BVP has a nontrivial solution. According to condition (H_1), consider the homogeneous BVP of (1.2) as follows:

$$\begin{cases} -\frac{1}{a(t)}{}^{H}D^{\alpha}_{1^+}x(t) = 0, & t \in (1,+\infty), \\ x(1) = x'(1) = 0, \qquad {}^{H}D^{\alpha-1}_{1^+}x(+\infty) = \int_1^{+\infty} g(t){}^{H}D^{\alpha-1}_{1^+}x(t)\frac{dt}{t}. \end{cases} \tag{1.3}$$

By using Lemma 2.2 (see the next section), we can check that BVP (1.3) has a nontrivial solution $x(t) = c(\ln t)^{\alpha-1}$, $c \in \mathbb{R}$. So, BVP (1.2) is a resonance problem.

In the present work, we are focused on establishing the existence theorems to deal with two types of Hadamard-type fractional BVPs on an infinite interval. The new features of this paper can be presented as follows. On the one hand, as far as we know, compared with the fractional BVPs on a finite interval, the BVPs on an infinite interval of FDEs have little been considered until now because the infinite interval lacks compactness. Thus, our paper enriches some existing results. On the other hand, most of the recent papers on Hadamard-type fractional BVPs discuss the non-resonance problems. In our work, we not only study the non-resonance problem but also consider the resonance problem. The main difficulties in this article are as follows. First, we have to construct suitable Banach spaces for problem (1.1) and (1.2). Second, we should give a new compactness judgment theorem. Third, the estimates on a priori bounds are more complicated.

The rest of this paper is organized as follows. In Sect. 2, we recall some preliminary definitions and lemmas. In Sect. 3, based on the monotone iterative method, we establish a theorem on the existence of positive solutions for problem (1.1). In Sect. 4, by using Mawhin's continuation theorem, we give an existence theorem for problem (1.2). Finally, the paper is concluded in Sect. 5.

2 Preliminaries

In this section, we recall some definitions and lemmas which are used throughout this paper. First, we present here the basic knowledge about the Hadamard-type fractional calculus. For more details, we refer the readers to [1, 28].

Definition 2.1 The Hadamard-type fractional integral of order $\alpha > 0$ of a function $f : [1, +\infty) \to \mathbb{R}$:

$$ {}^H I^\alpha_{1+} f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha - 1} f(s) \frac{ds}{s}, \quad (t > 1), $$

provided the integral exists.

Definition 2.2 The Hadamard-type fractional derivative of order $\alpha > 0$ of a function $f : [1, +\infty) \to \mathbb{R}$:

$$ {}^H D^\alpha_{1+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\ln \frac{t}{s} \right)^{n-\alpha-1} f(s) \frac{ds}{s}, \quad (t > 1), $$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α.

Lemma 2.1 *If* $\alpha, \beta > 0$, *then*

$$ {}^H I^\alpha_{1+} (\ln t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\ln t)^{\alpha+\beta-1}, \qquad {}^H D^\alpha_{1+} (\ln t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln t)^{\beta-\alpha-1}, $$

in particular, ${}^H D^\alpha_{1+} (\ln t)^{\alpha-j} = 0$, $j = 1, 2, \ldots, [\alpha] + 1$.

Lemma 2.2 *Let* $\alpha > 0$. *Assume that* $x \in C[1,\infty) \cap L^1[1,\infty)$, *then the solution of Hadamard-type fractional differential equation* ${}^H D^\alpha_{1+} x(t) = 0$ *can be denoted as*

$$ x(t) = \sum_{i=1}^n c_i (\ln t)^{\alpha-i}, $$

and the following formula holds:

$$ {}^H I^\alpha_{1+} {}^H D^\alpha_{1+} x(t) = x(t) + \sum_{i=1}^n c_i (\ln t)^{\alpha-i}, $$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n - 1 < \alpha < n$.

Next, we recall the results of coincidence degree theory due to Mawhin which can be found in [35, 36].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real Banach spaces. Define $L : \operatorname{dom} L \subset X \to Y$ to be a Fredholm operator with index zero, then there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q, \qquad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q,$$

and $L|_{\operatorname{dom} L\cap \operatorname{Ker} P} : \operatorname{dom} L \to \operatorname{Im} L$ is invertible. We denote its inverse by K_p. Let Ω be an open bounded subset of X and $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$. The map $N : X \to Y$ is called L-compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N = K_p(I - Q)N : \bar{\Omega} \to X$ is compact.

Theorem 2.1 *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied*:

(i) *$Lu \neq \lambda Nu$ for any $u \in (\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega$, $\lambda \in (0, 1)$;*
(ii) *$Nu \notin \operatorname{Im} L$ for any $u \in \operatorname{Ker} L \cap \partial\Omega$;*
(iii) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

3 The main result of (1.1)

Let

$$E = \left\{ x \in C\big([1, +\infty), \mathbb{R}\big) : \sup_{t\in[1,+\infty)} \frac{|x(t)|}{1 + (\ln t)^{\alpha-1}} < +\infty \right\},$$

endowed with the norm

$$\|x\|_E = \sup_{t\in[1,+\infty)} \frac{|x(t)|}{1 + (\ln t)^{\alpha-1}},$$

then $(E, \|\cdot\|_E)$ is a Banach space.

Lemma 3.1 *Suppose that $\int_1^{+\infty} g(t)(\ln t)^{\alpha-1} \frac{dt}{t} < \Gamma(\alpha)$. Then, for any $y \in C[1, +\infty)$ with $\int_1^{+\infty} y(s)\frac{ds}{s} < +\infty$, the unique solution of the following BVP*

$$\begin{cases} {}^H D^{\alpha}_{1^+} x(t) + y(t) = 0, \quad n-1 < \alpha \le n, t \in (1, +\infty), \\ x^{(m)}(1) = 0, \qquad {}^H D^{\alpha-1}_{1^+} x(+\infty) = \int_1^{+\infty} g(t)x(t)\frac{dt}{t}, \quad m = 0, 1, \dots, n-2, \end{cases} \tag{3.1}$$

can be given by

$$x(t) = \int_1^{+\infty} G(t, s)y(s)\frac{ds}{s}, \quad t \in [1, +\infty),$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \tag{3.2}$$

and

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\ln t)^{\alpha-1} - (\ln \frac{t}{s})^{\alpha-1}, & 1 \le s \le t < +\infty, \\ (\ln t)^{\alpha-1}, & 1 \le t \le s < +\infty, \end{cases}$$

$$G_2(t,s) = \frac{(\ln t)^{\alpha-1}}{\kappa} \int_1^{+\infty} g(t) G_1(t,s) \frac{dt}{t}.$$

Proof According to Lemma 2.2, the solution of (3.1) is

$$x(t) = c_1 (\ln t)^{\alpha-1} + c_2 (\ln t)^{\alpha-2} + \cdots + c_n (\ln t)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s},$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Considering the boundary conditions $x^{(m)}(1) = 0$, $m = 0, 1, \ldots, n-2$, we obtain $c_2 = c_3 = \cdots = c_n = 0$, that is,

$$x(t) = c_1 (\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}.$$

By Lemma 2.1, one has

$${}^H D_{1^+}^{\alpha-1} x(t) = c_1 \Gamma(\alpha) - \int_1^t y(s) \frac{ds}{s},$$

which shows

$${}^H D_{1^+}^{\alpha-1} x(+\infty) = c_1 \Gamma(\alpha) - \int_1^{+\infty} y(s) \frac{ds}{s}.$$

Combining the boundary condition ${}^H D_{1^+}^{\alpha-1} x(+\infty) = \int_1^{+\infty} g(t) x(t) \frac{dt}{t}$, we have

$$c_1 = \frac{1}{\Gamma(\alpha)} \left(\int_1^{+\infty} g(t) x(t) \frac{dt}{t} + \int_1^{+\infty} y(s) \frac{ds}{s} \right).$$

Therefore,

$$x(t) = \int_1^{+\infty} G_1(t,s) y(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \int_1^{+\infty} g(t) x(t) \frac{dt}{t}, \tag{3.3}$$

and then

$$\begin{aligned} \int_1^{+\infty} g(t) x(t) \frac{dt}{t} = {} & \int_1^{+\infty} g(t) \int_1^{+\infty} G_1(t,s) y(s) \frac{ds}{s} \frac{dt}{t} \\ & + \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} g(t) (\ln t)^{\alpha-1} \frac{dt}{t} \int_1^{+\infty} g(t) x(t) \frac{dt}{t}, \end{aligned}$$

which implies

$$\int_1^{+\infty} g(t) x(t) \frac{dt}{t} = \frac{\Gamma(\alpha)}{\kappa} \int_1^{+\infty} g(t) \int_1^{+\infty} G_1(t,s) y(s) \frac{ds}{s} \frac{dt}{t}. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$x(t) = \int_1^{+\infty} G_1(t,s)y(s)\frac{ds}{s} + \int_1^{+\infty} G_2(t,s)y(s)\frac{ds}{s} = \int_1^{+\infty} G(t,s)y(s)\frac{ds}{s}.$$

The proof is completed. □

Lemma 3.2 *The Green's function $G(t,s)$ defined by (3.2) satisfies the following properties*:

(i) *$G(t,s)$ is a continuous function for $(t,s) \in [1,+\infty) \times [1,+\infty)$*;

(ii) *$G(t,s)$ is nonnegative on $[1,+\infty) \times [1,+\infty)$*;

(iii) *$\frac{G(t,s)}{1+(\ln t)^{\alpha-1}} \leq \frac{1}{\kappa}$ for all $(t,s) \in [1,+\infty) \times [1,+\infty)$.*

Proof Easily, we can check that (i) and (ii) hold. To prove (iii), for $(t,s) \in [1,+\infty) \times [1,+\infty)$, it is clear that the following inequalities hold:

$$\frac{G_1(t,s)}{1+(\ln t)^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)},$$
$$\frac{G_2(t,s)}{1+(\ln t)^{\alpha-1}} \leq \frac{\int_1^{+\infty} G_1(t,s)g(t)\frac{dt}{t}}{\kappa} \leq \frac{\int_1^{+\infty} g(t)(\ln t)^{\alpha-1}\frac{dt}{t}}{\Gamma(\alpha)\kappa}.$$

Thus,

$$\frac{G(t,s)}{1+(\ln t)^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)} + \frac{\int_1^{+\infty} g(t)(\ln t)^{\alpha-1}\frac{dt}{t}}{\Gamma(\alpha)\kappa} = \frac{1}{\kappa}.$$

The proof is completed. □

Lemma 3.3 (see [25]) *Let $V = \{x \in E : \|x\|_E \leq r, r > 0\} \subset E$ be relatively compact in E if the following conditions hold*:

(i) *For any $x(t) \in V$, $\frac{x(t)}{1+(\ln t)^{\alpha-1}}$ is equicontinuous on any compact interval of $[1,+\infty)$*;

(ii) *For any $\varepsilon > 0$, there exists a constant $R = R(\varepsilon) > 0$ such that, for all $x(t) \in V$, $t_1, t_2 \geq R$, it holds*

$$\left|\frac{x(t_1)}{1+(\ln t_1)^{\alpha-1}} - \frac{x(t_2)}{1+(\ln t_2)^{\alpha-1}}\right| < \varepsilon.$$

Let

$$P = \{x \in E : x(t) \geq 0, t \in [1,+\infty)\}.$$

Obviously, $P \subset E$ is a cone. Define the operator $T : P \to E$ as follows:

$$Tx(t) = \int_1^{+\infty} G(t,s)a(s)f(s,x(s))\frac{ds}{s}, \quad t \in [1,+\infty).$$

Lemma 3.4 *Assume that* (A_1) *and* (A_2) *hold. Then $T : P \to P$ is completely continuous.*

Proof For any $x \in P$, it is obvious that $Tx(t) \geq 0$, i.e., $T : P \to P$. Take $\{x_n\}_{n=1}^{+\infty} \subset P$, $x \in P$, such that $x_n \to x$ as $n \to +\infty$, then there exists a constant $r_0 > 0$ such that $\sup_{n\in\mathbb{N}} \|x_n\|_E <$

r_0. Set $B_{r_0} = \sup\{f(t,(1+(\ln t)^{\alpha-1})x) : (t,x) \in [1,+\infty) \times [0,r_0]\}$. By ($A_1$) and ($A_2$), one has

$$\int_1^{+\infty} a(s)f\big(s,x(s)\big)\frac{ds}{s} \le B_{r_0}\int_1^{+\infty} a(s)\frac{ds}{s} < +\infty.$$

It follows from Lebesgue's dominated convergence and the continuity of $f(t,x(t))$ that

$$\int_1^{+\infty} a(s)f\big(s,x_n(s)\big)\frac{ds}{s} \to \int_1^{+\infty} a(s)f\big(s,x(s)\big)\frac{ds}{s}, \quad \text{as } n \to +\infty.$$

Thus,

$$\begin{aligned}\|Tx_n - Tx\|_E &= \sup_{t\in[1,+\infty)} \frac{|Tx_n - Tx|}{1+(\ln t)^{\alpha-1}} \\ &\le \frac{1}{\kappa}\left|\int_1^{+\infty} a(s)f\big(s,x_n(s)\big)\frac{ds}{s} - \int_1^{+\infty} a(s)f\big(s,x(s)\big)\frac{ds}{s}\right| \\ &\to 0, \quad \text{as } n \to +\infty,\end{aligned}$$

which shows $T : P \to P$ is continuous. In the following, we let Ω be any bounded subset of P and separate the proof into three steps to prove T is a compact operator. For simplicity of presentation, we let

$$\begin{aligned}&\sigma(\tau,t,s) = \frac{(\ln(t/s))^{\alpha-1}}{1+(\ln\tau)^{\alpha-1}}, \quad 1 \le s \le t < +\infty, 1 \le \tau < +\infty, \qquad \omega =: \int_1^{+\infty} a(t)\frac{dt}{t}, \\ &\delta(\tau,t) = \frac{(\ln t)^{\alpha-1}}{1+(\ln\tau)^{\alpha-1}}, \quad 1 \le \tau, t < +\infty, \qquad \varpi =: \frac{\int_1^{+\infty} g(t)(\ln t)^{\alpha-1}\frac{dt}{t}}{\Gamma(\alpha)(\Gamma(\alpha) - \int_1^{+\infty} g(t)(\ln t)^{\alpha-1}\frac{dt}{t})}, \\ &\rho_i(\tau,t,s) = \begin{cases} \frac{G(t,s)}{1+(\ln\tau)^{\alpha-1}}, & 1 \le \tau,t,s < +\infty, i = 0, \\ \frac{G_i(t,s)}{1+(\ln\tau)^{\alpha-1}}, & 1 \le \tau,t,s < +\infty, i = 1,2.\end{cases}\end{aligned}$$

Step 1. T is uniformly bounded on $\bar{\Omega}$. In fact, there exists a constant $r > 0$ such that $\|x\|_E \le r$ for any $x \in \bar{\Omega}$. Set $B_r = \sup\{f(t,(1+(\ln t)^{\alpha-1})x) : (t,x) \in [1,+\infty) \times [0,r]\}$. Then we have

$$\begin{aligned}\|Tx\|_E &= \sup_{t\in[1,+\infty)} \frac{1}{1+(\ln t)^{\alpha-1}}\left|\int_1^{+\infty} G(t,s)a(s)f\big(s,x(s)\big)\frac{ds}{s}\right| \\ &\le \frac{B_r\omega}{\kappa} < +\infty.\end{aligned}$$

Step 2. For any $x \in \bar{\Omega}$, Tx is equicontinuous on any compact intervals of $[1,+\infty)$. In fact, for any $x \in \bar{\Omega}$, $L > 1$, and $t_1, t_2 \in [1,L]$ with $t_1 < t_2$, one has

$$\begin{aligned}&\left|\frac{Tx(t_2)}{1+(\ln t_2)^{\alpha-1}} - \frac{Tx(t_1)}{1+(\ln t_1)^{\alpha-1}}\right| \\ &\quad = \left|\int_1^{+\infty} \rho_0(t_2,t_2,s)a(s)f\big(s,x(s)\big)\frac{ds}{s} - \int_1^{+\infty} \rho_0(t_1,t_1,s)a(s)f\big(s,x(s)\big)\frac{ds}{s}\right| \\ &\quad \le \int_1^{+\infty} \big|\rho_1(t_2,t_2,s) - \rho_1(t_1,t_1,s)\big|a(s)f\big(s,x(s)\big)\frac{ds}{s}\end{aligned}$$

$$
\begin{aligned}
&+\int_1^{+\infty}\left|\rho_2(t_2,t_2,s)-\rho_2(t_1,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
\le{}&\int_1^{+\infty}\left|\rho_1(t_2,t_2,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
&+\int_1^{+\infty}\left|\rho_1(t_1,t_1,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
&+\varpi\left|\delta(t_2,t_2)-\delta(t_1,t_1)\right|\int_1^{+\infty}a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}.
\end{aligned}
$$

Since the functions $\delta(\tau,t)$, $\sigma(\tau,t,s)$ are uniformly continuous on $[t_1,t_2]\times[t_1,t_2]$ and $[t_1,t_2]\times[t_1,t_2]\times[1,t_1]$, respectively, we have

$$
\begin{aligned}
&\int_1^{+\infty}\left|\rho_1(t_2,t_2,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
&\quad\le\int_1^{t_1}\left|\rho_1(t_2,t_2,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
&\qquad+\int_{t_1}^{t_2}\left|\rho_1(t_2,t_2,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
&\qquad+\int_{t_2}^{+\infty}\left|\rho_1(t_2,t_2,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\
&\quad\le\frac{B_r}{\Gamma(\alpha)}\int_1^{t_1}\bigl[\sigma(t_2,t_2,s)+\delta(t_2,t_2)-\sigma(t_2,t_1,s)-\delta(t_2,t_1)\bigr]a(s)\frac{ds}{s}\\
&\qquad+\frac{B_r}{\Gamma(\alpha)}\int_{t_1}^{t_2}\bigl[\sigma(t_2,t_2,s)+\delta(t_2,t_2)-\delta(t_2,t_1)\bigr]a(s)\frac{ds}{s}\\
&\qquad+\frac{B_r}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\bigl[\delta(t_2,t_2)-\delta(t_2,t_1)\bigr]a(s)\frac{ds}{s}\\
&\quad\to 0,\quad \text{as } t_1\to t_2,
\end{aligned}
\tag{3.5}
$$

and

$$
\varpi\left|\delta(t_2,t_2)-\delta(t_1,t_1)\right|\int_1^{+\infty}a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\to 0,\quad \text{as } t_1\to t_2.
\tag{3.6}
$$

Similarly, we can obtain

$$
\int_1^{+\infty}\left|\rho_1(t_1,t_1,s)-\rho_1(t_2,t_1,s)\right|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\to 0,\quad \text{as } t_1\to t_2.
\tag{3.7}
$$

Thus, from (3.5)–(3.7), we have

$$
\left|\frac{Tx(t_2)}{1+(\ln t_2)^{\alpha-1}}-\frac{Tx(t_1)}{1+(\ln t_1)^{\alpha-1}}\right|\to 0,\quad \text{as } t_1\to t_2.
$$

Step 3. For any $x\in\bar{\Omega}$, Tx is equiconvergent at infinity. In fact, for any $\varepsilon>0$, by (A_2), there exists a constant $\ell>1$ such that

$$
0<\int_\ell^{+\infty}a(s)\frac{ds}{s}<\varepsilon.
\tag{3.8}
$$

Because $\lim_{t\to+\infty}\sigma(t,t,\ell)=1$, $\lim_{t\to+\infty}\delta(t,t)=1$, then for above $\varepsilon>0$, there exist constants $\ell_1>1$, $\ell_2>\ell>1$ such that for any $t_1,t_2>\ell_1$ one has

$$\bigl|\delta(t_2,t_2)-\delta(t_1,t_1)\bigr|\le\bigl|1-\delta(t_2,t_2)\bigr|+\bigl|1-\delta(t_1,t_1)\bigr|<\varepsilon, \tag{3.9}$$

and for any $t_1,t_2>\ell_2$, $1\le s\le\ell$ one gets

$$\begin{aligned}\bigl|\sigma(t_2,t_2,s)-\sigma(t_1,t_1,s)\bigr| &\le\bigl|1-\sigma(t_2,t_2,s)\bigr|+\bigl|1-\sigma(t_1,t_1,s)\bigr|\\ &\le\bigl|1-\sigma(t_2,t_2,\ell)\bigr|+\bigl|1-\sigma(t_1,t_1,\ell)\bigr|<\varepsilon.\end{aligned} \tag{3.10}$$

We choose $\tilde\ell>\max\{\ell_1,\ell_2\}$, Then, for any $x\in\bar\Omega$, $t_2,t_1>\tilde\ell$ (without loss of generality we assume that $t_2>t_1$), we have

$$\begin{aligned}&\left|\frac{Tx(t_2)}{1+(\ln t_2)^{\alpha-1}}-\frac{Tx(t_1)}{1+(\ln t_1)^{\alpha-1}}\right|\\ &\quad=\left|\int_1^{+\infty}\rho_0(t_2,t_2,s)a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}-\int_1^{+\infty}\rho_0(t_1,t_1,s)a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\right|\\ &\quad\le\int_1^{+\infty}\bigl|\rho_1(t_2,t_2,s)-\rho_1(t_1,t_1,s)\bigr|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\ &\qquad+\varpi\bigl|\delta(t_2,t_2)-\delta(t_1,t_1)\bigr|\int_1^{+\infty}a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}.\end{aligned}$$

It follows from (3.8)–(3.10) that

$$\begin{aligned}&\int_1^{t_1}\bigl|\sigma(t_2,t_2,s)-\sigma(t_1,t_1,s)\bigr|a(s)\frac{ds}{s}\\ &\quad=\int_1^{\ell}\bigl|\sigma(t_2,t_2,s)-\sigma(t_1,t_1,s)\bigr|a(s)\frac{ds}{s}+\int_\ell^{t_1}\bigl|\sigma(t_2,t_2,s)-\sigma(t_1,t_1,s)\bigr|a(s)\frac{ds}{s}\\ &\quad\le\omega\varepsilon+2\int_\ell^{+\infty}a(s)\frac{ds}{s}=(\omega+2)\varepsilon,\\ &\int_1^{+\infty}\bigl|\rho_1(t_2,t_2,s)-\rho_1(t_1,t_1,s)\bigr|a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\\ &\quad\le\frac{B_r}{\Gamma(\alpha)}\int_1^{t_1}\bigl|\delta(t_2,t_2)-\sigma(t_2,t_2,s)-\delta(t_1,t_1)+\sigma(t_1,t_1,s)\bigr|a(s)\frac{ds}{s}\\ &\qquad+\frac{B_r}{\Gamma(\alpha)}\int_{t_1}^{t_2}\bigl|\delta(t_2,t_2)-\sigma(t_2,t_2,s)-\delta(t_1,t_1)\bigr|a(s)\frac{ds}{s}\\ &\qquad+\frac{B_r}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\bigl|\delta(t_2,t_2)-\delta(t_1,t_1)\bigr|a(s)\frac{ds}{s}\\ &\quad\le\frac{B_r\varepsilon}{\Gamma(\alpha)}(2\omega+2\varepsilon+3),\end{aligned} \tag{3.11}$$

and

$$\varpi\bigl|\delta(t_2,t_2)-\delta(t_1,t_1)\bigr|\int_1^{+\infty}a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\le B_r\omega\varpi\varepsilon. \tag{3.12}$$

By (3.11) and (3.12), for any $\varepsilon > 0$, there exists a sufficiently large number $R = R(\varepsilon) > 0$ such that, for any $x \in \bar{\Omega}$, $t_1, t_2 > R$,

$$\left| \frac{Tx(t_2)}{1 + (\ln t_2)^{\alpha-1}} - \frac{Tx(t_1)}{1 + (\ln t_1)^{\alpha-1}} \right| < \varepsilon.$$

Applying Lemma 3.3, $T : P \to P$ is completely continuous. □

Theorem 3.1 *Assume that* (A_1)–(A_2) *and the following conditions hold*:

(A_3) $f(t,x)$ *is continuous and nondecreasing on* x, $x \in P$;

(A_4) $f(t, (1 + (\ln t)^{\alpha-1})x) \leq \frac{\kappa a}{\omega}$ *for all* $(t,x) \in [1,+\infty) \times [0,a]$,

where a *is a positive constant. Then BVP* (1.1) *has the maximal positive solutions* x^* *and minimal positive solutions* y^* *in* $(0, a(\ln t)^{\alpha-1}]$, *which can be obtained by the following two iterative sequences*:

$$x_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, x_n(s)\bigr)\frac{ds}{s}, \quad t \in [1,+\infty), n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, y_n(s)\bigr)\frac{ds}{s}, \quad t \in [1,+\infty), n = 0,1,2,\ldots,$$

respectively, with the initial values $x_0(t) = a(\ln t)^{\alpha-1}$, $y_0(t) = 0$, $t \in [1,+\infty)$, *and they satisfy*

$$y_0 \leq y_1 \leq \cdots \leq y_n \leq \cdots \leq y^* \leq \cdots \leq x^* \leq \cdots \leq x_n \leq \cdots \leq x_1 \leq x_0.$$

Proof By Lemma 3.4, $T : P \to P$ is completely continuous. For any $x_1, x_2 \in P$ with $x_1 \leq x_2$, by condition (A_3) and the definition of T, we can see that $Tx_1 \leq Tx_2$. Set

$$\bar{P}_a = \bigl\{x \in P : \|x\|_E \leq a\bigr\}.$$

Then $T : \bar{P}_a \to \bar{P}_a$. In fact, for any $x \in \bar{P}_a$, then $\|x\|_E \leq a$, by (A_4), we have

$$f(t,x) = f\left(t, \bigl(1 + (\ln t)^{\alpha-1}\bigr)\frac{x(t)}{1 + (\ln t)^{\alpha-1}}\right) \leq \frac{\kappa a}{\omega}.$$

Thus,

$$\begin{aligned}\|Tx\|_E &= \sup_{t\in[1,+\infty)} \frac{1}{1 + (\ln t)^{\alpha-1}}\left|\int_1^{+\infty} G(t,s)a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s}\right| \\ &\leq \sup_{t\in[1,+\infty)} \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)(1 + (\ln t)^{\alpha-1})}\int_1^{+\infty} a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s} \\ &\quad + \sup_{t\in[1,+\infty)} \frac{(\ln t)^{\alpha-1}\varpi}{1 + (\ln t)^{\alpha-1}}\int_1^{+\infty} a(s)f\bigl(s,x(s)\bigr)\frac{ds}{s} \\ &\leq \frac{1}{\kappa}\int_1^{+\infty} a(s)\frac{ds}{s}\cdot\frac{\kappa a}{\omega} = a,\end{aligned}$$

which implies $T : \bar{P}_a \to \bar{P}_a$. Let $x_0(t) = a(\ln t)^{\alpha-1}$, $t \in [1,+\infty)$, then $x_0(t) \in \bar{P}_a$. Define the iterative sequence as follows:

$$x_{n+1}(t) = Tx_n(t), \quad t \in [1,+\infty), n = 0,1,2,\ldots.$$

Since $T:\bar{P}_a \to \bar{P}_a$ and T is completely continuous, we can derive

$$x_{n+1}(t) = Tx_n(t) \in \bar{P}_a, \quad t \in [1,+\infty), n = 0,1,2,\ldots,$$

and $\{x_n\}_{n=1}^{\infty}$ is a sequentially compact set. Then, by (A_4), we have

$$\begin{aligned} x_1(t) &= \int_1^{+\infty} G(t,s)a(s)f\bigl(s,x_0(s)\bigr)\frac{ds}{s} \\ &\le \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^{+\infty} a(s)f\bigl(s,x_0(s)\bigr)\frac{ds}{s} + \varpi(\ln t)^{\alpha-1}\int_1^{+\infty} a(s)f\bigl(s,x_0(s)\bigr)\frac{ds}{s} \\ &\le \frac{1}{\kappa}\int_1^{+\infty} a(s)\frac{ds}{s}(\ln t)^{\alpha-1}\frac{\kappa a}{\omega} = x_0(t). \end{aligned}$$

Therefore,

$$x_1(t) = Tx_0(t) \le x_0(t), \quad t \in [1,+\infty).$$

On account of T is a nondecreasing operator, we can derive a fact

$$x_{n+1}(t) \le x_n(t), \quad t \in [1,+\infty), n = 0,1,2,\ldots.$$

Thus, there exists $x^* \in \bar{P}_a$ such that $x_n \to x^*$ as $n \to \infty$ and $Tx^* = x^*$. Let $y_0(t) = 0$, $t \in [1,+\infty)$. Define the iterative sequence as follows:

$$y_{n+1}(t) = Ty_n(t), \quad t \in [1,+\infty), n = 0,1,2,\ldots.$$

Similarly, we have $\{y_n\}_{n=1}^{\infty} \subset \bar{P}_a$ is a sequentially compact set, and

$$y_{n+1}(t) \ge y_n(t) \ge \cdots \ge y_0(t) = 0, \quad t \in [1,+\infty), n = 0,1,2,\ldots.$$

Furthermore, there exists $y^* \in \bar{P}_a$ such that $y_n \to y^*$ as $n \to \infty$ and $Ty^* = y^*$. Since $f(t,0) \not\equiv 0$ on any subinterval of $[1,+\infty)$, it implies y^* is a positive solution of BVP (1.1). We now prove that $x^*(t)$ and $y^*(t)$ are the maximal and minimal solutions of BVP (1.1) in $(0, a(\ln t)^{\alpha-1}]$, respectively. Let $w(t)$ be any solution of BVP (1.1) with $0 \le w(t) \le a(\ln t)^{\alpha-1}$, that is,

$$y_0(t) = 0 \le w(t) \le a(\ln t)^{\alpha-1} = x_0(t), \quad t \in [1,+\infty).$$

Noting that T is nondecreasing, we have

$$y_1(t) = Ty_0(t) \le w(t) \le Tx_0(t) = x_1(t), \quad t \in [1,+\infty),$$

and

$$y_n(t) \le w(t) \le x_n(t), \quad t \in [1,+\infty), n = 1,2,\ldots.$$

From $\lim_{n\to\infty} y_n = y^*$, $\lim_{n\to\infty} x_n = x^*$, and the monotonicity of $\{x_n(t)\}$, $\{y_n(t)\}$, we obtain

$$y_0 \le y_1 \le \cdots \le y_n \le \cdots \le y^* \le \cdots \le x^* \le \cdots \le x_n \le \cdots \le x_1 \le x_0.$$

Therefore, y^* and x^* are respectively the minimal and maximal positive solutions of BVP (1.1) in $(0, a(\ln t)^{\alpha-1}]$. □

Example 3.1 Consider the boundary value problem

$$\begin{cases} {}^H D_{1^+}^{7/2} x(t) + \frac{1}{e^{\ln t}}[1 + \sin(\frac{\pi}{2} \cdot \frac{x(t)}{1+(\ln t)^{5/2}+x(t)})] = 0, \quad t \in (1, +\infty), \\ x(1) = x'(1) = x''(1) = 0, \qquad {}^H D_{1^+}^{5/2} x(+\infty) = \int_1^{+\infty} \frac{1}{2e^{\ln t}} x(t) \frac{dt}{t}. \end{cases} \tag{3.13}$$

Corresponding to problem (1.1), where

$$\begin{aligned} &n = 4, \qquad \alpha = \frac{7}{2}, \qquad a(t) = \frac{1}{e^{\ln t}}, \qquad g(t) = \frac{1}{2e^{\ln t}}, \\ &f\bigl(t, x(t)\bigr) = 1 + \sin\left(\frac{\pi}{2} \cdot \frac{x(t)}{1 + (\ln t)^{5/2} + x(t)}\right). \end{aligned} \tag{3.14}$$

By calculating, we have

$$\begin{aligned} &\kappa = \Gamma(7/2) - \int_1^{+\infty} \frac{1}{2e^{\ln t}} (\ln t)^{\alpha-1} \frac{dt}{t} = \frac{1}{2}\Gamma(7/2) = \frac{15}{16}\sqrt{\pi} > 0, \\ &\omega = \int_1^{+\infty} a(t)\frac{dt}{t} = \int_1^{+\infty} \frac{1}{e^{\ln t}} \frac{dt}{t} = 1. \end{aligned} \tag{3.15}$$

Let $a = 2$, then

$$f\bigl(t, \bigl(1 + (\ln t)^{5/2}\bigr)x\bigr) = 1 + \sin\left(\frac{\pi}{2} \cdot \frac{x(t)}{1 + x(t)}\right) \le 2 \le \frac{15}{8}\sqrt{\pi} = \frac{\kappa a}{\omega}. \tag{3.16}$$

From (3.14)–(3.16), we can see that (A_1)–(A_4) hold. By Theorem 3.1, BVP (3.13) has the positive maximal solution x^* and the minimal solution y^* in $(0, 2(\ln t)^{5/2}]$, which can be approximated by the following iterative sequences:

$$\begin{aligned} &x_{n+1}(t) \\ &\quad = \frac{16}{15\sqrt{\pi}}(\ln t)^{5/2} \int_1^{+\infty} \left[1 + \sin\left(\frac{\pi}{2} \cdot \frac{x_n(s)}{1 + (\ln s)^{5/2} + x_n(s)}\right)\right]\frac{ds}{s^2} \\ &\qquad - \frac{64}{225\pi}(\ln t)^{5/2} \int_1^{+\infty}\int_s^{+\infty} \bigl(\ln(t/s)\bigr)^{5/2}\frac{dt}{t^2}\left[1 + \sin\left(\frac{\pi}{2} \cdot \frac{x_n(s)}{1 + (\ln s)^{5/2} + x_n(s)}\right)\right]\frac{ds}{s^2} \\ &\qquad - \frac{8}{15\sqrt{\pi}} \int_1^t \bigl(\ln(t/s)\bigr)^{5/2}\left[1 + \sin\left(\frac{\pi}{2} \cdot \frac{x_n(s)}{1 + (\ln s)^{5/2} + x_n(s)}\right)\right]\frac{ds}{s^2}, \quad t \in [1, +\infty), \end{aligned}$$

$$\begin{aligned} &y_{n+1}(t) \\ &\quad= \frac{16}{15\sqrt{\pi}}(\ln t)^{5/2}\int_1^{+\infty}\left[1+\sin\left(\frac{\pi}{2}\cdot\frac{y_n(s)}{1+(\ln s)^{5/2}+y_n(s)}\right)\right]\frac{ds}{s^2} \\ &\qquad - \frac{64}{225\pi}(\ln t)^{5/2}\int_1^{+\infty}\int_s^{+\infty}\bigl(\ln(t/s)\bigr)^{5/2}\frac{dt}{t^2}\left[1+\sin\left(\frac{\pi}{2}\cdot\frac{y_n(s)}{1+(\ln s)^{5/2}+x_n(s)}\right)\right]\frac{ds}{s^2} \\ &\qquad - \frac{8}{15\sqrt{\pi}}\int_1^t\bigl(\ln(t/s)\bigr)^{5/2}\left[1+\sin\left(\frac{\pi}{2}\cdot\frac{y_n(s)}{1+(\ln s)^{5/2}+y_n(s)}\right)\right]\frac{ds}{s^2}, \quad t\in[1,+\infty), \end{aligned}$$

with the initial values $x_0(t) = 2(\ln t)^{5/2}$, $y_0(t) = 0$, $t \in [1,+\infty)$, respectively. It is easy to check that

$$\begin{aligned} x_1(t) &\le \frac{16}{15\sqrt{\pi}}(\ln t)^{2.5}\int_1^{+\infty}\bigl[1+\Xi(s)\bigr]\frac{ds}{s^2} \le \frac{16(2+\sqrt{3})}{30\sqrt{\pi}}(\ln t)^{2.5} < 2(\ln t)^{2.5}, \\ y_1(t) &= \frac{16}{15\sqrt{\pi}}(\ln t)^{2.5} - \frac{64}{225\pi}(\ln t)^{2.5}\int_1^{+\infty}\int_s^{+\infty}\left(\ln\frac{t}{s}\right)^{2.5}\frac{dt}{t^2}\frac{ds}{s^2} \\ &\quad - \frac{8}{15\sqrt{\pi}}\int_1^t\left(\ln\frac{t}{s}\right)^{2.5}\frac{ds}{s^2} \\ &> \frac{16}{15\sqrt{\pi}}(\ln t)^{2.5} - \frac{64}{225\pi}(\ln t)^{2.5}\int_1^{+\infty}\int_1^{+\infty}(\ln t)^{2.5}\frac{dt}{t^2}\frac{ds}{s^2} - \frac{8}{15\sqrt{\pi}}(\ln t)^{2.5} = 0, \end{aligned}$$

and

$$\begin{aligned} &x_1(t) - y_1(t) \\ &\quad= \frac{16}{15\sqrt{\pi}}(\ln t)^{2.5}\int_1^{+\infty}\Xi(s)\frac{ds}{s^2} - \frac{64}{225\pi}(\ln t)^{2.5}\int_1^{+\infty}\int_s^{+\infty}\left(\ln\frac{t}{s}\right)^{2.5}\frac{dt}{t^2}\Xi(s)\frac{ds}{s^2} \\ &\qquad - \frac{8}{15\sqrt{\pi}}\int_1^t\left(\ln\frac{t}{s}\right)^{2.5}\Xi(s)\frac{ds}{s^2} \\ &\quad> \frac{16}{15\sqrt{\pi}}(\ln t)^{2.5}\int_1^{+\infty}\Xi(s)\frac{ds}{s^2} - \frac{64}{225\pi}(\ln t)^{2.5}\int_1^{+\infty}\int_1^{+\infty}(\ln t)^{2.5}\frac{dt}{t^2}\Xi(s)\frac{ds}{s^2} \\ &\qquad - \frac{8}{15\sqrt{\pi}}\int_1^{+\infty}(\ln t)^{2.5}\Xi(s)\frac{ds}{s^2} \\ &\quad= 0, \end{aligned}$$

where $\Xi(s) = \sin(\frac{\pi}{2}\cdot\frac{2(\ln s)^{2.5}}{1+3(\ln s)^{2.5}})$. A tedious calculation can give two monotone sequences $\{x_n\}$ and $\{y_n\}$, $n = 1, 2, \ldots$.

4 The main result of (1.2)

Let

$$\begin{aligned} X &= \biggl\{x:[1,+\infty)\to\mathbb{R}\biggm| x, {}^H D_{1+}^{\alpha-2}x, {}^H D_{1+}^{\alpha-1}x \in C[1,+\infty), \sup_{t\in[1,+\infty)}\frac{|x(t)|}{1+(\ln t)^{\alpha-1}} < +\infty, \\ &\quad \sup_{t\in[1,+\infty)}\frac{|{}^H D_{1+}^{\alpha-2}x(t)|}{1+\ln t} < +\infty, \sup_{t\in[1,+\infty)}\bigl|{}^H D_{1+}^{\alpha-1}x(t)\bigr| < +\infty\biggr\}, \\ Y &= \biggl\{y:[1,+\infty)\to\mathbb{R}\biggm|\int_1^{+\infty}a(t)\bigl|y(t)\bigr|\frac{dt}{t} < +\infty\biggr\}. \end{aligned}$$

It is easy to check that X and Y are two Banach spaces, respectively, with the norms

$$\|x\|_X = \max\left\{\left\|\frac{x}{1+(\ln t)^{\alpha-1}}\right\|_\infty, \left\|\frac{{}^H D_{1+}^{\alpha-2}x}{1+\ln t}\right\|_\infty, \left\|{}^H D_{1+}^{\alpha-1}x\right\|_\infty\right\},$$
$$\|y\|_Y = \int_1^{+\infty} a(t)\left|y(t)\right|\frac{dt}{t},$$

where $\|x\|_\infty = \sup_{t\in[1,+\infty)}|x(t)|$.

Define the linear operator $L : \operatorname{dom} L \subset X \to Y$ and the nonlinear operator $N : X \to Y$ as follows:

$$Lx(t) = -\frac{1}{a(t)}{}^H D_{1+}^{\alpha}x(t), \quad x(t) \in \operatorname{dom} L,$$
$$Nx(t) = f\left(t, x(t), {}^H D_{1+}^{\alpha-2}x(t), {}^H D_{1+}^{\alpha-1}x(t)\right), \quad x(t) \in X,$$

where

$$\operatorname{dom} L = \left\{x(t) \in X \middle| {}^H D_{1+}^{\alpha}x(t) \in Y, x(t) \text{ satisfies boundary conditions of (1.2)}\right\}.$$

Then problem (1.2) is equivalent to the operator equation $Lx = Nx$, $x \in \operatorname{dom} L$.

Lemma 4.1 *Assume that* (H_1) *holds. Then the operator* $L : \operatorname{dom} L \subset X \to Y$ *satisfies*

$$\operatorname{Ker} L = \left\{x(t) \in \operatorname{dom} L \middle| x(t) = c(\ln t)^{\alpha-1}, c \in \mathbb{R}\right\}, \tag{4.1}$$
$$\operatorname{Im} L = \left\{y(t) \in Y \middle| \int_1^{+\infty} g(t)\int_t^{+\infty} a(s)y(s)\frac{ds}{s}\frac{dt}{t} = 0\right\}. \tag{4.2}$$

Proof For $Lx = -\frac{1}{a(t)}{}^H D_{1+}^{\alpha}x = 0$, by Lemma 2.2, we have

$$x(t) = c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + c_3(\ln t)^{\alpha-3}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Noting that $x(1) = x'(1) = 0$, we have

$$x(t) = c_1(\ln t)^{\alpha-1}.$$

So,

$$\operatorname{Ker} L \subset \left\{x \in \operatorname{dom} L \middle| x(t) = c(\ln t)^{\alpha-1}, c \in \mathbb{R}\right\}.$$

Conversely, take $x(t) = c(\ln t)^{\alpha-1}$, $c \in \mathbb{R}$. We can easily check that $-\frac{1}{a(t)}{}^H D_{1+}^{\alpha}x = 0$ and $x(t)$ satisfies the boundary conditions of (1.2). Hence,

$$\left\{x \in \operatorname{dom} L \middle| x(t) = c(\ln t)^{\alpha-1}, c \in \mathbb{R}\right\} \subset \operatorname{Ker} L.$$

That means (4.1) holds. For any $y \in \operatorname{Im} L$, there exists a function $x \in \operatorname{dom} L$ such that $Lx(t) = y(t)$. By Lemma 2.2 and the boundary conditions $x(1) = x'(1) = 0$, one has

$$x(t) = -{}^H I_{1+}^{\alpha}a(t)y(t) + c_1(\ln t)^{\alpha-1}.$$

Using the fact that ${}^{H}D_{1^+}^{\alpha-1}x(+\infty) = \int_1^{+\infty} g(t){}^{H}D_{1^+}^{\alpha-1}x(t)\frac{dt}{t}$, we have

$$\begin{aligned} {}^{H}D_{1^+}^{\alpha-1}x(+\infty) &= c_1\Gamma(\alpha) - \int_1^{+\infty} a(s)y(s)\frac{ds}{s} \\ &= \int_1^{+\infty} g(t)\left[c_1\Gamma(\alpha) - \int_1^{t} a(s)y(s)\frac{ds}{s}\right]\frac{dt}{t} \\ &= c_1\Gamma(\alpha) - \int_1^{+\infty} g(t)\int_1^{t} a(s)y(s)\frac{ds}{s}\frac{dt}{t}, \end{aligned}$$

that is,

$$\int_1^{+\infty} g(t)\int_t^{+\infty} a(s)y(s)\frac{ds}{s}\frac{dt}{t} = 0. \tag{4.3}$$

Thus,

$$\operatorname{Im} L \subset \left\{ y \in Y \,\middle|\, \int_1^{+\infty} g(t)\int_t^{+\infty} a(s)y(s)\frac{ds}{s}\frac{dt}{t} = 0 \right\}.$$

Conversely, let $y \in Y$ satisfy (4.3), take $x(t) = -{}^{H}I_{1^+}^{\alpha}a(t)y(t)$, we can check that $x \in \operatorname{dom} L$ and $Lx(t) = y(t)$. Then we obtain

$$\left\{ y \in Y \,\middle|\, \int_1^{+\infty} g(t)\int_t^{+\infty} a(s)y(s)\frac{ds}{s}\frac{dt}{t} = 0 \right\} \subset \operatorname{Im} L.$$

The proof is completed. □

Let

$$\Delta := \int_1^{+\infty} g(t)\int_t^{+\infty} a(s)\frac{ds}{s}\frac{dt}{t}.$$

Based on (H_1) and the nonnegativity of $g(t)$, $a(t)$, we get

$$\begin{aligned} 0 < \Delta &= \int_1^{+\infty} g(t)\int_t^{+\infty} a(s)\frac{ds}{s}\frac{dt}{t} \le \int_1^{+\infty} g(t)\int_1^{+\infty} a(s)\frac{ds}{s}\frac{dt}{t} \\ &= \int_1^{+\infty} a(t)\frac{dt}{t} < +\infty. \end{aligned}$$

Lemma 4.2 *Assume that* (H_1) *holds, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator of index zero. Set the linear operators* $P : X \to X$ *and* $Q : Y \to Y$ *defined as follows*:

$$(Px)(t) = \frac{1}{\Gamma(\alpha)}{}^{H}D_{1^+}^{\alpha-1}x(1)(\ln t)^{\alpha-1}, \qquad (Qy)(t) = \frac{1}{\Delta}\int_1^{+\infty} g(t)\int_t^{+\infty} a(s)y(s)\frac{ds}{s}\frac{dt}{t}.$$

Proof According to the definition of P, we can check that P is a continuous linear projector operator and satisfies $\operatorname{Im} P = \operatorname{Ker} L$, $X = \operatorname{Ker} P \oplus \operatorname{Ker} L$. By the definition of Q, we can see that Q is a continuous linear operator with $\dim \operatorname{Im} Q = 1$ and the following equations hold:

$$\begin{aligned}\big(Q^2y\big)(t) &= Q\big(Qy(t)\big) = \frac{1}{\Delta}\int_1^{+\infty} g(t)\int_t^{+\infty} a(s)Qy(s)\frac{ds}{s}\frac{dt}{t}\\ &= \frac{Qy(t)}{\Delta}\int_1^{+\infty} g(t)\int_t^{+\infty} a(s)\frac{ds}{s}\frac{dt}{t}\\ &= Qy(t).\end{aligned}$$

That is, Q is a projector operator. Obviously, we have $\operatorname{Im} L = \operatorname{Ker} Q$. For any $y \in Y$, then y can be expressed as $y = (y - Qy) + Qy$, i.e., $Y = \operatorname{Im} L + \operatorname{Im} Q$. In addition, for any $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, since $\operatorname{Im} L = \operatorname{Ker} Q$, we get $y = Qy = 0$, i.e., $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Thus, $Y = \operatorname{Im} Q \oplus \operatorname{Im} L$. Moreover, $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = co\dim \operatorname{Im} L = 1$. Therefore, L is a Fredholm operator with zero index. □

Lemma 4.3 *Suppose that* (H_1) *holds. Define a linear operator* $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *by*

$$(K_py)(t) = -\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} a(s)y(s)\frac{ds}{s}, \quad y \in \operatorname{Im} L.$$

Then K_p *is the inverse of* $L|_{\operatorname{dom} L\cap\operatorname{Ker} P}$ *and* $\|K_py\|_X \le \|y\|_Y$ *for any* $y \in \operatorname{Im} L$.

Proof For $y \in \operatorname{Im} L$, by the definition of K_p, we can check that $K_py \in \operatorname{dom} L \cap \operatorname{Ker} P$. Thus, K_p is well defined on $\operatorname{Im} L$. Now we show that $K_p = (L|_{\operatorname{dom} L\cap\operatorname{Ker} P})^{-1}$. In fact, it is easy to get $(LK_p)y(t) = y(t)$ for any $y \in \operatorname{Im} L$. For all $x(t) \in \operatorname{dom} L \cap \operatorname{Ker} P$, by Lemma 2.2, we have

$$(K_pL)x(t) = {}^HI_{1+}^{\alpha}\,{}^HD_{1+}^{\alpha}x(t) = x(t) + c(\ln t)^{\alpha-1}, \quad c \in \mathbb{R}.$$

Because $(K_pL)x(t) \in \operatorname{Ker} P$ and $c(\ln t)^{\alpha-1} \in \operatorname{Ker} L = \operatorname{Im} P$, we have $c(\ln t)^{\alpha-1} = -Px(t) = 0$. Then, $(K_pL)x(t) = x(t)$. Therefore, $K_p = (L|_{\operatorname{dom} L\cap\operatorname{Ker} P})^{-1}$. Also, we have the following inequalities:

$$\begin{aligned}\left\|\frac{K_py}{1+(\ln t)^{\alpha-1}}\right\|_\infty &= \sup_{t\in[1,+\infty)}\frac{|K_py|}{1+(\ln t)^{\alpha-1}}\\ &= \sup_{t\in[1,+\infty)}\frac{1}{\Gamma(\alpha)}\left|\int_1^t \frac{(\ln(t/s))^{\alpha-1}}{1+(\ln t)^{\alpha-1}}a(s)y(s)\frac{ds}{s}\right| \le \frac{1}{\Gamma(\alpha)}\|y\|_Y \le \|y\|_Y,\\ \left\|\frac{{}^HD_{1+}^{\alpha-2}K_py}{1+\ln t}\right\|_\infty &= \sup_{t\in[1,+\infty)}\frac{|{}^HD_{1+}^{\alpha-2}K_py|}{1+\ln t} = \sup_{t\in[1,+\infty)}\left|\int_1^t \frac{\ln(t/s)}{1+\ln t}a(s)y(s)\frac{ds}{s}\right| \le \|y\|_Y,\\ \left\|{}^HD_{1+}^{\alpha-1}K_py\right\|_\infty &= \sup_{t\in[1,+\infty)}\left|\int_1^t a(s)y(s)\frac{ds}{s}\right| \le \|y\|_Y.\end{aligned}$$

So, $\|K_py\|_X \le \|y\|_Y$ for all $y \in \operatorname{Im} L$. □

Lemma 4.4 *Let* $V = \{x \in X : \|x\|_X \le r, r > 0\} \subset X$. *Then* V *is relatively compact in* X *if it satisfies the following conditions*:

(i) *For any* $x(t) \in V$, $\frac{x(t)}{1+(\ln t)^{\alpha-1}}$, $\frac{{}^HD_{1+}^{\alpha-2}x(t)}{1+\ln t}$, *and* ${}^HD_{1+}^{\alpha-1}x(t)$ *are equicontinuous on any compact interval of* $[1,+\infty)$;

(ii) *For any* $\varepsilon > 0$, *there exists a constant* $S = S(\varepsilon) > 0$ *such that, for all* $x(t) \in V$, $t_1, t_2 \geq S$, *it holds*

$$\left|\frac{x(t_1)}{1+(\ln t_1)^{\alpha-1}} - \frac{x(t_2)}{1+(\ln t_2)^{\alpha-1}}\right| < \varepsilon, \qquad \left|\frac{{}^HD_{1+}^{\alpha-2}x(t_1)}{1+\ln t_1} - \frac{{}^HD_{1+}^{\alpha-2}x(t_2)}{1+\ln t_2}\right| < \varepsilon,$$
$$\left|{}^HD_{1+}^{\alpha-1}x(t_1) - {}^HD_{1+}^{\alpha-1}x(t_2)\right| < \varepsilon.$$

Proof Since X is a Banach space and $V \subset X$, it is sufficient to show that V is totally bounded. In fact, for any $S \in (1, +\infty)$, take

$$V_{[1,S]} = \{x(t) : x(t) \in V, t \in [1,S]\}, \qquad V_{[1,S]}^{\alpha-2} = \{{}^HD_{1+}^{\alpha-2}x(t) : x(t) \in V_{[1,S]}\},$$
$$V_{[1,S]}^{\alpha-1} = \{{}^HD_{1+}^{\alpha-1}x(t) : x(t) \in V_{[1,S]}\},$$

with the norms

$$\|x\|_\infty = \sup_{t\in[1,S]} \frac{|x(t)|}{1+(\ln t)^{\alpha-1}}, \qquad \left\|{}^HD_{1+}^{\alpha-2}x\right\|_\infty = \sup_{t\in[1,S]} \frac{|{}^HD_{1+}^{\alpha-2}x(t)|}{1+\ln t},$$
$$\left\|{}^HD_{1+}^{\alpha-1}x\right\|_\infty = \sup_{t\in[1,S]} \left|{}^HD_{1+}^{\alpha-1}x(t)\right|,$$

respectively. It is clear that $(V_{[1,S]}, \|x\|_\infty)$, $(V_{[1,S]}^{\alpha-2}, \|{}^HD_{1+}^{\alpha-2}x\|_\infty)$, and $(V_{[1,S]}^{\alpha-1}, \|{}^HD_{1+}^{\alpha-1}x\|_\infty)$ are Banach spaces. By using the Arzelà–Ascoli theorem, we can obtain that $V_{[1,S]}$, $V_{[1,S]}^{\alpha-2}$, and $V_{[1,S]}^{\alpha-1}$ are relatively compact under condition (i). Thus, $V_{[1,S]}$, $V_{[1,S]}^{\alpha-2}$, and $V_{[1,S]}^{\alpha-1}$ are totally bounded, i.e., for any $\varepsilon > 0$, there exist $\{x_i\}_{i=1}^n \subset V_{[1,S]}$, $\{y_j\}_{j=1}^m \subset V_{[1,S]}^{\alpha-2}$, and $\{z_k\}_{k=1}^l \subset V_{[1,S]}^{\alpha-1}$ such that

$$V_{[1,S]} \subset \bigcup_{i=1}^n B_\varepsilon(x_i), \qquad V_{[1,S]}^{\alpha-2} \subset \bigcup_{j=1}^m B_\varepsilon\left({}^HD_{1+}^{\alpha-2}y_j\right), \qquad V_{[1,S]}^{\alpha-1} \subset \bigcup_{k=1}^l B_\varepsilon\left({}^HD_{1+}^{\alpha-1}z_k\right), \quad (4.4)$$

where

$$B_\varepsilon(x_i) = \{x \in V_{[1,S]} : \|x - x_i\|_\infty < \varepsilon\},$$
$$B_\varepsilon\left({}^HD_{1+}^{\alpha-2}y_j\right) = \left\{{}^HD_{1+}^{\alpha-2}x(t) \in V_{[1,S]}^{\alpha-2} : \left\|{}^HD_{1+}^{\alpha-2}x - {}^HD_{1+}^{\alpha-2}y_j\right\|_\infty < \varepsilon\right\},$$
$$B_\varepsilon\left({}^HD_{1+}^{\alpha-1}z_k\right) = \left\{{}^HD_{1+}^{\alpha-1}x(t) \in V_{[1,S]}^{\alpha-1} : \left\|{}^HD_{1+}^{\alpha-1}x - {}^HD_{1+}^{\alpha-1}z_k\right\|_\infty < \varepsilon\right\}.$$

Set

$$V_{ijk} = \left\{x(t) \in V : x_{[1,S]} \in B_\varepsilon(x_i), {}^HD_{1+}^{\alpha-2}x_{[1,S]} \in B_\varepsilon\left({}^HD_{1+}^{\alpha-2}y_j\right), {}^HD_{1+}^{\alpha-1}x_{[1,S]} \in B_\varepsilon\left({}^HD_{1+}^{\alpha-1}z_k\right)\right\}.$$

Obviously, $V_{[1,S]} \subset \bigcup_{1\leq i\leq n, 1\leq j\leq m, 1\leq k\leq l} V_{ijk[1,S]}$. Take $x_{ijk} \in V_{ijk}$, then we claim that V can be covered by the balls $B_{4\varepsilon}(x_{ijk})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, l$, where

$$B_{4\varepsilon}(x_{ijk}) = \{x(t) \in V : \|x - x_{ijk}\|_X < 4\varepsilon\}.$$

Indeed, for $x(t) \in V$, by (4.4), there exist i, j, k such that

$$x_{[1,S]} \in B_\varepsilon(x_i), \quad {}^HD_{1+}^{\alpha-2}x_{[1,S]} \in B_\varepsilon\left({}^HD_{1+}^{\alpha-2}y_j\right), \qquad {}^HD_{1+}^{\alpha-1}x_{[1,S]} \in B_\varepsilon\left({}^HD_{1+}^{\alpha-1}z_k\right).$$

Then, for $t \in [1, S]$, we have

$$\begin{aligned}
\left|\frac{x(t)}{1+(\ln t)^{\alpha-1}} - \frac{x_{ijk}(t)}{1+(\ln t)^{\alpha-1}}\right| &\le \left|\frac{x(t)}{1+(\ln t)^{\alpha-1}} - \frac{x_i(t)}{1+(\ln t)^{\alpha-1}}\right| \\
&\quad + \left|\frac{x_i(t)}{1+(\ln t)^{\alpha-1}} - \frac{x_{ijk}(t)}{1+(\ln t)^{\alpha-1}}\right| \\
&< 2\varepsilon, \\
\left|\frac{{}^H D_{1^+}^{\alpha-2} x(t)}{1+\ln t} - \frac{{}^H D_{1^+}^{\alpha-2} x_{ijk}(t)}{1+\ln t}\right| &\le \left|\frac{{}^H D_{1^+}^{\alpha-2} x(t)}{1+\ln t} - \frac{{}^H D_{1^+}^{\alpha-2} y_j(t)}{1+\ln t}\right| \\
&\quad + \left|\frac{{}^H D_{1^+}^{\alpha-2} y_j(t)}{1+\ln t} - \frac{{}^H D_{1^+}^{\alpha-2} x_{ijk}(t)}{1+\ln t}\right| \\
&< 2\varepsilon, \\
\left|{}^H D_{1^+}^{\alpha-1} x(t) - {}^H D_{1^+}^{\alpha-1} x_{ijk}(t)\right| &\le \left|{}^H D_{1^+}^{\alpha-1} x(t) - {}^H D_{1^+}^{\alpha-1} z_k(t)\right| \\
&\quad + \left|{}^H D_{1^+}^{\alpha-1} z_k(t) - {}^H D_{1^+}^{\alpha-1} x_{ijk}(t)\right| \\
&< 2\varepsilon.
\end{aligned}$$

Combining this with condition (ii), we have

$$\begin{aligned}
&\left|\frac{x(t)}{1+(\ln t)^{\alpha-1}} - \frac{x_{ijk}(t)}{1+(\ln t)^{\alpha-1}}\right| \\
&\quad \le \left|\frac{x(t)}{1+(\ln t)^{\alpha-1}} - \frac{x(S)}{1+(\ln S)^{\alpha-1}}\right| + \left|\frac{x(S)}{1+(\ln S)^{\alpha-1}} - \frac{x_{ijk}(S)}{1+(\ln S)^{\alpha-1}}\right| \\
&\qquad + \left|\frac{x_{ijk}(S)}{1+(\ln S)^{\alpha-1}} - \frac{x_{ijk}(t)}{1+(\ln t)^{\alpha-1}}\right| \\
&\quad < 4\varepsilon, \quad t > S.
\end{aligned}$$

Using similar arguments as above, we can also get

$$\left|\frac{{}^H D_{1^+}^{\alpha-2} x(t)}{1+\ln t} - \frac{{}^H D_{1^+}^{\alpha-2} x_{ijk}(t)}{1+\ln t}\right| < 4\varepsilon, \qquad \left|{}^H D_{1^+}^{\alpha-1} x(t) - {}^H D_{1^+}^{\alpha-1} x_{ijk}(t)\right| < 4\varepsilon, \quad t > S.$$

Thus, $\|x - x_{ijk}\|_X < 4\varepsilon$. Therefore, V is totally bounded. □

Lemma 4.5 *Suppose that* (H_1) *holds,* $\Omega \subset X$ *is an open bounded subset with* $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$. *Then N is L-compact on* $\bar{\Omega}$.

Proof Since $\Omega \subset X$ is bounded, there exists a constant $l > 0$ such that $\|x\|_X \le l$, $\forall x \in \bar{\Omega}$. Then, by $f : [1, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies an a-Carathéodory condition, one has

$$\begin{aligned}
|QNx| &\le \frac{1}{\Delta} \int_1^{+\infty} g(t) \int_t^{+\infty} a(s) \left|Nx(s)\right| \frac{ds}{s} \frac{dt}{t} \\
&\le \frac{1}{\Delta} \int_1^{+\infty} g(t) \int_t^{+\infty} a(s) \varphi_l(s) \frac{ds}{s} \frac{dt}{t} \le \frac{1}{\Delta} \|\varphi_l\|_Y.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|QNx\|_Y &= \int_1^{+\infty} a(t)|QNx|\frac{dt}{t} \le \frac{\|\varphi_l\|_Y}{\Delta}\int_1^{+\infty} a(t)\frac{dt}{t} < +\infty, \\
\|K_p(I-Q)Nx\|_X &\le \|(I-Q)Nx\|_Y = \int_1^{+\infty} a(t)\big|(I-Q)Nx\big|\frac{dt}{t} \\
&\le \int_1^{+\infty} a(t)|Nx|\frac{dt}{t} + \int_1^{+\infty} a(t)|QNx|\frac{dt}{t} \\
&\le \|\varphi_l\|_Y + \frac{\|\varphi_l\|_Y}{\Delta}\int_1^{+\infty} a(t)\frac{dt}{t} := \tilde{l} < +\infty. \qquad (4.5)
\end{aligned}$$

Therefore, $QN(\bar{\Omega})$ and $K_p(I-Q)N(\bar{\Omega})$ are uniformly bounded. Now, we separate the proof into two steps. For simplicity of presentation, we let

$$\begin{aligned}
&h(t) = (I-Q)Nx(t), \quad t \in [1,+\infty), x(t) \in \bar{\Omega}, \\
&h_\mu(t,s) = \begin{cases} 1, & \mu = 1, 1 \le s \le t < +\infty, \\ \frac{(\ln(t/s))^{\mu-1}}{1+(\ln t)^{\mu-1}}, & \mu > 1, 1 \le s \le t < +\infty. \end{cases} \\
&H_\mu(t) = \int_1^t h_\mu(t,s)a(s)h(s)\frac{ds}{s}, \quad t \in [1,+\infty), \mu \ge 1.
\end{aligned}$$

Then we have

$$\|h\|_Y = \|(I-Q)Nx\|_Y \le \tilde{l} < +\infty, \quad 0 \le h_\mu(t,s) \le 1,$$

and

$$\big|H_\mu(t)\big| \le \int_1^t h_\mu(t,s)a(s)\big|h(s)\big|\frac{ds}{s} \le \int_1^{+\infty} a(s)\big|h(s)\big|\frac{ds}{s} = \|h\|_Y.$$

Step 1. For any $x \in \bar{\Omega}$, $K_p(I-Q)Nx$ is equicontinuous on any compact interval of $[1,+\infty)$. In fact, for any $T \in (1,+\infty)$ and $1 \le t_1 < t_2 \le T$. It follows from the uniform continuity of $h_\mu(t,s)$ on $[1,T] \times [1,T]$ and the absolute continuity of integral that

$$\begin{aligned}
\big|H_\mu(t_2) - H_\mu(t_1)\big| &= \left|\int_1^{t_2} h_\mu(t_2,s)a(s)h(s)\frac{ds}{s} - \int_1^{t_1} h_\mu(t_1,s)a(s)h(s)\frac{ds}{s}\right| \\
&= \left|\int_{t_1}^{t_2} h_\mu(t_2,s)a(s)h(s)\frac{ds}{s} + \int_1^{t_1}\big[h_\mu(t_2,s) - h_\mu(t_1,s)\big]a(s)h(s)\frac{ds}{s}\right| \\
&\le \left|\int_{t_1}^{t_2} a(s)\big|h(s)\big|\frac{ds}{s} + \int_1^{t_1}\big|h_\mu(t_2,s) - h_\mu(t_1,s)\big|a(s)\big|h(s)\big|\frac{ds}{s}\right| \\
&\to 0, \quad \text{as } t_1 \to t_2.
\end{aligned}$$

Then, as $t_1 \to t_2$, we get

$$\left|\frac{K_p(I-Q)Nx(t_2)}{1+(\ln t_2)^{\alpha-1}} - \frac{K_p(I-Q)Nx(t_1)}{1+(\ln t_1)^{\alpha-1}}\right| = \frac{1}{\Gamma(\alpha)}\big|H_\alpha(t_2) - H_\alpha(t_1)\big| \to 0,$$

$$\left|\frac{{}^H D_{1^+}^{\alpha-2} K_p(I-Q)Nx(t_2)}{1+\ln t_2} - \frac{{}^H D_{1^+}^{\alpha-2} K_p(I-Q)Nx(t_1)}{1+\ln t_1}\right| = \left|H_2(t_2)-H_2(t_1)\right| \to 0,$$
$$\left|{}^H D_{1^+}^{\alpha-1} K_p(I-Q)Nx(t_2) - {}^H D_{1^+}^{\alpha-1} K_p(I-Q)Nx(t_1)\right| = \left|H_1(t_2)-H_1(t_1)\right| \to 0.$$

Step 2. For any $x \in \bar{\Omega}$, $K_p(I-Q)Nx$ is equiconvergent at infinity. In fact, for any $x \in \bar{\Omega}$ and $\varepsilon > 0$, by (4.5), there exists a positive constant $L > 1$ such that

$$\int_L^{+\infty} a(s)\left|h(s)\right|\frac{ds}{s} < \varepsilon.$$

Since

$$\lim_{t\to\infty} h_\mu(t,L) = \lim_{t\to\infty} \frac{(\ln(t/L))^{\mu-1}}{1+(\ln t)^{\mu-1}} = 1, \quad (\mu > 1).$$

For above $\varepsilon > 0$, there exists a constant $\tilde{L}(\varepsilon) > L$ such that $1 - h_\mu(t,L) < \varepsilon$, $t > \tilde{L}(\varepsilon)$. Then, for any $t_2, t_1 > \tilde{L}(\varepsilon)$ (without loss of generality we assume that $t_2 > t_1$), we obtain

$$\begin{aligned}
\left|H_\mu(t_2) - H_\mu(t_1)\right| &= \left|\int_1^{t_2} h_\mu(t_2,s)a(s)h(s)\frac{ds}{s} - \int_1^{t_1} h_\mu(t_1,s)a(s)h(s)\frac{ds}{s}\right| \\
&= \left|\int_L^{t_2} h_\mu(t_2,s)a(s)h(s)\frac{ds}{s} - \int_L^{t_1} h_\mu(t_2,s)a(s)h(s)\frac{ds}{s}\right. \\
&\quad \left. + \int_1^{L} \left[h_\mu(t_2,s) - h_\mu(t_1,s)\right]a(s)h(s)\frac{ds}{s}\right| \\
&\le \int_1^{L} \left|h_\mu(t_2,s) - h_\mu(t_1,s)\right|\left|a(s)\right|\left|h(s)\right|\frac{ds}{s} + 2\int_L^{+\infty} a(s)\left|h(s)\right|\frac{ds}{s} \\
&\le \int_1^{L} \left[\left(1-h_\mu(t_2,s)\right) + \left(1-h_\mu(t_1,s)\right)\right]a(s)\left|h(s)\right|\frac{ds}{s} + 2\varepsilon \\
&\le 2\varepsilon\left(1+\|h\|_Y\right).
\end{aligned}$$

Thus, for any $t_2 > t_1 > \tilde{L}(\varepsilon)$, we have

$$\left|\frac{K_p(I-Q)Nx(t_2)}{1+(\ln t_2)^{\alpha-1}} - \frac{K_p(I-Q)Nx(t_1)}{1+(\ln t_1)^{\alpha-1}}\right| = \frac{1}{\Gamma(\alpha)}\left|H_\alpha(t_2)-H_\alpha(t_1)\right| \le \frac{2\varepsilon}{\Gamma(\alpha)}\left(1+\|h\|_Y\right),$$
$$\left|\frac{{}^H D_{1^+}^{\alpha-2} K_p(I-Q)Nx(t_2)}{1+\ln t_2} - \frac{{}^H D_{1^+}^{\alpha-2} K_p(I-Q)Nx(t_1)}{1+\ln t_1}\right| = \left|H_2(t_2)-H_2(t_1)\right| \le 2\varepsilon\left(1+\|h\|_Y\right),$$
$$\left|{}^H D_{1^+}^{\alpha-1} K_p(I-Q)Nx(t_2) - {}^H D_{1^+}^{\alpha-1} K_p(I-Q)Nx(t_1)\right| = \left|H_1(t_2)-H_1(t_1)\right| \le 2\varepsilon.$$

By Lemma 4.4, $K_p(I-Q)N : \bar{\Omega} \to X$ is compact. □

Theorem 4.1 *Suppose that* (H_1) *and the following conditions hold.*

(H_2) *There exist nonnegative functions* $b(t), c(t), d(t), e(t) \in Y$ *such that, for all* $t \in [1,+\infty)$ *and* $(u,v,w) \in \mathbb{R}^3$,

$$f(t,u,v,w) \le b(t)\frac{|u|}{1+(\ln t)^{\alpha-1}} + c(t)\frac{|v|}{1+\ln t} + d(t)|w| + e(t).$$

(H$_3$) *There exists a constant* $G > 0$ *such that, for all* $t \in [1, +\infty)$ *and* $x \in \operatorname{dom} L$, *if* $|^{H}D_{1^+}^{\alpha-1}x(t)| > G$, *then*

$$\int_1^{+\infty} g(t) \int_t^{+\infty} a(s) f\big(s, x(s), {}^{H}D_{1^+}^{\alpha-2}x(s), {}^{H}D_{1^+}^{\alpha-1}x(s)\big) \frac{ds}{s} \frac{dt}{t} \neq 0.$$

(H$_4$) *For any* $c \in \mathbb{R}$, *there exists a constant* $M > 0$ *such that, for* $|c| > M$,

$$c \int_1^{+\infty} g(t) \int_t^{+\infty} a(s) f\big(s, c(\ln s)^{\alpha-1}, c\Gamma(\alpha) \ln s, c\Gamma(\alpha)\big) \frac{ds}{s} \frac{dt}{t} > 0, \tag{4.6}$$

or

$$c \int_1^{+\infty} g(t) \int_t^{+\infty} a(s) f\big(s, c(\ln s)^{\alpha-1}, c\Gamma(\alpha) \ln s, c\Gamma(\alpha)\big) \frac{ds}{s} \frac{dt}{t} < 0. \tag{4.7}$$

Then BVP (1.2) *has at least one solution in* X *provided that*

$$\big(3 + \big(1/\Gamma(\alpha)\big)\big)\big(\|b\|_Y + \|c\|_Y + \|d\|_Y\big) < 1.$$

To prove Theorem 4.1, *we establish the following lemmas.*

Lemma 4.6 *Assume that* (H$_1$)–(H$_3$) *hold, set*

$$\Omega_1 = \big\{x \in \operatorname{dom} L \backslash \operatorname{Ker} L : Lx = \lambda Nx, \lambda \in (0, 1)\big\}.$$

Then Ω_1 *is bounded in* X.

Proof For $x \in \Omega_1$, then $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$. That is, $QNx = 0$. By (H$_3$), there exists a constant $t_0 \in [1, +\infty)$ such that $|^{H}D_{1^+}^{\alpha-1}x(t_0)| \leq G$. Since $Lx = \lambda Nx$, we obtain

$$x(t) = -\frac{\lambda}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} a(s) f\big(s, x(s), {}^{H}D_{1^+}^{\alpha-2}x(s), {}^{H}D_{1^+}^{\alpha-1}x(s)\big) \frac{ds}{s} + c(\ln t)^{\alpha-1},$$

and so

$${}^{H}D_{1^+}^{\alpha-1}x(t) = -\lambda \int_1^t a(s) f\big(s, x(s), {}^{H}D_{1^+}^{\alpha-2}x(s), {}^{H}D_{1^+}^{\alpha-1}x(s)\big) \frac{ds}{s} + c\Gamma(\alpha).$$

Then

$${}^{H}D_{1^+}^{\alpha-1}x(t) = -\lambda \int_{t_0}^t a(s) f\big(s, x(s), {}^{H}D_{1^+}^{\alpha-2}x(s), {}^{H}D_{1^+}^{\alpha-1}x(s)\big) \frac{ds}{s} + {}^{H}D_{1^+}^{\alpha-1}x(t_0).$$

Therefore,

$$\begin{aligned} \big|{}^{H}D_{1^+}^{\alpha-1}x\big| &\leq \int_1^{+\infty} a(s) \big|f\big(s, x(s), {}^{H}D_{1^+}^{\alpha-2}x(s), {}^{H}D_{1^+}^{\alpha-1}x(s)\big)\big| \frac{ds}{s} + \big|{}^{H}D_{1^+}^{\alpha-1}x(t_0)\big| \\ &\leq \|Nx\|_Y + G. \end{aligned}$$

On the other hand, by (H_2), we have

$$\begin{aligned}\|Nx\|_Y &= \int_1^{+\infty} a(s)\left|f\left(s, x(s), {}^H D_{1^+}^{\alpha-2}x(s), {}^H D_{1^+}^{\alpha-1}x(s)\right)\right|\frac{ds}{s} \\ &\le \left(\|b\|_Y + \|c\|_Y + \|d\|_Y\right)\|x\|_X + \|e\|_Y,\end{aligned} \tag{4.8}$$

and from the definition of P, we get

$$\begin{aligned}&\left\|\frac{Px}{1+(\ln t)^{\alpha-1}}\right\|_\infty \le \frac{|{}^H D_{1^+}^{\alpha-1}x(1)|}{\Gamma(\alpha)} \le \frac{\|Nx\|_Y + G}{\Gamma(\alpha)}, \\ &\left\|\frac{{}^H D_{1^+}^{\alpha-2}Px}{1+\ln t}\right\|_\infty \le \left|{}^H D_{1^+}^{\alpha-1}x(1)\right| \le \|Nx\|_Y + G, \\ &\left\|{}^H D_{1^+}^{\alpha-1}Px\right\|_\infty = \left|{}^H D_{1^+}^{\alpha-1}x(1)\right| \le \|Nx\|_Y + G.\end{aligned}$$

So,

$$\begin{aligned}\|Px\|_X &= \max\left\{\left\|\frac{Px}{1+(\ln t)^{\alpha-1}}\right\|_\infty, \left\|\frac{{}^H D_{1^+}^{\alpha-2}Px}{1+\ln t}\right\|_\infty, \left\|{}^H D_{1^+}^{\alpha-1}Px\right\|_\infty\right\} \\ &\le \left\|\frac{Px}{1+(\ln t)^{\alpha-1}}\right\|_\infty + \left\|\frac{{}^H D_{1^+}^{\alpha-2}Px}{1+\ln t}\right\|_\infty + \left\|{}^H D_{1^+}^{\alpha-1}Px\right\|_\infty \\ &\le \left(2 + \left(1/\Gamma(\alpha)\right)\right)\left(\|Nx\|_Y + G\right).\end{aligned} \tag{4.9}$$

By Lemma 4.3, one has

$$\left\|(I-P)x\right\|_X = \left\|K_p L(I-P)x\right\|_X \le \left\|L(I-P)x\right\|_Y = \|Lx\|_Y \le \|Nx\|_Y. \tag{4.10}$$

Then we obtain from (4.8)–(4.10)

$$\begin{aligned}\|x\|_X &= \left\|Px + (I-P)x\right\|_X \le \|Px\|_X + \left\|(I-P)x\right\|_X \\ &= \left(2 + \left(1/\Gamma(\alpha)\right)\right)\left(\|Nx\|_Y + G\right) + \|Nx\|_Y \\ &\le \left(3 + \left(1/\Gamma(\alpha)\right)\right)\left(\|b\|_Y + \|c\|_Y + \|d\|_Y\right)\|x\|_X \\ &\quad + \left(3 + \left(1/\Gamma(\alpha)\right)\right)\|e\|_Y + \left(2 + \left(1/\Gamma(\alpha)\right)\right)G.\end{aligned}$$

It follows that

$$\|x\|_X \le \frac{(3 + (1/\Gamma(\alpha)))\|e\|_Y + (2 + (1/\Gamma(\alpha)))G}{1 - (3 + (1/\Gamma(\alpha)))(\|b\|_Y + \|c\|_Y + \|d\|_Y)}.$$

Consequently, Ω_1 is bounded in X. □

Lemma 4.7 *Assume that* (H_1) *and* (H_4) *hold, set*

$$\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}.$$

Then Ω_2 *is bounded in* X.

Proof For $x \in \Omega_2$, then x can be rewritten as $x = c(\ln t)^{\alpha-1}$, $c \in \mathbb{R}$. Because $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$, then $QNx = 0$, that is,

$$\int_1^{+\infty} g(t) \int_t^{+\infty} a(s) f\bigl(s, c(\ln s)^{\alpha-1}, c\Gamma(\alpha)\ln s, c\Gamma(\alpha)\bigr) \frac{ds}{s}\frac{dt}{t} = 0.$$

By (H_4), we get $|c| \le M$. Thus, $\|x\|_X \le \Gamma(\alpha) M$, that is, Ω_2 is bounded in X. □

Lemma 4.8 *Assume that* (H_1) *and* (H_4) *hold, set*

$$\Omega_3 = \bigl\{x \in \operatorname{Ker} L : \vartheta \lambda Jx + (1-\lambda) QNx = 0, \lambda \in [0,1]\bigr\}.$$

Then Ω_3 *is bounded in* X, *where* $\vartheta = \pm 1$ *is such that* $\vartheta = 1$ *for* (4.6) *holds and* $\vartheta = -1$ *for* (4.7) *holds*, $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is the linear isomorphism defined by*

$$J\bigl(c(\ln t)^{\alpha-1}\bigr) = c, \quad \forall c \in \mathbb{R}.$$

Proof Without loss of generality, we suppose that (4.7) holds, then for any $x \in \Omega_3$, there exist constants $c \in \mathbb{R}$, $\lambda \in [0,1]$ such that $x(t) = c(\ln t)^{\alpha-1}$ and $-\lambda Jx + (1-\lambda) QNx = 0$. Namely,

$$\lambda c = \frac{(1-\lambda)}{\Delta} \int_1^{+\infty} g(t) \int_t^{+\infty} a(s) f\bigl(s, c(\ln s)^{\alpha-1}, c\Gamma(\alpha)\ln s, c\Gamma(\alpha)\bigr) \frac{ds}{s}\frac{dt}{t}.$$

For $\lambda = 1$, then $c = 0$. Otherwise, if $|c| > M$, by (H_4) one gets

$$0 \le \lambda c^2 = \frac{(1-\lambda)c}{\Delta} \int_1^{+\infty} g(t) \int_t^{+\infty} a(s) f\bigl(s, c(\ln s)^{\alpha-1}, c\Gamma(\alpha)\ln s, c\Gamma(\alpha)\bigr) \frac{ds}{s}\frac{dt}{t} < 0.$$

It is a contradiction. So, Ω_3 is bounded in X. If (4.6) holds, by a similar method, we can see that Ω_3 is bounded. □

Proof of Theorem 4.1 Set Ω to be a bounded open subset of X such that $\bigcup_{i=1}^{3} \bar{\Omega}_i \subset \Omega$. By Lemma 4.5, N is L-compact on $\bar{\Omega}$. According to Lemmas 4.6 and 4.7, we have

(i) $Lx \neq \lambda Nx$ for any $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \in \operatorname{Im} L$ for any $x \in \operatorname{Ker} L \cap \partial\Omega$.

Next, we show that (iii) of Theorem 2.1 is satisfied. Therefore, we define

$$H(x, \lambda) = \vartheta \lambda Jx + (1-\lambda) QNx,$$

where ϑ is defined as before. By the preceding lemma, we derive $H(x,\lambda) \neq 0$, $x \in \operatorname{Ker} L \cap \partial\Omega$. According to the homotopy property of degree, it follows that

$$\begin{aligned} \deg\{QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\{\vartheta J, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

Then we conclude from Theorem 2.1 that the operator function $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$, thus, problem (1.2) has at least one solution in X. □

Example 4.1 Consider the following fractional boundary value problem:

$$\begin{cases} {}^{H}D_{1^+}^{2.5}x(t) + \frac{1}{e^{\ln t}}\big[\frac{1}{5e^{\ln t}}\frac{\sin x(t)}{1+(\ln t)^{1.5}} + \frac{1}{10e^{\ln t}}\frac{\sin {}^{H}D_{1^+}^{0.5}x(t)}{1+\ln t} \\ \quad + \frac{4|{}^{H}D_{1^+}^{1.5}x(t)|}{25e^{\ln t}} + \frac{1}{5e^{\ln t}}\big] = 0, \quad t \in (1,+\infty), \\ x(1) = x'(1) = 0, \qquad {}^{H}D_{1^+}^{1.5}x(+\infty) = \int_1^{+\infty} \frac{1}{e^{\ln t}}\,{}^{H}D_{1^+}^{1.5}x(t)\frac{dt}{t}. \end{cases} \tag{4.11}$$

Corresponding to BVP (1.2), where

$$\alpha = \frac{5}{2}, \qquad a(t) = g(t) = \frac{1}{e^{\ln t}},$$

$$f\big(t, x(t), {}^{H}D_{1^+}^{0.5}x(t), {}^{H}D_{1^+}^{1.5}x(t)\big) = \frac{1}{5e^{\ln t}}\frac{\sin x(t)}{1+(\ln t)^{1.5}} + \frac{1}{10e^{\ln t}}\frac{\sin {}^{H}D_{1^+}^{0.5}x(t)}{1+\ln t} + \frac{4|{}^{H}D_{1^+}^{1.5}x(t)|}{25e^{\ln t}} + \frac{1}{5e^{\ln t}}, \quad t \in (1,+\infty).$$

Let

$$b(t) = \frac{1}{5e^{\ln t}}, \qquad c(t) = \frac{1}{10e^{\ln t}}, \qquad d(t) = \frac{4}{25e^{\ln t}}, \qquad e(t) = \frac{1}{5e^{\ln t}},$$

and choose $G = M = 7$, we can check that (H_1)–(H_4) hold. Then, by Theorem 4.1, BVP (4.11) has at least one solution.

5 Conclusion

In this paper, by means of the monotone iterative technique and Mawhin's continuation theorem, we have proved the existence of solutions for two types of higher-order Hadamard-type FDEs with integral boundary conditions on an infinite interval. There are relatively few articles which study the existence of solutions for Hadamard-type fractional BVPs on an infinite interval. It is a very interesting topic and there is some work to be done in the future such as: investigating the existence and uniqueness of solutions for Hadamard-type fractional BVPs with p-Laplacian operator on an infinite interval; studying the Hyers–Ulam stability for Hadamard-type fractional non-resonance BVPs with p-Laplacian operator, and so on.

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