Kang et al. *Boundary Value Problems* (2018) 2018:139
https://doi.org/10.1186/s13661-018-1057-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A global nonexistence of solutions for a quasilinear viscoelastic wave equation with acoustic boundary conditions

Yong Han Kang[1], Jong Yeoul Park[2] and Daewook Kim[3*]

*Correspondence:
kdw@seowon.ac.kr
[3]Department of Mathematics and Education, Seowon University, Cheongju, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we consider a quasilinear viscoelastic wave equation with acoustic boundary conditions. Under some appropriate assumption on the relaxation function g, the function Φ, $p > \max\{\rho + 2, m, q, 2\}$, and the initial data, we prove a global nonexistence of solutions for a quasilinear viscoelastic wave equation with positive initial energy.

MSC: 35L70; 35B40; 76EXX

Keywords: Global nonexistence of solution; Blow-up of solution; Quasilinear viscoelastic wave equation; Acoustic boundary conditions

1 Introduction

In this paper, we are concerned with the following a quasi-nonlinear viscoelastic wave equation with acoustic boundary conditions:

$$\begin{aligned} &\left|u_t(t)\right|^\rho u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\,ds \\ &\quad + \left|u_t(t)\right|^{m-2} u_t(t) = \left|u(t)\right|^{p-2} u(t) \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{1}$$

$$u(t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{2}$$

$$\frac{\partial u(t)}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,ds + \Phi\bigl(u_t(t)\bigr) = h(x)y_t(t) \quad \text{on } \Gamma_1 \times (0,\infty), \tag{3}$$

$$u_t(t) + f(x)y_t(t) + q(x)y(t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{4}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{5}$$

$$y(x,0) = y_0(x) \quad \text{on } \Gamma_1, \tag{6}$$

where Ω is a regular and bounded domain of $\mathbb{R}^n$ ($n \geq 1$), and $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here Γ_0, Γ_1 are closed and disjoint and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative to Γ. The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive nonincreasing function, the function $\Phi : \mathbb{R} \to \mathbb{R}$ is a monotone and continuous, and the functions $f, q, h : \Gamma_1 \to \mathbb{R}_+$ are essentially bounded and $q(x) \geq q_0 > 0$.

System (1)–(6) is a model of a quasilinear viscoelastic wave equation with acoustic boundary conditions. The acoustic boundary conditions were introduced by Morse and

Ingard [14] in 1968 and developed by Beale and Rosencrans in [1], where the authors proved the global existence and regularity of the nonlinear problem. When $|u_t(t)|^\rho$ is not a constant, system (1)–(6) can model materials whose density depends on the velocity u_t. The physical application of the above system is the problem of noise suppression in structural acoustic systems, which is one of great interests in physics and engineering. Also reducing the level of pressure in a helicopter's cabin and suppressing the noise in the interior of an acoustic chamber are based on some special type of boundary conditions like those described in system (1)–(6), (see [4, 5] and another case [9]).

Boukhatem and Benabderrahmane [2, 3] studied the existence, blow-up, and decay of solutions for viscoelastic wave equations with acoustic boundary conditions. Recently, many authors have treated wave/beam equations with acoustic boundary conditions, see [7, 8, 10, 12, 13, 15, 16] and the references therein. Graber and Haid-Houari [5] studied the blow-up solutions for a nonlinear wave equation with porous acoustic boundary conditions:

$$
\begin{aligned}
&u_{tt}(t) - \Delta u(t) + \alpha(x)u(t) + \phi(u_t) = j_1\bigl(u(t)\bigr) \quad \text{in } \Omega \times (0,\infty),\\
&u(t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
&u_t(t) + f(x)z_t(t) + g(x)z(t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty),\\
&\frac{\partial u(t)}{\partial \nu} - h(x)\eta\bigl(z_t(t)\bigr) + \rho\bigl(u_t(t)\bigr) = j_2\bigl(u(t)\bigr) \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega,\\
&z(x,0) = z_0(x) \quad \text{on } \Gamma_1,
\end{aligned}
$$

where $\alpha : \Omega \to \mathbb{R}$ and $f, g, h : \overline{\Gamma}_1 \to \mathbb{R}$ are given functions. Also the functions j_1 and j_2 are of a polynomial structure as follows: $j_1(s) = |s|^{p-2}s$, $j_2(s) = |s|^{k-2}sk$, $p \geq 2$, the functions ρ and ϕ are monotone, continuous, and there exist four positive constants m_q, M_q, c_r, and C_r such that $m_q|s|^q \leq \rho(s)s \leq M_q|s|^q$, $c_r|s|^r \leq \phi(s)s \leq C_r|s|^r$. In addition, Di et al. [4] studied a viscoelastic wave equation with nonlinear boundary source term:

$$
\begin{aligned}
&\bigl|u_t(t)\bigr|^\rho u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\,ds = 0 \quad \text{in } \Omega \times (0,\infty),\\
&u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
&\frac{\partial u}{\partial \nu}(t) - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)\,ds = f\bigl(u(t)\bigr) \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega,
\end{aligned}
$$

where $\rho \geq 1$ and Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\Gamma := \partial\Omega$. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, and $\operatorname{meas}(\Gamma_0) > 0$. Here, ν is the unit outward normal to Γ, and g, f are given functions satisfying suitable conditions. They introduced a family of potential wells and proved the invariance of some sets. Then they established the existence and nonexistence of a global weak solution with small initial energy under suitable assumptions on $g(\cdot), f(\cdot)$, initial data, and the parameters in the equation. Also they showed the global existence of a weak solution for the problem with critical initial conditions $I(u_0) \geq 0$ and $e(0) = d$. Furthermore, Song [17] studied the nonlinear viscoelastic wave equation

$$\begin{aligned}
&\left|u_t(t)\right|^{\rho} u_{tt}(t) - \Delta u(t) + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau \\
&\quad + \left|u_t(t)\right|^{m-2} u_t(t) = \left|u(t)\right|^{p-2} u(t) \quad \text{in } \Omega \times [0,T], \\
&u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T], \\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega,
\end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, $m \geq 2$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ a positive nonincreasing function, and

$$2 < p, \rho < \infty, \quad \text{if } n = 1, 2, \qquad 2 < p, \rho \leq \frac{2(n-1)}{n-2} \quad \text{if } n \geq 3.$$

The author proved the global nonexistence of positive initial energy solutions for a viscoelastic wave equation. Recently Jeong et al. [6] investigated the quasilinear wave equation with acoustic boundary conditions

$$\begin{aligned}
&u_{tt}(t) - \Delta u_t(t) - \operatorname{div}\left(\left|\nabla u(t)\right|^{\alpha-2} \nabla u(t)\right) - \operatorname{div}\left(\left|\nabla u_t(t)\right|^{\beta-2} \nabla u_t(t)\right) + a\left|u_t(t)\right|^{m-2} u_t(t) \\
&\quad = \left|u(t)\right|^{p-2} u(t) \quad \text{in } \Omega \times (0,\infty), \\
&u(t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
&\frac{\partial u_t(t)}{\partial \nu} + \left|\nabla u(t)\right|^{\alpha-2} \frac{\partial u(t)}{\partial \nu} + \left|\nabla u_t(t)\right|^{\beta-2} \frac{\partial u_t(t)}{\partial \nu} = h(x) y_t(t) \quad \text{on } \Gamma_1 \times (0,\infty), \\
&u_t(t) + f(x) y_t(t) + q(x) y(t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\
&u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\
&y(x,0) = y_0(x) \quad \text{on } \Gamma_1,
\end{aligned}$$

where $a, b > 0$, $\alpha, \beta, m, p > 2$, Ω is a regular and bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ and $\partial\Omega(= \Gamma) = \Gamma_0 \cup \Gamma_1$. The functions $f, q, h : \Gamma_1 \to \mathbb{R}_+$ are essentially bounded. They studied the global nonexistence of solutions for a quasilinear wave equation with acoustic boundary conditions. Motivated by the previous works [5, 17], we consider problem (1)–(6). Under suitable assumptions on the relaxation function g, the nonlinear function $\Phi(\cdot)$, $p > \max\{\rho + 2, m, q, 2\}$, the initial data, and the parameters in the system, we prove the nonexistence of a weak solution with small positive initial energy.

2 Blow-up result

In this section, we present some material which will be used throughout this work. First, we introduce the set

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_0\right\},$$

and endow $H^1_{\Gamma_0}(\Omega)$ with the Hilbert structure induced by $H^1(\Omega)$. We have that $H^1_{\Gamma_0}(\Omega)$ is a Hilbert space. For simplicity, we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{p,\Gamma} = \|\cdot\|_{L^p(\Gamma)}$, $1 \leq p \leq \infty$.

We present some assumptions and preliminaries needed in the proof of our main result.

We make the following assumptions:

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$1 - \int_0^\infty g(s)\,ds = l > 0, \qquad g(t) \geq 0, \qquad g'(t) \leq 0, \quad \forall t \geq 0. \tag{7}$$

(H2) For the nonlinear terms, we have

$$2 < p \leq \frac{2(n-1)}{n-2} \quad \text{if } n \geq 3 \quad \text{and} \quad p > 2 \quad \text{if } n = 1, 2, \tag{8}$$

$$2 < \rho \leq \frac{2}{n-2} \quad \text{if } n \geq 3 \quad \text{and} \quad \rho > 0 \quad \text{if } n = 1, 2. \tag{9}$$

(H3) $\Phi : \mathbb{R} \to \mathbb{R}$ is monotone, continuous, and there exist positive constants m_q and M_q such that

$$m_q |s|^q \leq \Phi(s)s \leq M_q |s|^q, \quad \forall s \in \mathbb{R}. \tag{10}$$

(H4) The functions f, q, h are essentially bounded such that

$$f(x) > 0, \qquad q(x) > 0 \quad \text{and} \quad h(x) > 0, \quad \forall x \in \Gamma_1.$$

We state, without a proof, a local existence which can be established by combining arguments of [4, 5].

Let assumptions (H1)–(H4) hold, $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, and $y_0 \in L^2(\Gamma_1)$. Then problem (1)–(6) admits a weak local solution (u, y) such that, for some $T > 0$,

$$\begin{aligned}
&u \in L^\infty\bigl([0,T); H^1_{\Gamma_0}(\Omega)\bigr),\\
&u_t \in L^\infty\bigl([0,T); L^2(\Omega)\bigr) \cap L^m\bigl([0,T); \Omega\bigr) \cap L^q\bigl([0,T); \Gamma_1\bigr),\\
&y \in L^2\bigl([0,T); \Gamma_1\bigr).
\end{aligned}$$

To obtain the global nonexistence result, we need the following lemmas.

Lemma 2.1 *Assume that* (H1)–(H4) *hold. Let $u(t)$ be a solution of problem* (1)–(6). *Then the energy functional $E(t)$ of problem* (1)–(6) *is nonincreasing. Moreover, the following energy inequality holds*:

$$\begin{aligned}
E'(t) &= \frac{1}{2}\bigl(g' \circ \nabla u\bigr)(t) - \frac{1}{2} g(t) \bigl\| \nabla u(t) \bigr\|^2 - \bigl\| u_t(t) \bigr\|_m^m \\
&\quad - \int_{\Gamma_1} h(x) f(x) y_t^2(t)\,d\Gamma - \int_{\Gamma_1} \Phi\bigl(u_t(t)\bigr) u_t(t)\,d\Gamma \\
&\leq 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
E(t) &= \frac{1}{\rho+2} \bigl\| u_t(t) \bigr\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right) \bigl\| \nabla u(t) \bigr\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\
&\quad - \frac{1}{p} \bigl\| u(t) \bigr\|_p^p + \frac{1}{2} \int_{\Gamma_1} h(x) q(x) y^2(t)\,d\Gamma,
\end{aligned} \tag{12}$$

and

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)\big\|\nabla u(t) - \nabla u(s)\big\|^2\,ds. \tag{13}$$

Lemma 2.2 *Suppose that* (H1)–(H4) *hold. Let $u(t)$ be the solution of problem* (1)–(6). *Furthermore, assume that*

$$E(0) < E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)B_1^{-\frac{2p}{p-2}}$$

and

$$\|\nabla u_0\| \geq B_1^{-\frac{p}{p-2}},$$

where $B_1 = B/l^{\frac{1}{2}}$ and B is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Then there exists a constant $\beta > B_1^{-\frac{p}{p-2}}$ such that

$$\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 \geq \beta^2, \quad \forall t > 0 \tag{14}$$

and

$$\big\|u(t)\big\|_p \geq B_1\beta, \quad \forall t > 0. \tag{15}$$

Proof From (6) and the embedding theorem, we have

$$\begin{aligned}
E(t) &= \frac{1}{\rho+2}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\
&\quad - \frac{1}{p}\big\|u(t)\big\|_p^p + \frac{1}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma \\
&\geq \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 - \frac{1}{p}\big\|u(t)\big\|_p^p \\
&\geq \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 - \frac{1}{p}B_1^p l^{\frac{p}{2}}\big\|\nabla u(t)\big\|^p \\
&\geq \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 - \frac{1}{p}B_1^p\left(\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2\right)^{\frac{p}{2}} \\
&= \frac{1}{2}\xi^2 - \frac{B_1^p}{p}\xi^p := G(\xi),
\end{aligned} \tag{16}$$

where $\xi = ((1 - \int_0^t g(s)\,ds)\|\nabla u(t)\|^2)^{\frac{1}{2}}$. It is easy to see that $G(\xi)$ takes its maximum for $\xi = \xi^* = B_1^{-\frac{p}{p-2}}$, which is strictly increasing for $0 < \xi < \xi^*$, strictly decreasing for $\xi > \xi^*$, $G(\xi) \to -\infty$ as $\xi \to \infty$, and

$$G(\xi^*) = \left(\frac{1}{2} - \frac{1}{p}\right)B_1^{-\frac{2p}{p-2}} = E_1.$$

Since $E(0) < E_1$, there exists $\beta > \xi^*$ such that $G(\beta) = E(0)$. Set $\xi_0 = \|\nabla u(0)\|$, by (16), we see that

$$G(\xi_0) \leq E(0) = G(\beta),$$

which implies that

$$\|\nabla u_0\| = \xi_0 > \beta.$$

To prove (14), we suppose on the contrary that

$$\left(\left(1 - \int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2\right)^{\frac{1}{2}} < \beta$$

for some $t = t_0 > 0$. By the continuity of $(1 - \int_0^t g(s)\,ds)\|\nabla u(t)\|^2$, we may choose t_0 such that

$$\beta > \left(\left(1 - \int_0^{t_0} g(s)\,ds\right)\left\|\nabla u(t_0)\right\|^2\right)^{\frac{1}{2}} > \xi^*.$$

Then it follows from (16) that

$$E(t_0) \geq G\left(\left(\left(1 - \int_0^{t_0} g(s)\,ds\right)\left\|\nabla u(t_0)\right\|^2\right)^{\frac{1}{2}}\right) > G(\beta) = E(0),$$

which contradicts Lemma 2.1. Hence (14) is proved. Now we will prove (15). From (12), (13), (14), and Lemma 2.1, we deduce that

$$\begin{aligned}
\frac{1}{p}\left\|u(t)\right\|_p^p &= \frac{1}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\
&\quad + \frac{1}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma - E(t) \\
&\geq \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2 - E(0) \\
&\geq \frac{1}{2}\beta^2 - E(0) = \frac{1}{2}\beta^2 - G(\beta) \\
&= \frac{1}{2}\beta^2 - \left(\frac{1}{2}\beta^2 - \frac{B_1^p}{p}\beta^p\right) = \frac{B_1^p}{p}\beta^p \quad \forall t > 0.
\end{aligned}$$

Thus the proof of Lemma 2.2 is complete. □

Theorem 2.1 *Let* $2 < m < p$, $2 \leq q < p$ *and assume that* (H1)–(H4) *hold. Suppose that* $\rho < p - 2$, $0 < \varepsilon_0 < \frac{p}{2} - 1$, *and*

$$\int_0^\infty g(s)\,ds < \frac{\frac{p}{2} - CM_q\frac{\lambda^q}{q} - 1 - \varepsilon_0}{\frac{p}{2} - 1 - CM_q\frac{\lambda^q}{q} + \frac{1}{2p}} \tag{17}$$

are satisfied, then there exists no global solution of problem (1)–(6) *if*

$$E(0) < \left(1 - CM_q \frac{\lambda^q}{q} \frac{2}{p-2} - \frac{1}{p(p-2)} \frac{1-l}{l}\right)\left(\frac{1}{2} - \frac{1}{p}\right) B_1^{-\frac{2p}{p-2}}, \tag{18}$$

and

$$\|\nabla u_0\| > B_1^{-\frac{p}{p-2}}. \tag{19}$$

Proof Assume that the solution $u(t)$ of (1)–(6) is global. We set

$$H(t) = E_2 - E(t), \tag{20}$$

where the constant $E_2 \in (E(0), E_1)$ shall be chosen later. By Lemma 2.1, the function $H(t)$ is increasing. Then, for $t \geq s \geq 0$,

$$\begin{aligned} 0 < H(0) &\leq H(s) \leq H(t) \\ &= E_2 - \frac{1}{\rho+2} \|u_t(t)\|_{\rho+2}^{\rho+2} - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t) \\ &\quad + \frac{1}{p}\|u(t)\|_p^p - \frac{1}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \end{aligned} \tag{21}$$

Thus from (14) we get

$$\begin{aligned} H(t) &\leq E_2 - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{p}\|u(t)\|_p^p \\ &\leq E_1 - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{p}\|u(t)\|_p^p \\ &\leq E_1 - \frac{1}{2} B_1^{-\frac{2p}{p-2}} + \frac{1}{p}\|u(t)\|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) B_1^{-\frac{2p}{p-2}} - \frac{1}{2} B_1^{-\frac{2p}{p-2}} + \frac{1}{p}\|u(t)\|_p^p \\ &\leq \frac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{22}$$

Now, we define

$$\begin{aligned} L(t) &= H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx \\ &\quad - \frac{\varepsilon}{2}\int_{\Gamma_1} h(x)f(x)y^2(t)\,d\Gamma - \varepsilon\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma, \end{aligned} \tag{23}$$

where the constants $0 < \sigma < 1$, $\varepsilon > 0$ shall be chosen later.

Taking a derivative of (23), using (7)–(10) and Lemma 2.1, we have

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} + \varepsilon\int_\Omega |u_t(t)|^\rho u_{tt}(t)u(t)\,dx \\ &\quad - \varepsilon\int_{\Gamma_1} h(x)f(x)y(t)y_t(t)\,d\Gamma - \varepsilon\int_{\Gamma_1} h(x)u_t(t)y(t)\,d\Gamma \end{aligned}$$

$$
\begin{aligned}
&- \varepsilon \int_{\Gamma_1} h(x)u(t)y_t(t)\,d\Gamma \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} \\
&\quad + \varepsilon \int_\Omega u(t)\left[\Delta u(t) - \int_0^t g(t-s)\Delta u(s)\,ds - \big|u_t(t)\big|^{m-2}u_t(t) + \big|u(t)\big|^{p-2}u(t)\right]dx \\
&\quad - \varepsilon \int_{\Gamma_1} h(x)f(x)y(t)y_t(t)\,d\Gamma - \varepsilon \int_{\Gamma_1} h(x)u_t(t)y(t)\,d\Gamma \\
&\quad - \varepsilon \int_{\Gamma_1} h(x)u(t)y_t(t)\,d\Gamma \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} \\
&\quad - \varepsilon \int_\Omega \big|\nabla u(t)\big|^2\,dx + \varepsilon \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\
&\quad - \varepsilon \int_\Omega \big|u_t(t)\big|^{m-2}u_t(t)u(t)\,dx + \varepsilon\big\|u(t)\big\|_p^p \\
&\quad + \varepsilon \int_{\Gamma_1} u(t)\left(\frac{\partial u(t)}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,ds\right)d\Gamma \\
&\quad - \varepsilon \int_{\Gamma_1} h(x)y(t)\big(f(x)y_t(t) + u_t(t)\big)\,d\Gamma \\
&\quad - \varepsilon \int_{\Gamma_1} h(x)u(t)y_t(t)\,d\Gamma \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} - \varepsilon\big\|\nabla u(t)\big\|^2 \\
&\quad + \varepsilon \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \varepsilon \int_\Omega \big|u_t(t)\big|^{m-2}u_t(t)u(t)\,dx \\
&\quad + \varepsilon\big\|u(t)\big\|_p^p - \varepsilon \int_{\Gamma_1} \Phi\big(u_t(t)\big)u(t)\,d\Gamma + \varepsilon \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma.
\end{aligned}
\tag{24}
$$

Exploiting Hölder's and Young's inequalities, for any ε_1 $(0 < \varepsilon_1 < 1)$, we obtain

$$
\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\
&\quad = \int_0^t g(t-s)\int_\Omega \nabla u(t)\big(\nabla u(s) - \nabla u(t)\big)\,dx\,ds + \left(\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 \\
&\quad \geq -\frac{p(1-\varepsilon_1)}{2}(g\circ\nabla u)(t) + \left(1 - \frac{1}{2p(1-\varepsilon_1)}\right)\left(\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2.
\end{aligned}
\tag{25}
$$

Thus from (24) and (25), we arrive at

$$
\begin{aligned}
L'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} - \frac{\varepsilon p(1-\varepsilon_1)}{2}(g\circ\nabla u)(t) \\
&\quad - \varepsilon\left[1 - \left(1 - \frac{1}{2p(1-\varepsilon_1)}\right)\int_0^t g(s)\,ds\right]\big\|\nabla u(t)\big\|^2
\end{aligned}
$$

$$- \varepsilon \int_\Omega \left|u_t(t)\right|^{m-2} u_t(t)u(t)\,dx - \varepsilon \int_{\Gamma_1} \Phi\bigl(u_t(t)\bigr)u(t)\,d\Gamma + \varepsilon \left\|u(t)\right\|_p^p + \varepsilon \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \tag{26}$$

Consequently, from (11), (12), (20), and (26), we deduce that

$$\begin{aligned} L'(t) \geq{}& (1-\sigma)H^{-\sigma}(t)\biggl[\frac{1}{2}g(t)\left\|\nabla u(t)\right\|^2 - \frac{1}{2}\bigl(g' \circ \nabla u\bigr)(t) + \left\|u_t(t)\right\|_m^m \\ &+ \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \int_{\Gamma_1} \Phi\bigl(u_t(t)\bigr)u_t(t)\,d\Gamma\biggr] \\ &+ \frac{\varepsilon}{\rho+1}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} - \varepsilon\biggl[1 - \biggl(1 - \frac{1}{2p(1-\varepsilon_1)}\biggr)\int_0^t g(s)\,ds\biggr]\left\|\nabla u(t)\right\|^2 \\ &- \frac{\varepsilon p(1-\varepsilon_1)}{2}(g\circ\nabla u)(t) - \varepsilon\int_\Omega \left|u_t(t)\right|^{m-2}u_t(t)u(t)\,dx \\ &+ \varepsilon\left\|u(t)\right\|_p^p - \varepsilon\int_{\Gamma_1}\Phi\bigl(u_t(t)\bigr)u(t)\,d\Gamma + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \end{aligned} \tag{27}$$

From this relation and using

$$\begin{aligned} \varepsilon(1-\varepsilon_1)pH(t) ={}& \varepsilon(1-\varepsilon_1)pE_2 - \frac{\varepsilon p(1-\varepsilon_1)}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} \\ &- \frac{\varepsilon p(1-\varepsilon_1)}{2}\biggl(1 - \int_0^t g(s)\,ds\biggr)\left\|\nabla u(t)\right\|^2 \\ &- \frac{\varepsilon p(1-\varepsilon_1)}{2}(g\circ\nabla u)(t) + \varepsilon(1-\varepsilon_1)\left\|u(t)\right\|_p^p \\ &- \frac{\varepsilon p(1-\varepsilon_1)}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma, \end{aligned}$$

it follows that

$$\begin{aligned} L'(t) \geq{}& (1-\sigma)H^{-\sigma}(t)\left\|u_t(t)\right\|_m^m + (1-\sigma)H^{-\sigma}(t)\int_{\Gamma_1}\Phi\bigl(u_t(t)\bigr)u_t(t)\,d\Gamma \\ &+ \varepsilon(1-\varepsilon_1)pH(t) - \varepsilon(1-\varepsilon_1)pE_2 \\ &+ \varepsilon\biggl(\frac{1}{\rho+1} + \frac{p(1-\varepsilon_1)}{\rho+2}\biggr)\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} \\ &+ \varepsilon\varepsilon_1\left\|u(t)\right\|_p^p - \varepsilon\int_\Omega\left|u_t(t)\right|^{m-2}u_t(t)u(t)\,dx \\ &- \varepsilon\int_{\Gamma_1}\Phi\bigl(u_t(t)\bigr)u(t)\,d\Gamma + \varepsilon\biggl(\frac{(1-\varepsilon_1)p}{2}+1\biggr)\int_{\Gamma_1}h(x)q(x)y^2(t)\,d\Gamma \\ &+ \varepsilon\biggl(\frac{(1-\varepsilon_1)p}{2}-1\biggr)\biggl(1-\int_0^t g(s)\,ds\biggr)\left\|\nabla u(t)\right\|^2 \\ &- \frac{\varepsilon}{2p(1-\varepsilon_1)}\int_0^t g(s)\,ds\left\|\nabla u(t)\right\|^2 \\ \geq{}& (1-\sigma)H^{-\sigma}(t)\left\|u_t(t)\right\|_m^m + pH(t)\varepsilon(1-\varepsilon_1) \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\left(\frac{1}{\rho+1}+\frac{p(1-\varepsilon_1)}{\rho+2}\right)\|u_t(t)\|_{\rho+2}^{\rho+2}\\
&+\varepsilon\left(\frac{(1-\varepsilon_1)p}{2}-1\right)\left(1-\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2\\
&-\frac{\varepsilon}{2p(1-\varepsilon_1)}\int_0^t g(s)\,ds\|\nabla u(t)\|^2-\varepsilon(1-\varepsilon_1)pE_2+\varepsilon\varepsilon_1\|u(t)\|_p^p\\
&-\varepsilon\int_\Omega|u_t(t)|^{m-2}u_t(t)u(t)\,d\Gamma-\varepsilon\int_{\Gamma_1}\Phi(u_t(t))u(t)\,d\Gamma\\
&+(1-\sigma)H^{-\sigma}(t)\int_{\Gamma_1}\Phi(u_t(t))u_t(t)\,d\Gamma\\
&+\varepsilon\left(\frac{(1-\varepsilon_1)p}{2}+1\right)\int_{\Gamma_1}h(x)q(x)y^2(t)\,d\Gamma. \qquad (28)
\end{aligned}
$$

From Hölder's and Young's inequalities, the condition $m < p$, (22), and the embedding theorem ($L^p(\Omega) \hookrightarrow L^m(\Omega)$), we obtain

$$
\begin{aligned}
\int_\Omega|u_t(t)|^{m-2}u_t(t)u(t)\,d\Gamma &\le \left(\int_\Omega|u_t(t)|^m\,dx\right)^{\frac{m-1}{m}}\left(\int_\Omega|u(t)|^m\,dx\right)^{\frac{1}{m}}\\
&\le \|u_t(t)\|_m^{m-1}\|u(t)\|_m\\
&\le C\|u_t(t)\|_m^{m-1}\|u(t)\|_p\\
&\le C\|u_t(t)\|_m^{m-1}\|u(t)\|_p^{1-\frac{p}{m}}\|u(t)\|_p^{\frac{p}{m}}\\
&\le C\|u(t)\|_p^{1-\frac{p}{m}}\left(\varepsilon_1\|u(t)\|_p^p+C(\varepsilon_1)\|u_t(t)\|_m^m\right)\\
&\le CH(t)^{\frac{1}{p}-\frac{1}{m}}\left(\varepsilon_2\|u(t)\|_p^p+C(\varepsilon_2)\|u_t(t)\|_m^m\right), \qquad (29)
\end{aligned}
$$

where C is a generic positive constant which might change from line to line and $\varepsilon_2 > \varepsilon_1 p^{1/p-1/m}$.

Here we choose

$$
0<\sigma<\min\left(\frac{1}{\rho+2}-\frac{1}{p},\frac{1}{m}-\frac{1}{p},\frac{1}{2}-\frac{1}{p}\right) \qquad (30)
$$

and take $\alpha = \frac{m-p}{pm} + \sigma = -(\frac{1}{m} - \frac{1}{p}) + \sigma < 0$. Then the properties (21) of the function $H(t)$ show that

$$
H(t)^{\frac{1}{p}-\frac{1}{m}}=H(t)^{-\sigma}H(t)^{\alpha}\le H(t)^{-\sigma}H(0)^{\alpha}.
$$

Thus from inequality (30) it follows

$$
\int_\Omega|u_t(t)|^{m-2}u_t(t)u(t)\,d\Gamma\le CH(t)^{-\sigma}H(0)^{\alpha}\left(\varepsilon_2\|u(t)\|_p^p+C(\varepsilon_2)\|u_t(t)\|_m^m\right). \qquad (31)
$$

Moreover, from (10), it is clear that

$$
\int_{\Gamma_1}\Phi(u_t(t))u_t(t)\,d\Gamma\ge m_q\|u_t(t)\|_{q,\Gamma_1}^q \qquad (32)
$$

and the following Young's inequality

$$XY \le \frac{\lambda^{\gamma} X^{\gamma}}{\gamma} + \frac{\lambda^{-\beta} Y^{\beta}}{\beta},$$

$X, Y \ge 0$, $\lambda > 0$, $\gamma, \beta \in \mathbb{R}_+$ such that $\frac{1}{\gamma} + \frac{1}{\beta} = 1$, then from (10), we get

$$\begin{aligned} \int_{\Gamma_1} \Phi\big(u_t(t)\big) u(t)\, d\Gamma &\le M_q \int_{\Gamma_1} \big|u_t(t)\big|^{q-1} \big|u(t)\big|\, d\Gamma \\ &\le M_q \frac{\lambda^q}{q} \big\|u(t)\big\|_{q,\Gamma_1}^q + M_q \frac{q-1}{q} \lambda^{-\frac{q}{q-1}} \big\|u_t(t)\big\|_{q,\Gamma_1}^q. \end{aligned} \tag{33}$$

Thus from (28) and (31)–(33), we deduce

$$\begin{aligned} L'(t) \ge{}& (1-\sigma) H^{-\sigma}(t) \big\|u_t(t)\big\|_m^m + \varepsilon \left[\frac{1}{\rho+1} + \frac{p(1-\varepsilon_1)}{\rho+1} \right] \big\|u_t(t)\big\|_{\rho+2}^{\rho+2} + \varepsilon (1-\varepsilon_1) p H(t) \\ &+ \varepsilon \left[\left(\frac{(1-\varepsilon_1)p}{2} - 1 \right) \left(1 - \int_0^t g(s)\, ds \right) - \frac{1}{2p(1-\varepsilon_1)} \int_0^t g(s)\, ds \right] \big\|\nabla u(t)\big\|^2 \\ &- \varepsilon (1-\varepsilon_1) p E_2 + \varepsilon \varepsilon_1 \big\|u(t)\big\|_p^p \\ &- \varepsilon C H(t)^{-\sigma} H(0)^{\alpha} \big(\varepsilon_2 \big\|u(t)\big\|_p^p + C(\varepsilon_2) \big\|u_t(t)\big\|_m^m \big) \\ &+ (1-\sigma) H^{-\sigma}(t) m_q \big\|u_t(t)\big\|_{q,\Gamma_1}^q \\ &- \varepsilon M_q \frac{\lambda^q}{q} \big\|u(t)\big\|_{q,\Gamma_1}^q - \varepsilon M_q \frac{q-1}{q} \lambda^{-\frac{q}{q-1}} \big\|u_t(t)\big\|_{q,\Gamma_1}^q \\ &+ \varepsilon \left(\frac{(1-\varepsilon_1)p}{2} + 1 \right) \int_{\Gamma_1} h(x) q(x) y^2(t)\, d\Gamma \\ ={}& H^{-\sigma}(t) \big[1 - \sigma - \varepsilon C H^{\alpha}(0) C(\varepsilon_2) \big] \big\|u_t(t)\big\|_m^m + \varepsilon \left[\frac{1}{\rho+1} + \frac{p(1-\varepsilon_1)}{\rho+2} \right] \big\|u_t(t)\big\|_{\rho+2}^{\rho+2} \\ &+ \varepsilon (1-\varepsilon_1) p H(t) - \varepsilon (1-\varepsilon_1) p E_2 \\ &+ \varepsilon \left[\left(\frac{(1-\varepsilon_1)p}{2} - 1 \right) \left(1 - \int_0^t g(s)\, ds \right) - \frac{1}{2p(1-\varepsilon_1)} \int_0^t g(s)\, ds \right] \big\|\nabla u(t)\big\|^2 \\ &+ \varepsilon \big[\varepsilon_1 - \varepsilon_2 C H^{-\sigma}(t) H^{\alpha}(0) \big] \big\|u(t)\big\|_p^p - \varepsilon M_q \frac{\lambda^q}{q} \big\|u(t)\big\|_{q,\Gamma_1}^q \\ &+ \left[(1-\sigma) H^{-\sigma}(t) m_q - \varepsilon M_q \frac{q-1}{q} \lambda^{-\frac{q}{q-1}} \right] \big\|u_t(t)\big\|_{q,\Gamma_1}^q \\ &+ \varepsilon \left(\frac{(1-\varepsilon_1)p}{2} + 1 \right) \int_{\Gamma_1} h(x) q(x) y^2(t)\, d\Gamma, \quad \forall t \ge T_0. \end{aligned} \tag{34}$$

We also use the embedding theorem. Let us recall the inequality (C denotes a generic positive constant)

$$\big\|u(t)\big\|_{q,\Gamma_1} \le C \big\|u(t)\big\|_{H^s(\Omega)},$$

where $q \geq 1$ and $0 \leq s < 1$, $s \geq \frac{N}{2} - \frac{N-1}{q} > 0$ and the interpolation and Poincaré's inequality (see [11])

$$\|u(t)\|_{H^s(\Omega)} \leq C\|u(t)\|^{1-s}\|\nabla u(t)\|^s \leq C\|u(t)\|_p^{1-s}\|\nabla u(t)\|^s.$$

If $s < \frac{2}{q}$, using again Young's inequality, we obtain

$$\|u(t)\|_{q,\Gamma_1}^q \leq C\left[\left(\|u(t)\|_p^p\right)^{\frac{q(1-s)\mu}{p}} + \left(\|\nabla u(t)\|^2\right)^{\frac{qs\theta}{2}}\right]$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Here we choose $\theta = \frac{2}{qs}$ to get $\mu = \frac{2}{2-qs}$. Therefore the previous inequality

$$\|u(t)\|_{q,\Gamma_1}^q \leq C\left[\left(\|u(t)\|_p^p\right)^{\frac{2q(1-s)}{(2-qs)p}} + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2\right]. \tag{35}$$

Now, choosing s such that

$$0 < s \leq \frac{2(p-q)}{q(p-2)},$$

we get

$$\frac{2q(1-s)}{(2-qs)p} \leq 1. \tag{36}$$

Once inequality (36) is satisfied, we use the classical algebraic inequality

$$\chi^\nu \leq (\chi + 1) \leq \left(1 + \frac{1}{w}\right)(\chi + w), \quad \forall \chi \geq 0, 0 < \nu \leq 1, w \geq 0, \tag{37}$$

with $\chi = \|u(t)\|_p^p$, $d = 1 + \frac{1}{H(0)}$, $w = H(0)$, and $\nu = \frac{2q(1-s)}{(2-qs)p}$ to get the following estimate:

$$\left(\|u(t)\|_p^p\right)^{\frac{2q(1-s)}{(2-qs)p}} \leq d\left(\|u(t)\|_p^p + H(0)\right) \leq d\left(\|u(t)\|_p^p + H(t)\right), \quad \forall t \geq 0. \tag{38}$$

From (35) and (38), we have

$$\|u(t)\|_{q,\Gamma_1}^q \leq C\left(\|u(t)\|_p^p + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + 2H(t)\right). \tag{39}$$

Inserting estimate (39) into (34) and using (14), we arrive at

$$\begin{aligned}
L'(t) \geq{}& H^{-\sigma}(t)\left[1 - \sigma - \varepsilon C H^\alpha(0) C(\varepsilon_2)\right]\|u_t(t)\|_m^m \\
&+ \varepsilon\left[\frac{1}{\rho+1} + \frac{(1-\varepsilon_1)p}{\rho+2}\right]\|u_t(t)\|_{\rho+2}^{\rho+2} + \varepsilon(1-\varepsilon_1)pH(t) \\
&+ \varepsilon\left[\left(\frac{p(1-\varepsilon_1)}{2} - 1\right)\left(1 - \int_0^t g(s)\,ds\right) - \frac{1}{2p(1-\varepsilon_1)}\int_0^t g(s)\,ds\right] \\
&\times \|\nabla u(t)\|^2 - \varepsilon(1-\varepsilon_1)pE_2 + \varepsilon\left[\varepsilon_1 - \varepsilon_2 C H^{-\sigma}(t) H^\alpha(0)\right]\|u(t)\|_p^p
\end{aligned}$$

$$
\begin{aligned}
& - \varepsilon C M_q \frac{\lambda^q}{q} \left[\|u(t)\|_p^p + \left(1 - \int_0^t g(s)\,ds\right) \|\nabla u(t)\|^2 + 2H(t) \right] \\
& + \left[(1-\sigma) H^{-\sigma}(t) m_q - \varepsilon M_q \frac{q-1}{1} \lambda^{\frac{q}{q-1}} \right] \|u_t(t)\|_{q,\Gamma_1}^q \\
& + \varepsilon \left[\frac{(1-\varepsilon_1)p}{2} + 1 \right] \int_{\Gamma_1} h(x) q(x) y^2(t)\,d\Gamma \\
= {} & H^{-\sigma}(t) \left[1 - \sigma - \varepsilon C H^{\alpha}(0) C(\varepsilon_2)\right] \|u_t(t)\|_m^m + \varepsilon \left[\frac{1}{\rho+1} + \frac{(1-\varepsilon_1)p}{\rho+2} \right] \|u_t(t)\|_{\rho+2}^{\rho+2} \\
& + \varepsilon \left[(1-\varepsilon_1)p - 2 C M_q \frac{\lambda^q}{q} \right] H(t) - \varepsilon (1-\varepsilon_1) p E_2 \\
& + \varepsilon \left[\left(\frac{(1-\varepsilon_1)p}{2} - C M_q \frac{\lambda^q}{q} - 1 \right) \left(1 - \int_0^t g(s)\,ds\right) \right. \\
& \left. - \frac{1}{2(1-\varepsilon_1)p} \int_0^t g(s)\,ds \right] \|\nabla u(t)\|^2 \\
& + \varepsilon \left[\varepsilon_1 - \varepsilon_2 C H^{-\sigma}(t) H^{\alpha}(0) - C M_q \frac{\lambda^q}{q} \right] \|u(t)\|_p^p \\
& + \left[(1-\sigma) H^{-\sigma}(t) m_q - \varepsilon M_q \frac{q-1}{q} \lambda^{-\frac{q}{q-1}} \right] \|u(t)\|_{q,\Gamma_1}^q \\
& + \varepsilon \left[\frac{(1-\varepsilon_1)p}{2} + 1 \right] \int_{\Gamma_1} h(x) q(x) y^2(t)\,d\Gamma \\
\geq {} & H^{-\sigma}(t) \left[1 - \sigma - \varepsilon C H^{\alpha}(0) C(\varepsilon_2)\right] \|u_t(t)\|_m^m + \varepsilon \left[\frac{1}{\rho+1} + \frac{(1-\varepsilon_1)p}{\rho+2} \right] \|u_t(t)\|_{\rho+2}^{\rho+2} \\
& + \varepsilon \left[(1-\varepsilon_1)p - 2 C M_q \frac{\lambda^q}{q} \right] H(t) - \varepsilon (1-\varepsilon_1) p E_2 \\
& + \frac{\varepsilon}{1 - \int_0^t g(s)\,ds} \left[\left(\frac{(1-\varepsilon_1)p}{2} - C M_q \frac{\lambda^q}{q} - 1 \right) l - \frac{1}{2(1-\varepsilon_1)p} (1-l) \right] \beta^2 \\
& + \varepsilon \left[\varepsilon_1 - \varepsilon_2 C H^{-\sigma}(t) H^{\alpha}(0) - C M_q \frac{\lambda^q}{q} \right] \|u(t)\|_p^p \\
& + \left[(1-\sigma) H^{-\sigma}(t) m_q - \varepsilon M_q \frac{q-1}{q} \lambda^{-\frac{q}{q-1}} \right] \|u(t)\|_{q,\Gamma_1}^q \\
& + \varepsilon \left[\frac{(1-\varepsilon_1)p}{2} + 1 \right] \int_{\Gamma_1} h(x) q(x) y^2(t)\,d\Gamma.
\end{aligned}
\tag{40}
$$

Since

$$
\int_0^\infty g(s)\,ds < \frac{\frac{p}{2} - 1 - C M_q \frac{\lambda^q}{q} - \varepsilon_0}{\frac{p}{2} - 1 - C M_q \frac{\lambda^q}{q} + \frac{1}{2p}}, \quad p > 2,
$$

we have

$$
\left(\frac{p}{2} - 1 - C M_q \frac{\lambda^q}{q} \right) \left(1 - \int_0^\infty g(s)\,ds \right) - \frac{1}{2p} \int_0^\infty g(s)\,ds > \varepsilon_0 > 0.
$$

It is easy to see that there exist $\varepsilon_1^* > 0$ and $T_0 > 0$ such that, for $0 < \varepsilon_1 < \varepsilon_1^* := 1 - \frac{2(1+\varepsilon_0)}{p}$, $0 < \varepsilon_0 < \frac{p}{2} - 1$, and $t > T_0$,

$$\frac{[(\frac{(1-\varepsilon_1)p}{2} - CM_q\frac{\lambda^q}{q} - 1)l - \frac{1}{2(1-\varepsilon_1)p}(1-l)]\beta^2}{1-\int_0^t g(s)\,ds} > \frac{(\frac{(1-\varepsilon_1)p}{2} - CM_q\frac{\lambda^q}{q} - 1)l - \frac{1}{2(1-\varepsilon_1)p}(1-l)}{1-\int_0^\infty g(s)\,ds}B_1^{-\frac{2p}{p-2}}.$$

Now, we may choose $\varepsilon_1 > 0$ sufficiently small and $E_2 \in (E(0), E_1)$ sufficiently near $E(0)$ such that

$$\frac{(\frac{(1-\varepsilon_1)p}{2} - CM_q\frac{\lambda^q}{q} - 1)l - \frac{1}{2(1-\varepsilon_1)p}(1-l)}{1-\int_0^\infty g(s)\,ds}B_1^{-\frac{2p}{p-2}} - p(1-\varepsilon_1)E_2 > 0, \tag{41}$$

since

$$\begin{aligned} E(0) < E_2 &< \left(\frac{1}{2} - CM_q\frac{\lambda^q}{q}\frac{1}{p} - \frac{1}{p} - \frac{1}{2p^2}\frac{1-l}{l}\right)B_1^{-\frac{2p}{p-2}} \\ &= \left(1 - CM_q\frac{\lambda^q}{q}\frac{2}{p-2} - \frac{1}{p(p-2)}\frac{1-l}{l}\right)\left(\frac{1}{2}-\frac{1}{p}\right)B_1^{-\frac{2p}{p-2}} < \left(\frac{1}{2}-\frac{1}{p}\right)B_1^{-\frac{2p}{p-2}}. \end{aligned}$$

From (40) and (41), we arrive at

$$\begin{aligned} L'(t) \ge{}& H^{-\sigma}(t)\bigl[1-\sigma-\varepsilon CH^{\alpha}(0)C(\varepsilon_2)\bigr]\bigl\|u_t(t)\bigr\|_m^m + \varepsilon\left[\frac{1}{\rho+1} + \frac{(1-\varepsilon_1)p}{\rho+2}\right]\bigl\|u_t(t)\bigr\|_{\rho+2}^{\rho+2} \\ &+ \varepsilon\left[(1-\varepsilon_1)p - 2CM_q\frac{\lambda^q}{q}\right]H(t) \\ &+ \varepsilon\left[\varepsilon_1 - \varepsilon_2 CH^{-\sigma}(t)H^{\alpha}(0) - CM_q\frac{\lambda^q}{q}\right]\bigl\|u(t)\bigr\|_p^p \\ &+ \left[(1-\sigma)H^{-\sigma}(t)m_q - \varepsilon M_q\frac{q-1}{q}\lambda^{-\frac{q}{q-1}}\right]\bigl\|u(t)\bigr\|_{q,\Gamma_1}^q \\ &+ \varepsilon\left[\frac{(1-\varepsilon_1)p}{2} + 1\right]\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \end{aligned} \tag{42}$$

At this point, for $\varepsilon_2 CH^{-\sigma}(t)H^{\alpha}(0) < \varepsilon_1 < \min\{1, \varepsilon_2 p^{1/m-1/p}\}$, we may take λ sufficiently small such that

$$\begin{aligned} &(1-\varepsilon_1)p - 2CM_q\frac{\lambda^q}{q} > 0, \\ &\varepsilon_1 - \varepsilon_2 CH^{-\sigma}(t)H^{\alpha}(0) - CM_q\frac{\lambda^q}{q} > 0. \end{aligned}$$

Once again, we choose ε small enough such that

$$\begin{aligned} &1-\sigma-\varepsilon CH^{\alpha}(0)C(\varepsilon_2) > 0, \\ &(1-\sigma)H^{-\sigma}(t)m_q - \varepsilon M_q\frac{q-1}{q}\lambda^{-\frac{q}{q-1}} > 0. \end{aligned}$$

Then from (42) there exists a positive constant $K_1 > 0$ such that following inequality holds:

$$L'(t) \geq \varepsilon K_1 \biggl(H(t) + \|u(t)\|_{\rho+2}^{\rho+2} + \|u(t)\|_p^p + \|\nabla u(t)\|^2 + \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma \biggr). \tag{43}$$

On the other hand, from definition (23) and since $f, h > 0$, we have

$$L(t) \leq H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1} \int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx K_1 - \varepsilon \int_{\Gamma_1} h(x)q(x)y(t)\,d\Gamma.$$

Consequently, the above estimate leads to

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) \leq C(\varepsilon, \sigma, \rho) \biggl[& H(t) + \biggl(\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx \biggr)^{\frac{1}{1-\sigma}} \\ & + \biggl(\int_{\Gamma_1} h(x)q(x)y(t)\,d\Gamma \biggr)^{\frac{1}{1-\sigma}} \biggr]. \end{aligned} \tag{44}$$

We now estimate (see [17])

$$\begin{aligned} \biggl(\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx \biggr)^{\frac{1}{1-\sigma}} &\leq C \|u_t(t)\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}} \|u(t)\|_p^{\frac{1}{1-\sigma}} \\ &\leq C \|u_t(t)\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\mu} \|u(t)\|_p^{\frac{1}{1-\sigma}\theta}, \end{aligned}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Choose $\mu = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1$, then

$$\frac{\theta}{1-\sigma} = \frac{\rho+2}{(1-\sigma)(\rho+2) - (\rho+1)}.$$

From (30), we know

$$\frac{\theta}{1-\sigma} < p. \tag{45}$$

Then from (22) we deduce

$$\begin{aligned} \|u(t)\|_p^{\frac{\theta}{1-\sigma}} = \|u(t)\|_p^{p-(p-\frac{\theta}{1-\sigma})} = \|u(t)\|_p^p \|u(t)\|_p^{-k} \\ \leq C\|u(t)\|_p^p H(0)^{-\frac{k}{p}}, \end{aligned} \tag{46}$$

where $k = p - \frac{\theta}{1-\sigma}$ is a positive constant. Thus from (46) we obtain

$$\biggl(\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx \biggr)^{\frac{1}{1-\sigma}} \leq C\|u_t(t)\|_{\rho+2}^{\rho+2} + \|u(t)\|_p^p H(0)^{-\frac{k}{p}}. \tag{47}$$

On the other hand, by the same method as in [13], we obtain

$$\begin{aligned} \biggl| \int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma \biggr| &= \biggl| \int_{\Gamma_1} \frac{h(x)q(x)}{q(x)} u(t)y(t)\,d\Gamma \biggr| \\ &\leq \frac{\|h\|_{L^\infty}^{\frac{1}{2}} \|q\|_{L^\infty}^{\frac{1}{2}}}{q_0} \biggl(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma \biggr)^{\frac{1}{2}} \biggl(\int_{\Gamma_1} u^2(t)\,d\Gamma \biggr)^{\frac{1}{2}}. \end{aligned}$$

Using the embedding $L^p(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ and Young's inequality, we get

$$\begin{aligned}\left|\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right| &= \left|\int_{\Gamma_1} \frac{h(x)q(x)}{q(x)}u(t)y(t)\,d\Gamma\right| \\ &\le \tilde{C}\frac{\|h\|_{L^\infty}^{\frac{1}{2}}\|q\|_{L^\infty}^{\frac{1}{2}}}{q_0}\left(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right)^{\frac{1}{2}}\left(\int_{\Gamma_1} u^q(t)\,d\Gamma\right)^{\frac{1}{q}}.\end{aligned}$$

Consequently, there exists a positive constant $\tilde{C}_1 = \tilde{C}_1(\|h\|_{L^\infty}, \|q\|_{L^\infty}, q_0, \sigma)$ such that

$$\begin{aligned}&\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \\ &\quad\le \tilde{C}_1\left(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right)^{\frac{1}{2(1-\sigma)}}\left(\int_{\Gamma_1} u^q(t)\,d\Gamma\right)^{\frac{1}{q(1-\sigma)}}.\end{aligned}$$

Applying Young's inequality to the right-hand side of the preceding inequality, there exists a positive constant, also denoted by $\tilde{C}_2$, such that

$$\begin{aligned}&\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \\ &\quad\le \tilde{C}_2\left(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right)^{\frac{\theta}{2(1-\sigma)}}\left(\int_{\Gamma_1} \left|u(t)\right|^q\,d\Gamma\right)^{\frac{\tau}{q(1-\sigma)}}\end{aligned} \tag{48}$$

for $\frac{1}{\tau} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\sigma)$, hence $\tau = \frac{2(1-\sigma)}{1-2\sigma}$ to get

$$\begin{aligned}&\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \\ &\quad\le C\left[\left(\int_{\Gamma_1} \left|u(t)\right|^q\,d\Gamma\right)^{\frac{2}{q(1-\sigma)}} + \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right].\end{aligned} \tag{49}$$

By using (30) and the algebraic inequality (37) with $\chi = \int_{\Gamma_1} |u(t)|^p\,d\Gamma$, $d = 1 + \frac{1}{H(0)}$, $w = H(0)$, and $\nu = \frac{2}{p(1-2\sigma)}$, condition (30) on σ ensures that $0 < \nu < 1$, and we get

$$\chi^\nu \le d\bigl(\chi + H(0)\bigr) \le d\bigl(\chi + H(t)\bigr).$$

Therefore from (49) there exists a positive constant $\tilde{C}$ such that, for all $t \ge 0$,

$$\begin{aligned}&\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \\ &\quad\le \tilde{C}\left(H(t) + \left\|u(t)\right\|_{q,\Gamma_1}^q + \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right).\end{aligned} \tag{50}$$

Thus from (39) and (50) we get

$$\begin{aligned}
&\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \\
&\quad\le C\left(H(t)+\|u(t)\|_p^p+\left(1-\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2+\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right) \\
&\quad\le C\left(H(t)+\|u(t)\|_p^p+\|\nabla u(t)\|^2+\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right),
\end{aligned} \tag{51}$$

where C is a positive constant. Therefore, from (44), (47), and (51), we arrive at

$$\begin{aligned}
L^{\frac{1}{1-\sigma}}(t) &\le \overline{C}\left[H(t)+\left(\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx\right)^{\frac{1}{1-\sigma}}+\left(\int_{\Gamma_1} h(x)q(x)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}}\right] \\
&\le \overline{C}\left[H(t)+\|u_t(t)\|_{\rho+2}^{\rho+2}+\|u(t)\|_p^p+\|\nabla u(t)\|^2+\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right],
\end{aligned} \tag{52}$$

where $\overline{C}$ is a constant depending on $\varepsilon, \sigma, \rho, \tilde{C}, C$. Consequently, combining (43) and (52), for some $\xi > 0$, we get

$$L'(t) \ge \xi L^{\frac{1}{1-\sigma}}(t), \quad \forall t \ge 0.$$

For ε sufficiently small, there exists some constant T_1 such that

$$\begin{aligned}
L(T_1) &= H^{1-\sigma}(T_1)+\frac{\varepsilon}{\rho+1}\int_\Omega |u_t(T_1)|^\rho u_t(T_1)u(T_1)\,dx \\
&\quad -\frac{\varepsilon}{2}\int_{\Gamma_1} h(x)f(x)y^2(T_1)\,d\Gamma-\varepsilon\int_{\Gamma_1} h(x)u(t)y(T_1)\,d\Gamma \\
&> 0.
\end{aligned}$$

Hence we get

$$L'(t) \ge \xi L^{\frac{1}{1-\sigma}}(t) > 0, \quad \forall t \ge T_1. \tag{53}$$

A simple integration of (53) over (T_1, t) yields

$$L^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(T_1)-\frac{\xi\sigma}{1-\sigma}t}, \quad \forall t \ge T_1.$$

Hence $L(t)$ blows up in finite time

$$T^* \le \frac{1-\sigma}{\xi\sigma L^{\frac{\sigma}{1-\sigma}}(T_1)}.$$

Thus the proof of Theorem 2.1 is complete. □

3 Conclusion

In this paper, we consider a quasilinear viscoelastic wave equation with acoustic boundary condition. Under some appropriate assumption on the relaxation function g, the functionΦ, $p > \max\{\rho + 2, m, q, 2\}$, and the initial data, we prove a global nonexistence of solutions for a quasilinear viscoelastic wave equation with positive initial energy. Actually, the principle result of the paper, Theorem 2.1, is a global nonexistence result in the case where the interior source term $|u|^{p-2}u$ dominates both the interior and boundary damping terms, $|u_t|^{m-2}u_t$ and $\Phi(u_t) \sim |u_t|^{q-2}u_t$, in an appropriate sense under the added assumption that the initial total energy is sufficiently small.

4 Abbreviations

A quasilinear viscoelastic wave equation with acoustic boundary condition is considered; Some assumptions and needed lemmas are presented; The nonexistence of the weak solution with small positive initial energy is proved by suitable assumptions on the relaxation function g, the nonlinear function $\Phi(\cdot)$, $p > \max\{\rho + 2, m, q, 2\}$, the initial data, and the parameters in the system.

Acknowledgements
The authors are thankful to the honorable reviewers and editors for their valuable reviewing of the manuscript.

Funding
The first author's work was supported by the National Research Foundation of Korea (Grant # NRF-2016R1D1A1B03930361). The corresponding author's work was supported by the National Research Foundation of Korea (Grant # NRF-2016R1C1B1016288).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the manuscript.

Author details
[1]Institute of Basic Liberal Educations, Catholic University of Daegu, Gyeongsan, Republic of Korea. [2]Department of Mathematics, Pusan National University, Busan, Republic of Korea. [3]Department of Mathematics and Education, Seowon University, Cheongju, Republic of Korea.

Publisher's Note

Received: 4 June 2018 Accepted: 31 August 2018 Published online: 14 September 2018

References

1. Beale, J.T., Rosencrans, S.I.: Acoustic boundary conditions. Bull. Am. Math. Soc. **80**, 1276–1278 (1974)
2. Boukhatem, Y., Benabderrahmane, B.: Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. Nonlinear Anal. **97**, 191–209 (2014)
3. Boukhatem, Y., Benabderrahmane, B.: Polynomial decay and blow up of solutions for variable coefficients viscoelastic wave equation with acoustic boundary conditions. Acta Math. Sin. Engl. Ser. **32**(2), 153–174 (2016)
4. Di, H., Shang, Y., Peng, X.: Global existence and nonexistence of solutions for a viscoelastic wave equation with nonlinear boundary source term. Math. Nachr. **289**(11–12), 1408–1432 (2016)
5. Graber, P.J., Said-Houari, B.: On the wave equation with semilinear porous acoustic boundary conditions. J. Differ. Equ. **252**, 4898–4941 (2012)
6. Jeong, J.M., Park, J.Y., Kang, Y.H.: Global nonexistence of solutions for a quasilinear wave equation with acoustic boundary conditions. Bound. Value Probl. **2017**, 42 (2017). https://doi.org/10.1186/s13661-017-0773-1
7. Jeong, J.M., Park, J.Y., Kang, Y.H.: Energy decay rates for the semilinear wave equation with memory boundary condition and acoustic boundary conditionds. Comput. Math. Appl. **73**, 1975–1986 (2017)
8. Kang, Y.H.: Energy decay rate for the Kelvin–Voigt type wave equation with Balakrishnan–Taylor damping and acoustic boundary. East Asian Math. J. **32**(3), 355–364 (2016)

9. Kumar, S., Kumar, D., Singh, J.: Fractional modelling arising in unidirectional propagation of long waves in dispersive media. Adv. Nonlinear Anal. **5**(4), 383–394 (2016)
10. Lee, M.J., Park, J.Y., Kang, Y.H.: Exponential decay rate for a quasilinear von Karman equation of memory type with acoustic boundary conditions. Bound. Value Probl. **2015**, 122 (2015). https://doi.org/10.1186/s13661-015-0381-x
11. Lions, J.L., Magenes, E.: Probelèms aux limits non homogènes et applications, vols. 1, 2. Dunod, Paris (1968)
12. Liu, W., Sun, Y.: General decay of solutions for a weak viscoelastic equation with acoustic boundary conditions. Z. Angew. Math. Phys. **65**(1), 125–134 (2013)
13. Messaoudi, S.A.: Blow up and global existence in a nonlinear viscoelastic wave equation. Math. Nachr. **260**, 58–66 (2003)
14. Morse, P.M., Ingard, K.U.: Theoretical Acoustics. McGraw-Hill, New York (1968)
15. Park, J.Y., Park, S.H.: General decay for quasilinear viscoelastic equations with nonlinear weak damping. J. Math. Phys. **50**, 1–10 (2009)
16. Park, J.Y., Park, S.H.: Decay rate estimates for wave equations of memory type with acoustic boundary conditions. Nonlinear Anal. **74**(3), 993–998 (2011)
17. Song, H.: Global nonexistence of positive initial energy solutions for a viscoelastic wave equation nonlinear analysis. Nonlinear Anal. **125**, 260–269 (2015)