Zhong and Wang *Boundary Value Problems* (2018) 2018:137
https://doi.org/10.1186/s13661-018-1056-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions of conformable fractional differential equations with integral boundary conditions

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Abstract

In this paper, we discuss the existence of positive solutions of the conformable fractional differential equation $T_\alpha x(t) + f(t, x(t)) = 0$, $t \in [0, 1]$, subject to the boundary conditions $x(0) = 0$ and $x(1) = \lambda \int_0^1 x(t)\,\mathrm{d}t$, where the order α belongs to $(1, 2]$, $T_\alpha x(t)$ denotes the conformable fractional derivative of a function $x(t)$ of order α, and $f : [0, 1] \times [0, \infty) \mapsto [0, \infty)$ is continuous. By use of the fixed point theorem in a cone, some criteria for the existence of at least one positive solution are established. The obtained conditions are generally weaker than those derived by using the classical norm-type expansion and compression theorem. A concrete example is given to illustrate the possible application of the obtained results.

Keywords: Conformable fractional derivatives; Integral boundary value problems; Positive solutions; Fixed point theorems

1 Introduction

The fractional derivative is a generalization of the classical one to an arbitrary order, and the question of what is a fractional derivative was first raised by L'Hôpital in a letter to Leibniz in 1695. Since then, fractional calculus has been extensively studied, and it has been applied to almost every field of science, engineering, and mathematics in the last four decades [1–10]. It is worth emphasizing that there exist a number of definitions of fractional derivatives in the literature, and the different definitions are constructed to satisfy various constraints.

Recently, in [11] Khalil et al. introduced a new well-behaved definition of a fractional derivative termed the conformable fractional derivative. The new definition has drawn much interest from many researchers. And some results have been obtained on the properties of the conformable fractional derivative [11–13]. Several applications and generalizations of the definition were also discussed in [14–20], among which [14] indicated that several specific conformable fractional models are consistent with experimental date, and which [15] interpreted the physical and geometrical meaning of the conformable fractional derivative. Although the definite meaning indicates potential applications of the conformable fractional derivative in physics and engineering, it is worth noting that the investigation of the theory of conformable fractional differential equations has only entered an initial stage.

Initial value problems of conformable fractional differential equations were discussed in [21–23]; and analytical solutions to some specific conformable fractional partial differential equations were studied in [24–30]. For the discussion of boundary value problems (**BVPs** for short) of conformable fractional differential equations, some theoretical developments have also been achieved. In particular, Lyapunov type inequalities for some conformable boundary value problems were established in [31, 32]; a regular conformable fractional Sturm–Liouville eigenvalue problem was considered in [33]; solvability of some two-point fractional **BVP** was considered in [34–36] by using topological transversality theorem; a type of three-point fractional **BVP** was studied in [37] by means of fixed point theorems; and a class of periodic **BVP** was discussed in [38] by virtue of methods of lower and upper solutions. Applying approximation methods of operators and fixed point theorems, Xiaoyu Dong et al. [39] investigated the existence of positive solutions to a specific type of two-point **BVP** of p-Laplacian.

Motivated by the above-mentioned results and techniques in treating those **BVPs** of the conformable fractional differential equations, we then turn to investigating the existence of positive solutions for the **BVP** as follows:

$$\begin{cases} T_\alpha x(t) + f(t, x(t)) = 0, \quad t \in [0,1], \\ x(0) = 0, \qquad x(1) = \mathcal{L}(x), \end{cases} \tag{1.1}$$

where α belongs to $(1,2]$, T_α denotes the conformable fractional derivative of order α, the function $f : [a,\infty) \times [0,\infty) \mapsto [0,\infty)$ is continuous, and $\mathcal{L}(x) = \lambda \int_0^1 x(t)\,\mathrm{d}t$ for which the parameter λ is a positive number.

In the context of the conformable fractional derivatives, to the best of our knowledge, there have been very few results in the literature for the existence of positive solution to the conformable fractional differential equations with integral boundary conditions. It is worth pointing out that the obtained Green function in this work is singular, while the Green functions of **BVPs** of some new fractional derivatives with nonsingular kernels are nonsingular [40, 41].

The rest of paper is organized as follows. Section 2 preliminarily provides some definitions and lemmas which are crucial to the following discussion. In Sect. 3, we establish some criteria for the existence of at least one positive solution to the **BVP** (1.1) by means of the fixed point theorem in a cone. The obtained conditions are generally weaker than those derived by using the classical norm-type expansion and compression theorem [42]. Finally, an example is given to illustrate the possible application of the obtained results.

2 Preliminaries

In this section, we preliminarily provide some definitions and lemmas which play a key role in the following discussion.

Definition 2.1 ([11, 12]) Let α be in $(0,1]$. The conformable fractional derivative of a function $f : [0,\infty) \mapsto \mathbb{R}$ of order α is defined by

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$

If $T_\alpha f(t)$ exists on $(0,b)$, then $T_\alpha f(0) = \lim_{t\to 0} T_\alpha f(t)$.

Definition 2.2 ([11, 12]) Let α be in $(n, n+1]$. The conformable fractional derivative of a function $f : [0,\infty) \mapsto \mathbb{R}$ of order α is defined by

$$T_\alpha f(t) = T_\beta f^{(n)}(t) \quad \text{for which } \beta = \alpha - n.$$

Definition 2.3 ([12]) Let α be in $(n, n+1]$. The fractional integral of a function $f : [0,\infty) \mapsto \mathbb{R}$ of order α is defined by

$$I_\alpha f(t) = \frac{1}{n!}\int_0^t (t-s)^n s^{\alpha-n-1} f(s)\,\mathrm{d}s.$$

Lemma 2.1 ([11, 12]) *Let α be in $(n, n+1]$. If f is a continuous function on $[0,\infty)$, then, for all $t > 0$, $T_\alpha I_\alpha f(t) = f(t)$.*

Lemma 2.2 ([11, 12]) *Let α be in $(n, n+1]$. Then $T_\alpha t^k = 0$ for t in $[0,1]$ and $k = 1, 2, \ldots, n$.*

Lemma 2.3 ([12]) *Let α be in $(n, n+1]$. If $T_\alpha f(t)$ is continuous on $[0,\infty)$, then*

$$I_\alpha T_\alpha f(t) = f(t) + c_0 + c_1 t + \cdots + c_n t^n$$

for some real numbers c_k, $k = 1, 2, \ldots, n$.

By Lemma 2.3, we next present an integral presentation of the solution for the **BVP** of the linearized equation related to the **BVP** (1.1).

$$\begin{cases} T_\alpha x(t) + h(t) = 0, & t \in [0,1], \alpha \in (1,2], \\ x(0) = 0, & x(1) = \mathcal{L}(x). \end{cases} \tag{2.1}$$

Lemma 2.4 *Let h be in $C[0,1]$. If $\lambda \neq 2$, then the **BVP** (2.1) exists a unique solution defined on $[0,1]$ given by*

$$x(t) = \int_0^1 \mathcal{K}(t,s) h(s)\,\mathrm{d}s, \tag{2.2}$$

where

$$\mathcal{K}(t,s) = \mathcal{G}(t,s) + \mathcal{H}(t,s), \tag{2.3}$$

$$\mathcal{G}(t,s) = \begin{cases} (1-t)s^{\alpha-1}, & 0 \le s \le t \le 1, \\ ts^{\alpha-2}(1-s), & 0 < t \le s \le 1, \end{cases} \tag{2.4}$$

and

$$\mathcal{H}(t,s) = \frac{2\lambda t}{2-\lambda}\int_0^1 \mathcal{G}(\tau,s)\,\mathrm{d}\tau. \tag{2.5}$$

Proof By the continuity of h and Lemma 2.3, it follows from Eq. (2.1) that

$$x(t) = c_0 + c_1 t - I_\alpha h(t).$$

This, together the boundary conditions, implies $c_0 = 0$ and

$$c_1 = I_\alpha h(1) + \mathcal{L}(x).$$

Hence

$$\begin{aligned} x(t) &= -I_\alpha h(t) + tI_\alpha h(1) + t\mathcal{L}(x) \\ &= -\int_0^t (t-s)s^{\alpha-2}h(s)\,\mathrm{d}s + \int_0^t t(1-s)s^{\alpha-2}h(s)\,\mathrm{d}s \\ &\quad + \int_t^1 t(1-s)s^{\alpha-2}h(s)\,\mathrm{d}s + t\mathcal{L}(x), \end{aligned}$$

which yields

$$x(t) = \int_0^1 \mathcal{G}(t,s)h(s)\,\mathrm{d}s + t\mathcal{L}(x). \tag{2.6}$$

Applying the transformation $\mathcal{L}$ to both sides of Eq. (2.6), we get

$$\mathcal{L}(x) = \frac{2}{2-\lambda}\int_0^1 \mathcal{L}\big(\mathcal{G}(t,s)\big)h(s)\,\mathrm{d}s. \tag{2.7}$$

Substituting the above expression into (2.6), we obtain the desired result. □

The functions $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ have several important properties as follows.

Lemma 2.5 *For any* (t,s) *in* $(0,1] \times (0,1]$,

$$0 \le q(t)\mathcal{G}(s,s) \le \mathcal{G}(t,s) \le \mathcal{G}(s,s). \tag{2.8}$$

Furthermore, if λ *belongs to* $[0,2)$, *then*

$$0 \le q(t)\mathcal{H}(1,s) \le \mathcal{H}(t,s) \le \mathcal{H}(1,s), \tag{2.9}$$

$$0 \le q(t)\mathcal{M}(s) \le \mathcal{K}(t,s) \le \mathcal{M}(s), \tag{2.10}$$

where $q(t) = t(1-t)$, *and* $\mathcal{M}(s) = \mathcal{G}(s,s) + \mathcal{H}(1,s)$.

Proof By the definition of $\mathcal{G}$, for $0 \le s \le t \le 1$,

$$\mathcal{G}(t,s) = (1-t)s^{\alpha-1} \le (1-s)s^{\alpha-1} = \mathcal{G}(s,s),$$

and for $0 < t \le s \le 1$,

$$\mathcal{G}(t,s) = ts^{\alpha-2}(1-s) = \frac{t}{s}(1-s)s^{\alpha-1} \le (1-s)s^{\alpha-1} = \mathcal{G}(s,s).$$

Thus $\mathcal{G}(t,s) \le \mathcal{G}(s,s)$ for (t,s) in $(0,1] \times (0,1]$.

Moreover, observe that, for $0 < s \le t \le 1$,

$$\mathcal{G}(t,s) = (1-t)s^{\alpha-1} \ge (1-t)s^{\alpha-1}(1-s) \ge q(t)\mathcal{G}(s,s) \ge 0$$

and that, for $0 < t \le s \le 1$,

$$\mathcal{G}(t,s) = ts^{\alpha-2}(1-s) \ge ts^{\alpha-1}(1-s) \ge q(t)\mathcal{G}(s,s) \ge 0.$$

Hence

$$0 \le q(t)\mathcal{G}(s,s) \le \mathcal{G}(t,s) \le \mathcal{G}(s,s)$$

for (t,s) in $(0,1] \times (0,1]$. Furthermore, the above inequality and the definitions of $\mathcal{H}$ and $\mathcal{K}$ clearly yield the inequalities (2.9) and (2.10). The proof is complete. □

The key tool in our approach is the following well-known fixed point theorem in a cone [42, 43].

Lemma 2.6 *Let $\mathfrak{B}$ be a Banach space, $\mathcal{P} \subseteq \mathfrak{B}$ a cone, and Ω_1, Ω_2 two bounded open balls of $\mathfrak{B}$ centered at the origin with $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $\Phi : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ is a completely continuous operator such that*

(C1) $\|\Phi x\| \le \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_1$.

(C2) *There exists $\psi \in \mathcal{P} \setminus \{0\}$ such that $x \ne \Phi x + \lambda\psi$ for $x \in \mathcal{P} \cap \partial\Omega_2$ and $\lambda > 0$.*

Then Φ has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The same conclusion remains valid if (C1) *holds on $\mathcal{P} \cap \partial\Omega_2$ and* (C2) *holds on $\mathcal{P} \cap \partial\Omega_1$.*

3 Main results

In order to utilize the fixed point theorem to discuss the existence of solutions of the boundary value problem, we now make the basic assumption and define some sets of functions in $C[a,b]$ and operators.

(H) The function f is nonnegative, and continuous on $[0,1] \times [0,\infty)$, and the parameter λ belongs to $[0,2)$.

Let $\mathfrak{B} = C[0,1]$ be the classical Banach space with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Furthermore, define the cone $\mathcal{P}$ in $\mathfrak{B}$ by

$$\mathcal{P} = \{x \in \mathfrak{B} | x(t) \ge q(t)\|x\| \text{ for } t \in [0,1]\}.$$

Here the function $q(t)$ is defined as in Lemma 2.5.

Given a positive number r, define the subset Ω_r of $\mathfrak{B}$ by

$$\Omega_r = \{x \in \mathfrak{B} : \|x\| < r\}.$$

Also, define the operator from the space $\mathfrak{B}$ to itself by

$$(\Phi x)(t) = \int_0^1 \mathcal{K}(t,s)f(t,x(s))\,\mathrm{d}s. \tag{3.1}$$

Under the hypothesis (H), the operator is well defined and has the following property.

Lemma 3.1 *If the hypothesis* (H) *holds, then* $\Phi(\mathcal{P}) \subset \mathcal{P}$.

Proof For any x in $\mathcal{P}$, the definition of Φ and the inequality (2.10) imply that

$$(\Phi x)(t) = \int_0^1 \mathcal{K}(t,s) f\bigl(t, x(s)\bigr)\,\mathrm{d}s \geq q(t) \int_0^1 \mathcal{M}(s) f\bigl(t, x(s)\bigr)\,\mathrm{d}s$$

and that

$$(\Phi x)(t)) = \int_0^1 \mathcal{K}(t,s) f\bigl(t, x(s)\bigr)\,\mathrm{d}s \leq \int_0^1 \mathcal{M}(s) f\bigl(t, x(s)\bigr)\,\mathrm{d}s,$$

which yield

$$(\Phi x)(t) \geq q(t)\|\Phi x\|.$$

Hence $\Phi x \in \mathcal{P}$. We thus complete the proof. □

We further discuss the complete continuity of the operator Φ. To this end, denote the operator Φ by

$$\Phi = \Phi_1 + \Phi_2, \tag{3.2}$$

where the operators Φ_1 and Φ_2 are defined, respectively, by

$$(\Phi_1 x)(t) = \int_0^1 \mathcal{G}(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \tag{3.3}$$

and

$$(\Phi_2 x)(t) = \int_0^1 \mathcal{H}(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s. \tag{3.4}$$

And then we claim that Φ_1 and $\Phi_2 : \mathcal{P} \mapsto \mathcal{P}$ are completely continuous operators. Indeed, by an argument similar to the proof of Lemma 3.1, using the inequalities (2.8) and (2.9) we first infer that $\Phi_1(\mathcal{P}) \subset \mathcal{P}$ and $\Phi_2(\mathcal{P}) \subset \mathcal{P}$.

Furthermore, observe that the kernel $\mathcal{G}(t,s)$ of Φ_1 is singular on $[0,1] \times [0,1]$, and that the complete continuity of the operator Φ_1 was verified in [34, 39] by using approximations of the operator. As for the operator Φ_2, its kernel $\mathcal{H}(t,s)$ is continuous on $[0,1] \times [0,1]$, and using the standard argument, we can easily check that it is also completely continuous. Thus we obtain the following lemma.

Lemma 3.2 *If the hypothesis* (H) *holds, then the operator* $\Phi : \mathcal{P} \mapsto \mathcal{P}$ *completely continuous.*

The next lemma transforms the BVP (1.1) into an equivalent fixed point problem.

Lemma 3.3 *If the hypothesis* (H) *holds, then a function* x *in* $C[0,1]$ *is a positive solution of the* ***BVP*** (1.1) *if and only if it is a fixed point of* Φ *in* $\mathcal{P}$.

Proof Let x be a fixed point of Φ in $\mathcal{P}$, then

$$x(t) = \int_0^1 \mathcal{K}(t,s)f\big(s,x(s)\big)\,\mathrm{d}s = -I_\alpha f\big(t,x(t)\big) + tI_\alpha f\big(1,x(1)\big) + t\mathcal{L}(x), \tag{3.5}$$

and thus, by the continuity of f, Lemma 2.1, 2.2 and 2.3,

$$T_\alpha x(t) + f\big(t,x(t)\big) = 0.$$

Moreover, the equality (3.5) directly implies $x(0) = 0$ and $x(1) = \mathcal{L}(x)$. Therefore x is a positive solution of the **BVP** (1.1).

On the other hand, if x is a positive solution of the **BVP** (1.1), then Lemma 2.4 implies $\Phi x = x$. Moreover, by the same type of argument as for the proof of Lemma 3.1, we also get $x(t) \geq q(t)\|x\|$ for $t \in [0,1]$. Hence x is a fixed point of Φ in $\mathcal{P}$. We consequently complete the proof. $\square$

Before presenting the main results, we further introduce some notations as follows:

$$f_0 = \lim_{x\to 0}\min_{t\in[0,1]}\frac{f(t,x)}{x} \quad \text{and} \quad f^\infty = \lim_{x\to\infty}\max_{t\in[0,1]}\frac{f(t,x)}{x};$$
$$f^0 = \lim_{x\to 0}\max_{t\in[0,1]}\frac{f(t,x)}{x} \quad \text{and} \quad f_\infty = \lim_{x\to\infty}\min_{t\in[0,1]}\frac{f(t,x)}{x};$$
$$\Lambda_1 = \left(q(\delta)\int_\delta^{1-\delta}\mathcal{M}(s)\,\mathrm{d}s\right)^{-1} \quad \text{and} \quad \Lambda_2 = \left(\int_0^1 \mathcal{M}(s)\,\mathrm{d}s\right)^{-1}.$$

Here δ is a positive number given in $(0,\frac{1}{2})$. The functions $\mathcal{M}(s)$ and $q(t)$ are defined as in Lemma 2.5.

Now we are in a position to give and show the main results.

Theorem 3.1 *Assume that the hypothesis* (H) *holds. If* $f_0 > \Lambda_1$ *and* $f^\infty < \frac{\Lambda_2}{2}$, *then the* ***BVP*** (1.1) *has at least one positive solution.*

Proof The assertion will be proven by Lemma 2.6. Observe that Lemma 3.2 ensures that the operator $\Phi : \mathcal{P} \to \mathcal{P}$ is completely continuous.

We first verify that the operator Φ satisfies the condition (C2) in Lemma 2.6. Since $f_0 > \Lambda_1$, there exists a positive number r_1 such that

$$f(t,x) \geq \Lambda_1 x \quad \text{for } t \in [0,1] \text{ and } 0 \leq x \leq r_1.$$

Thus

$$f\big(t,x(t)\big) \geq \Lambda_1 x(t) \quad \text{for } t \in [0,1] \text{ and } x \in \mathcal{P} \cap \partial\Omega_{r_1}.$$

Now, choose the function $\psi \equiv 1$, and obviously, ψ belongs to $\mathcal{P} \setminus \{0\}$. We next show that, for the specified ψ,

$$x \neq \Phi x + \lambda\psi$$

for $x \in \mathcal{P} \cap \partial\Omega_{r_1}$ and $\lambda > 0$. If such were not the case, then there exist a function $x_0 \in \mathcal{P} \cap \partial\Omega_{r_1}$ and a positive number λ_0 such that

$$x_0 = \Phi x_0 + \lambda_0 \psi.$$

Let $\bar{x}_0 = \min_{t\in[\delta,1-\delta]} x_0(t)$. Then, by the inequality (2.10), for each t in $[\delta, 1-\delta]$,

$$\begin{aligned} x_0(t) &= \int_0^1 \mathcal{K}(t,s) f\bigl(s, x_0(s)\bigr)\,\mathrm{d}s + \lambda_0 \\ &\geq \Lambda_1 q(t) \int_\delta^{1-\delta} \mathcal{M}(s) x_0(s)\,\mathrm{d}s + \lambda_0 \\ &\geq \Lambda_1 q(\delta) \int_\delta^{1-\delta} \mathcal{M}(s)\,\mathrm{d}s \cdot \bar{x}_0 + \lambda_0 \\ &= \bar{x}_0 + \lambda_0. \end{aligned}$$

Thus, $\bar{x}_0 \geq \bar{x}_0 + \lambda_0$. This is a contradiction. Hence the operator Φ satisfies the condition (C2) in Lemma 2.6.

We now show that the operator Φ satisfies the condition (C1) in Lemma 2.6. From the assumption $f^\infty < \frac{\Lambda_2}{2}$, it follows that there exists a positive number γ_1 such that

$$f(t,x) \leq \frac{\Lambda_2}{2} x \quad \text{for } t \in [0,1] \text{ and } x \geq \gamma_1, \tag{3.6}$$

Now let $\gamma_2 = \max\{f(t,x) : t \in [0,1], x \in [0,\gamma_1]\}$. Then the inequality (3.6) yields

$$f(t,x) \leq \frac{\Lambda_2}{2} x + \gamma_2 \quad \text{for } t \in [0,1] \text{ and } x \geq 0. \tag{3.7}$$

Set $r_2 = \max\{2r_1, 2\gamma_2 \int_0^1 \mathcal{M}(s)\,\mathrm{d}s\}$ and let $x \in \mathcal{P} \cap \partial\Omega_{r_2}$. Then Lemma 2.5 and the inequality (3.7) imply

$$\begin{aligned} \|\Phi x\| &= \max_{t\in[0,1]} \int_0^1 \mathcal{K}(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \\ &\leq \int_0^1 \mathcal{M}(s) \biggl(\frac{\Lambda_2}{2} x(s) + \gamma_2\biggr)\,\mathrm{d}s \\ &\leq \frac{\Lambda_2}{2} \int_0^1 \mathcal{M}(s)\,\mathrm{d}s \|x\| + \gamma_2 \int_0^1 \mathcal{M}(s)\,\mathrm{d}s \\ &\leq \|x\|. \end{aligned}$$

Hence the operator Φ satisfies condition (C1) in Lemma 2.6. Consequently, the operator Φ has at least one fixed point $x \in \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, and Lemma 3.3 ensures that x is one positive solution of the **BVP** (1.1). The proof is complete. □

Theorem 3.2 *Assume that the hypothesis* (H) *holds. If* $f^0 < \Lambda_2$ *and* $f_\infty > \Lambda_1$, *then the* ***BVP*** (1.1) *has at least one positive solution.*

Proof The assertion will be shown by Lemma 2.6. Note that the complete continuity of the operator Φ is guaranteed by Lemma 3.2. We only need to prove that the operator Φ satisfies the conditions (C1) and (C2) in Lemma 2.6.

Since $f^0 < \Lambda_2$ and $f_\infty > \Lambda_1$, there exist two positive numbers r_1 and γ_1 such that

$$f(t,u) \le \Lambda_2 x \quad \text{for } t \in [0,1] \text{ and } 0 \le x \le r_1, \tag{3.8}$$

$$f(t,x) \ge \Lambda_1 x \quad \text{for } t \in [0,1] \text{ and } x \ge \gamma_1. \tag{3.9}$$

It follows from Lemma 2.5 and the inequality (3.8) that, for $x \in \mathcal{P} \cap \partial\Omega_{r_1}$,

$$\|\Phi x\| = \max_{t\in[0,1]} \int_0^1 \mathcal{K}(t,s) f\bigl(s,x(s)\bigr)\,\mathrm{d}s \le \Lambda_2 \int_0^1 \mathcal{M}(s)x(s)\,\mathrm{d}s \le \|x\|.$$

Thus the operator Φ satisfies the condition (C1) in Lemma 2.6.

It remains to show that the operator Φ also satisfies the condition (C2) in Lemma 2.6. To this end, let $r_2 = \max\{2r_1, \gamma_1 q^{-1}(\delta)\}$. If $x \in \mathcal{P} \cap \partial\Omega_{r_2}$, then

$$x(t) \ge q(t)\|x\| \ge q(\delta) r_2 \ge \gamma_1 \quad \text{for } t \in [\delta, 1-\delta],$$

and hence, by the inequality (3.9),

$$f\bigl(t,x(t)\bigr) \ge \Lambda_1 x(t) \quad \text{for } t \in [\delta, 1-\delta] \text{ and } x \in \mathcal{P} \cap \partial\Omega_{r_2}.$$

Now, choose the function $\psi \equiv 1$, and clearly, ψ belongs to $\mathcal{P} \setminus \{0\}$. We then claim that

$$x \ne \Phi x + \lambda\psi$$

for $x \in \mathcal{P} \cap \partial\Omega_{r_2}$ and $\lambda > 0$. Indeed, if the preceding assertion is not true, then there exist a function $x_0 \in \mathcal{P} \cap \partial\Omega_{r_2}$ and a positive number λ_0 such that

$$x_0 = \Phi x_0 + \lambda_0 \psi.$$

Let $\bar{x}_0 = \min_{t\in[\delta,1-\delta]} x_0(t)$. Then, by the inequality (2.10), for each t in $[\delta, 1-\delta]$,

$$\begin{aligned} x_0(t) &= \int_0^1 \mathcal{K}(t,s) f\bigl(s,x_0(s)\bigr)\,\mathrm{d}s + \lambda_0 \\ &\ge \Lambda_1 q(t) \int_\delta^{1-\delta} \mathcal{M}(s)x_0(s)\,\mathrm{d}s + \lambda_0 \\ &\ge \Lambda_1 q(\delta) \int_\delta^{1-\delta} \mathcal{M}(s)\,\mathrm{d}s \cdot \bar{x}_0 + \lambda_0 \\ &= \bar{x}_0 + \lambda_0. \end{aligned}$$

Therefore, $\bar{x}_0 \ge \bar{x}_0 + \lambda_0$. This contradiction ensures that the operator Φ satisfies the condition (C2) in Lemma 2.6. Therefore, in the light of Lemma 2.6, we conclude that the operator Φ has at least one fixed point $x \in \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, and by Lemma 3.3, the fixed point x is one positive solution of the **BVP** (1.1). The proof is complete. □

Remark 3.1 The conditions in Lemma 2.6 are weaker than those in the classical norm-type expansion and compression theorem [42], and accordingly, it is generally difficult to utilize the latter to prove Theorem 3.1 and 3.2.

By Theorem 3.1 and 3.2, we directly obtain the following corollary.

Corollary 3.1 *If* $f_0 = \infty$ *and* $f^\infty = 0$, *or if* $f^0 = 0$ *and* $f_\infty = \infty$, *then the* ***BVP*** (1.1) *has at least one positive solution*.

3.1 An illustrative example

Let $\mathfrak{D} = [0,1] \times [0,\infty)$, $f(t,x) = (t+1)(2+\sin x)$, and $\lambda \in [0,2)$. Then the function f is nonnegative, and continuous on $\mathfrak{D}$. Furthermore,

$$f_0 = \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \lim_{x\to 0}\left(\frac{2}{x} + \frac{\sin x}{x}\right) = \infty$$

and

$$f^\infty = \lim_{x\to \infty} \max_{t\in[0,1]} \frac{f(t,x)}{x} = \lim_{x\to \infty}\left(\frac{4}{x} + \frac{2\sin x}{x}\right) = 0.$$

Hence, the corresponding conditions in Corollary 3.1 are satisfied for the above specified function and parameters, which implies that to the boundary value problem (1.1) there exists at least one positive solution defined on $[0,1]$.

4 Conclusion

By using the fixed point theorem in a cone, we establish some criteria for the existence of at least one positive solution to the conformable fractional differential equations with integral boundary conditions. The obtained conditions are generally weaker than those derived by using the classical norm-type expansion and compression theorem and are easy to satisfy and check. We will further investigate boundary value problems of fractional differential equations with nonsingular kernel in the future.

Acknowledgements
The authors are grateful to the referees for carefully reading the paper and for their comments and suggestions.

Funding
The paper is supported by the Natural Science Foundation of Hunan Province of China (Grant no. 11JJ3007).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Publisher's Note

Received: 14 March 2018 Accepted: 2 September 2018 Published online: 12 September 2018

References

1. Oldham, K.B., Spanier, J.: Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order. Academic Press, New York (1974)
2. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integral and Derivatives: Theory and Applications. Gordon & Breach, Switzerland (1993)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Diethelm, K.: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics, vol. 204. Springer, Berlin (2010)
5. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
6. Lundstrom, B.N., Higgs, M.H., Spain, W.J., Fairhall, A.L.: Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. **11**, 1335–1342 (2008)
7. West, B.J.: Colloquium: Fractional calculus view of complexity: a tutorial. Rev. Mod. Phys. **86**, 1169–1184 (2014)
8. Bai, Z., Chen, Y., Lian, H., Sun, S.: On the existence of blow up solutions for a class of fractional differential equations. Fract. Calc. Appl. Anal. **17**, 1175–1187 (2014)
9. Bai, Z.: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. **72**, 916–924 (2010)
10. Bai, Z., Zhang, S., Sun, S., Yin, C.: Monotone iterative method for fractional differential equations. Electron. J. Differ. Equ. **2016**, 6 (2016)
11. Khalil, R., Horani, M.A., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
12. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
13. Horani, M.A., Khalil, R.: Total fractional differentials with applications to exact fractional differential equations. Int. J. Comput. Math. **2017** (2017). https://doi.org/10.1080/00207160.2018.1438602
14. Yang, S., Wang, L., Zhang, S.: Conformable derivative: application to non-Darcian flow in low-permeability porous media. Appl. Math. Lett. **79**, 105–110 (2018)
15. Zhao, D., Luo, M.: General conformable fractional derivative and its physical interpretation. Calcolo **54**, 903–917 (2017)
16. Zhou, H.W., Yang, S., Zhang, S.Q.: Conformable derivative approach to anomalous diffusion. Physica A **491**, 1001–1013 (2018)
17. Anderson, D.R., Ulness, D.J.: Newly defined conformable derivatives. Adv. Dyn. Syst. Appl. **10**, 109–137 (2015)
18. Anderson, D.R., Ulness, D.J.: Properties of the Katugampola fractional derivative with potential application in quantum mechanics. J. Math. Phys. **56**, 063502 (2015)
19. Weberszpil, J., Helaël-Neto, J.A.: Variational approach and deformed derivatives. Physica A **450**, 217–227 (2016)
20. Katugampola, U.N.: A new fractional derivative with classical properties. e-print. arXiv:1410.6535
21. Souahi, A., Ben Makhlouf, A., Hammami, M.A.: Stability analysis of conformable fractional-order nonlinear systems. Indag. Math. **28**, 1265–1274 (2017)
22. Abdeljawad, T., Horani, M.A., Khalil, R.: Fractional semigroups of operators. J. Semigroup Theory Appl. **2015**, 7 (2015)
23. Bayour, B., Torres, D.F.M.: Existence of solution to a local fractional nonlinear differential equation. J. Comput. Appl. Math. **312**, 127–133 (2017)
24. Eslami, M.: Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations. Appl. Math. Comput. **285**, 141–148 (2016)
25. Ekici, M., Mirzazadeh, M., Eslami, M., et al.: Optical soliton perturbation with fractional-temporal evolution by first integral method with conformable fractional derivatives. Optik **127**, 10659–10669 (2016)
26. Hosseini, K., Mayeli, P., Ansari, R.: Bright and singular soliton solutions of the conformable time-fractional Klein–Gordon equations with different nonlinearities. Waves Random Complex Media **28**, 411–425 (2018)
27. Nuruddeen, R.I.: Multiple soliton solutions for the (3 + 1) conformable space-time fractional modified Korteweg–de-Vries equations. J. Ocean Eng. Sci. **3**, 11–18 (2018)
28. Korkmaz, A.: Explicit exact solutions to some one-dimensional conformable time fractional equations. Waves Random Complex Media **2017** (2017). https://doi.org/10.1080/17455030.2017.1416702
29. Cenesiz, Y., Baleanu, D., Kurt, A., Tasbozan, O.: New exact solutions of Burgers' type equations with conformable derivative. Waves Random Complex Media **27**, 103–116 (2017)
30. Zhou, Q., Sonmezoglu, A., Ekici, M., Mirzazadeh, M.: Optical solitons of some fractional differential equations in nonlinear optics. J. Mod. Opt. **64**, 2345–2349 (2017)
31. Abdeljawad, T., Alzabut, J., Jarad, F.: A generalized Lyapunov-type inequality in the frame of conformable derivatives. Adv. Differ. Equ. **2017**, 321 (2017) https://doi.org/10.1186/s13662-017-1383-z
32. Abdeljawad, T., Ravi, P., Agarwal, R.P., Alzabut, J., Jarad, F., Zbekler, A.: Lyapunov-type inequalities for mixed non-linear forced differential equations within conformable derivatives. J. Inequal. Appl. **2018**, 143 (2018)
33. Al-Rifae, M., Abdeljawad, T.: Fundamental results of conformable Sturm–Liouville eigenvalue problems. Complexity **2017**, Article ID 3720471 (2017)
34. Dong, X., Bai, Z., Zhang, W.: Positive solutions for nonlinear eigenvalue problems with conformable fractional differential derivatives. J. Shandong Univ. Sci. Tech. Nat. Sci. (Chin. Ed.) **35**, 85–90 (2016)
35. Song, Q., Dong, X., Bai, Z., Chen, B.: Existence for fractional Dirichlet boundary value problem under barrier strip conditions. J. Nonlinear Sci. Appl. **10**, 3592–3598 (2017)
36. He, L., Dong, X., Bai, Z., Chen, B.: Solvability of some two-point fractional boundary value problems under barrier strip conditions. J. Funct. Spaces **2017**, Article ID 1465623 (2017)
37. Batarfi, H., Losada, J., Nieto, J.J., Shammakh, W.: Three-point boundary value problems for conformable fractional differential equations. J. Funct. Spaces **2015**, Article ID 706383 (2015)
38. Asawasamrit, S., Ntouyas, S.K., Thiramanus, P., Tariboon, J.: Periodic boundary value problems for impulsive conformable fractional integrodifferential equations. Bound. Value Probl. **2016**, 122 (2016)
39. Dong, X., Bai, Z.: Positive solutions to boundary value problems of p-Laplacian with fractional derivative. Bound. Value Probl. **2017**, 5 (2017)
40. Abdeljawad, T.: Fractional operators with exponential kernels and a Lyapunov type inequality. Adv. Differ. Equ. **2017**, 313 (2017)

41. Abdeljawad, T.: A Lyapunov type inequality for fractional operators with nonsingular Mittag–Leffler kernel. J. Inequal. Appl. **2017**, 130 (2017). https://doi.org/10.1186/s13660-017-1400-5
42. Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)
43. Lan, K., Webb, J.R.L.: Positive solutions of semilinear differential equations with singularities. J. Differ. Equ. **148**, 407–421 (1998)