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Properties of solutions for a reaction–diffusion equation with nonlinear absorption and nonlinear nonlocal Neumann boundary condition

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Abstract

In this paper, the authors consider the reaction–diffusion equation with nonlinear absorption and nonlinear nonlocal Neumann boundary condition. They prove that the solution either exists globally or blows up in finite time depending on the initial data, the weighting function on the border, and nonlinear indexes in the equation by using the comparison principle.

Keywords: Reaction-diffusion equation; Global existence; Nonlinear absorption; Blow-up; Neumann boundary condition

1 Introduction

In this paper, we consider the initial boundary value problem for the following nonlocal reaction–diffusion equation with nonlinear absorption:

$$u_t = \Delta u + au^p \int_\Omega u^q(y,t)\,dy - bu^m, \quad x \in \Omega, 0 < t < T, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \int_\Omega k(x,y)u^l(y,t)\,dy, \quad x \in \partial\Omega, 0 < t < T, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $p, q, m, l > 0$, Ω is a bounded domain in R^n ($n \geq 1$) with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$. Here a, b are positive constants, $k(x,y)$ is a positive continuous bounded function defined for $x \in \partial\Omega$, $y \in \Omega$. Furthermore, we assume that $u_0(x) \geq 0$ and satisfies the compatibility conditions

$$\frac{\partial u_0}{\partial \nu} = \int_\Omega k(x,y)u_0^l(y,t)\,dy, \quad x \in \partial\Omega.$$

It is found that lots of physical phenomena could be formulated into nonlocal mathematical models and studied by many authors (see [1–5] and the related references). Problem (1.1)–(1.3) can be used to describe, for example, heat conduction in solid media with nonlinear absorption terms and nonlinear boundary currents. In the last few decades, there has been a large amount of literature devoted to the study of properties of solutions to reaction–diffusion equation with nonlocal source with homogeneous Dirichlet boundary conditions or with nonlinear boundary conditions (see [6–9] and the related references). In particular, the following nonlocal reaction–diffusion equation with nonlinear absorption

$$u_t = \Delta u + \int_\Omega u^p\,dx - cu^q, \quad x \in \Omega, t > 0, \tag{1.4}$$

$$u(x,t) = \int_\Omega f(x,y)u^l(y,t)\,dy, \quad x \in \partial\Omega, t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.6}$$

was studied by Mu in [10]. They discussed that the weighting function on the border and nonlinear index influenced global and non-global existence of solutions. The conditions on the existence and nonexistence of global positive solutions are given and the uniform blow-up estimates for the blow-up solution are established. They focus on the reaction–diffusion equation with nonlocal source, nonlinear absorption, and nonlocal Dirichlet boundary condition; however, the authors did not give any result about equation (1.4) with nonlinear nonlocal Neumann boundary condition. The nonlinear nonlocal Neumann boundary condition can be considered as some cross boundary flow. Thus, this paper will extend the above work to the reaction–diffusion equation (1.1) with nonlinear nonlocal Neumann boundary condition and obtain the corresponding results.

In addition, in [11] Zhou and Yang considered the local reaction–diffusion equation with the weighted coefficient

$$u_t = \Delta u + c(x,t)u^p \int_\Omega u^q(y,t)\,dy, \quad x \in \Omega, t > 0, \tag{1.7}$$

$$\frac{\partial u}{\partial \nu} = \int_\Omega k(x,y)u^l(y,t)\,dy, \quad x \in \partial\Omega, 0 < t < T, \tag{1.8}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.9}$$

The authors discussed the effect of the behavior for the weighted function on the properties of the solution and obtained that $p + q \le 1$, $l \le 1$, then the solutions exist globally for any nonnegative initial data. They found that $p + q < 1$, $l > 1$, then the solutions exist globally for small initial data. They proved that $p + q > 1$, $l > 0$, then the solutions blow up in finite time for any positive initial data. They also assumed that $\min\{p, q\} > 1$, $l > 0$, then the solutions blow up in finite time for large initial data. This paper just studied the reaction–diffusion equation with nonlocal source and Neumann boundary condition, but they did not consider the equation with nonlinear absorption. We will show that the effect of the nonlinear absorption of (1.1) plays a substantial role in determining whether a solution blows up or not.

However, a nonlocal reaction–diffusion equation with nonlinear absorption and coupled with nonlinear nonlocal Neumann boundary condition, to our knowledge, has not been well studied. Furthermore, because this kind of problem is widely used in physics and engineering, it is necessary to study it. Motivated by those of the above works, we will get blow-up and global existence criteria for problem (1.1) with nonlinear absorption and nonlocal nonlinear Neumann boundary, which are not only different from the situations with the Dirichlet boundary condition, but also different from the situations with problem (1.7)–(1.9) without absorption term. We will show that the nonlinear absorption and the nonlinear term $u^l(y,t)$ in the boundary condition of (1.1) play substantial roles in determining whether a solution blows up or not. In fact, we will prove that if $p+q<m$, $l\le 1$, the solution exists globally for any $k(x,y)$ and any nonnegative initial data. We notice that if $p+q>m>1$, $l>0$, a blow-up occurs in finite time if the initial data $u_0(x)$ satisfies $\int_\Omega u_0(x)\varphi(x)\,dx>1$. We also find that if $p+q=m>1$, then problem (1.1)–(1.3) has blow-up solutions in finite time as well as global solutions.

Because we will study the nonlocal reaction–diffusion equation with nonlinear absorption and coupled with nonlinear nonlocal Neumann boundary condition, we have some new difficulties to overcome. First, we establish a complete proof for the local existence of the solution of (1.1)–(1.3) which was not proved in [10] and [11]. Second, due to the appearance of the Neumann boundary condition, some approaches used in [10] cannot be extended to handle our problem; for example, the treatment of boundary integral $-\int_\Omega u\frac{\partial\varphi}{\partial\nu}\,dS$ during the calculation of auxiliary function $J'(t)$, the selection of ODE, and the condition of auxiliary function $\zeta(x)$. Third, for problem (1.7)–(1.9), there is no global solution when $p+q>1$, $l>0$. However, for our problem (1.1)–(1.3), if $p+q=m>1$, $l\ge 1$, a solution still may exist globally. Thus, compared with [11], we can see that the absorption plays an important role in the properties of solutions.

The structure of this paper is as follows. In Sect. 2, we establish and prove the comparison principle and local existence. In Sect. 3, by using the comparison principle and super-subsolution method, we establish the conditions for blow-up in finite time and global existence.

2 The comparison principle and local existence

In this section we start with the definition of supersolution and subsolution of problem (1.1)–(1.3). Then we will prove the comparison principle and give the local existence of solutions for (1.1)–(1.3).

First, for convenience, we set $Q_T=\Omega\times(0,T)$, $S_T=\partial\Omega\times(0,T)$, and $\Gamma_T=S_T\cup\overline{\Omega}\times 0$, $T>0$.

Next, the definitions of supersolution, subsolution, and solution for (1.1)–(1.3) will be given.

Definition 2.1 We say that a nonnegative function $\underline{u}\in C^{2,1}(Q_T)\cap C(Q_T\cup\Gamma_T)$ is a subsolution of (1.1)–(1.3) if

$$\underline{u}_t\le\Delta\underline{u}+a\underline{u}^p\int_\Omega\underline{u}^q(y,t)\,dy-b\underline{u}^m,\quad (x,t)\in Q_T,\tag{2.1}$$

$$\frac{\partial\underline{u}}{\partial\nu}\le\int_\Omega k(x,y)\underline{u}^l(y,t)\,dy,\quad (x,t)\in S_T,\tag{2.2}$$

$$\underline{u}(x,0)\le u_0(x),\quad x\in\Omega,\tag{2.3}$$

and similarly we say that a nonnegative function $\overline{u}(x,t)\in C^{2,1}(Q_T)\cap C(Q_T\cup\Gamma_T)$ is a supersolution of (1.1)–(1.3) in Q_T and satisfies (2.1)–(2.3) in the reverse order. We say that $u(x,t)$ is a solution of problem (1.1)–(1.3) in Q_T, if $u(x,t)$ is both a subsolution and a supersolution of (1.1)–(1.3) in Q_T.

The following comparison principle plays an important role in the proof of our main results.

Lemma 2.1 *Let $\underline{u}(x,t)$ and $\overline{u}(x,t)$ be a subsolution and a supersolution of (1.1)–(1.3) in Q_T, respectively, with $\underline{u}(x,0) \le \overline{u}(x,0)$ in Ω. Suppose that $\overline{u}(x,0) > 0$ or $\underline{u}(x,0) > 0$ in $Q_T \cup \Gamma_T$ if* $\min(p,q,m,l) < 1$. *Then $\underline{u}(x,t) \le \overline{u}(x,t)$ in $Q_T \cup \Gamma_T$.*

Proof Set $\phi(x,t) \in C^{2,1}(\overline{Q}_t)$ $(0 < t < T)$ is a nonnegative function which satisfies the homogeneous Neumann boundary condition

$$\frac{\partial \phi}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Multiplying (2.1) by ϕ and integrating over Q_t, we obtain that the subsolution $\underline{u}(x,t)$ satisfies

$$\begin{aligned}\int_\Omega \underline{u}(x,t)\phi(x,t)\,dx \le{}& \int_\Omega \underline{u}(x,0)\phi(x,0)\,dx \\ &+ \int_0^t \int_\Omega \Big[\underline{u}(x,\tau)\phi_\tau(x,\tau) + \underline{u}(x,\tau)\Delta\phi(x,\tau) \\ &+ a\phi(x,\tau)\underline{u}^p(x,\tau)\int_\Omega \underline{u}^q(y,\tau)\,dy - b\phi(x,\tau)\underline{u}^m(x,\tau)\Big]\,dx\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} \phi(x,\tau)\left(\int_\Omega k(x,y)\underline{u}^l(y,\tau)\,dy\right)dS\,d\tau. \end{aligned} \tag{2.4}$$

On the other hand, the supersolution $\overline{u}(x,t)$ satisfies

$$\begin{aligned}\int_\Omega \overline{u}(x,t)\phi(x,t)\,dx \ge{}& \int_\Omega \overline{u}(x,0)\phi(x,0)\,dx \\ &+ \int_0^t \int_\Omega \Big[\overline{u}(x,\tau)\phi_\tau(x,\tau) + \overline{u}(x,\tau)\Delta\phi(x,\tau) \\ &+ a\phi(x,\tau)\overline{u}^p(x,\tau)\int_\Omega \overline{u}^q(y,\tau)\,dy - b\phi(x,\tau)\overline{u}^m(x,\tau)\Big]\,dx\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} \phi(x,\tau)\int_\Omega k(x,y)\overline{u}^l(y,\tau)\,dy\,dS\,d\tau. \end{aligned} \tag{2.5}$$

Let $\omega(x,t) = \underline{u} - \overline{u}$, subtracting (2.5) from (2.4) and using mean value theorem, we get

$$\begin{aligned}\int_\Omega \omega(x,t)\phi(x,t)\,dx \le{}& \int_\Omega \omega(x,0)\phi(x,0)\,dx \\ &+ \int_0^t \int_\Omega \omega(x,\tau)\bigg(\phi_\tau(x,\tau) + \Delta\phi(x,\tau) \\ &+ ap{\theta_1}^{p-1}\phi(x,\tau)\int_\Omega \underline{u}^q(y,\tau)\,dy - bm{\theta_2}^{m-1}\bigg)\,dx\,d\tau \\ &+ \int_0^t \int_\Omega aq\overline{u}^p\phi(x,\tau)\int_\Omega \omega(y,\tau){\theta_3}^{q-1}(y,\tau)\,dy\,dx\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} l\phi(x,\tau)\left(\int_\Omega k(x,y){\theta_4}^{l-1}\omega(y,\tau)\,dy\right)dS\,d\tau, \end{aligned} \tag{2.6}$$

where $\theta_i(x,t)$ ($i = 1,2,3,4$) are some positive continuous functions between $\underline{u}$ and $\overline{u}$ in $\overline{Q}_t$, if $\min(p,q,l,m) < 1$ and some nonnegative continuous functions in $\overline{Q}_t$ otherwise.

The function $\phi(x,t)$ is defined as a solution of the following problem:

$$\phi_\tau(x,\tau) + \Delta\phi(x,\tau) + ap{\theta_1}^{p-1}\phi(x,\tau)\int_\Omega \underline{u}^q(y,\tau)\,dy - bm{\theta_2}^{m-1} = 0, \quad (x,\tau) \in Q_t, \tag{2.7}$$

$$\frac{\partial\phi}{\partial\nu} = 0, \quad (x,\tau) \in S_t, \tag{2.8}$$

$$\phi(x,0) = \mu(x), \quad x \in \Omega, \tag{2.9}$$

where $\mu(x) \in C_0^\infty(\Omega)$, $0 \le \mu(x) \le 1$. By virtue of the comparison principle, the solution $\phi(x,t)$ is nonnegative and bounded (see [12] for example). Denote a solution of (2.7)–(2.9) as $\phi_n(x,\tau)$. Then, by the standard theory for linear parabolic equations, we know that $\phi_n \in C^{2,1}(\overline{Q}_t)$, $0 \le \phi_n(x,\tau) \le 1$ in $\overline{Q}_t$. Putting $\phi = \phi_n$ in (2.6) and $\omega(x,0) \le 0$, we get

$$\int_\Omega \omega(x,t)\phi(x,t)\,dx \le K\int_0^t\int_\Omega \omega_+(x,\tau)\,dx\,d\tau, \tag{2.10}$$

where we denote $\omega_+ = \max(0,\omega)$ and choose

$$K = aq\sup_{Q_t}\underline{u}^p(x,\tau)\phi(x,\tau)\theta_3^{q-1}(x,\tau) + l|\partial\Omega|\sup_{\partial\Omega\times Q_t}k(x,y)\sup_{Q_t}\theta_4^{l-1}(x,\tau)\sup_{S_t}\phi(x,\tau).$$

Inequality (2.10) holds for each function $\mu(x)$,

$$\int_\Omega \omega(x,t)\mu_n(x,t)\,dx \le K\int_0^t\int_\Omega \omega_+(x,\tau)\,dx\,d\tau,$$

so we can choose a sequence $\mu_n(x) \in C_0^\infty(\Omega)$ converging in $L^1(\Omega)$ to the function

$$\mu(x) = \begin{cases} 1, & \omega(x,t) > 0, \\ 0, & \omega(x,t) \le 0. \end{cases}$$

Substituting $\mu_n(x)$ instead of $\mu(x)$ in (2.10) and letting $n \to \infty$, we obtain

$$\int_\Omega \omega(x,t)\mu_n(x,t)\,dx = \int_\Omega \omega_+(x,t)\,dx \le K\int_0^t\int_\Omega \omega_+(x,\tau)\,dx\,d\tau. \tag{2.11}$$

By using Gronwall's inequality, we have $\omega_+(x,t) \le 0$, which means that $\underline{u}(x,t) \le \overline{u}(x,t)$ in $Q_T \cup \Gamma_T$. □

Next, we will establish the local existence of solution for (1.1)–(1.3) using representation formula and the contraction mapping argument.

Let $\{\varepsilon_m\}$ be decreasing to 0 sequence such that $0 < \varepsilon_m < 1$. For $\varepsilon = \varepsilon_m$, let $u_{0\varepsilon}(x)$ be the functions with the following properties (the existence of $u_{0\varepsilon}$, see [13]):

$$u_{0\varepsilon}(x) \in C^1(\overline{\Omega}), \qquad u_{0\varepsilon}(x) \ge \varepsilon, \qquad u_{0\varepsilon_i}(x) \ge u_{0\varepsilon_j}(x), \qquad \varepsilon_i \ge \varepsilon_j,$$
$$u_{0\varepsilon}(x) \to u_0(x) \quad \text{as } \varepsilon \to 0, \qquad \frac{\partial u_{0\varepsilon}(x)}{\partial\nu} = \int_\Omega k(x,y)u_{0\varepsilon}^l(y)\,dy \quad \text{for } x \in \partial\Omega.$$

Due to the nonlinearities in (1.1) and (1.2), the Lipschitz condition is not satisfied if $\min(p, q, m, l) < 1$, and thus we need to consider the auxiliary problem

$$u_t = \Delta u + au^p \int_\Omega u^q(y,t)\,dy - bu^m + b\varepsilon^m, \quad x \in \Omega, 0 < t < T, \tag{2.12}$$

$$\frac{\partial u}{\partial \nu} = \int_\Omega k(x,y)u^l(y,t)\,dy, \quad x \in \partial\Omega, 0 < t < T, \tag{2.13}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.14}$$

where $\varepsilon = \varepsilon_m$.

Theorem 2.1 *For some small values of T, problem (2.12)–(2.14) has a unique solution in Q_T.*

Proof Let $G_N(x,y;t-\tau)$ be the Green function for the following heat equation:

$$u_t - \Delta u = 0, \quad x \in \Omega, t > 0,$$

with homogeneous Neumann boundary condition. We denote that the function $G_N(x,y;t-\tau)$ has the following properties (see [14]):

$$G_N(x,y;t-\tau) \geq 0, \quad x,y \in \Omega, 0 \leq \tau < t < T, \tag{2.15}$$

$$\int_\Omega G_N(x,y;t-\tau)\,dy = 1, \quad x \in \Omega, 0 \leq \tau < t < T. \tag{2.16}$$

Then $u_\varepsilon(x,t)$ is a solution of (2.12)–(2.14) in Q_T if and only if

$$\begin{aligned} u_\varepsilon(x,t) = {} & \int_\Omega G_N(x,y;t-\tau)u_{0\varepsilon}(y)\,dy \\ & + \int_0^t \int_\Omega G_N(x,y;t-\tau)au_\varepsilon^p \int_\Omega u_\varepsilon^q(x,\tau)\,dx\,dy\,d\tau \\ & + b\int_0^t \int_\Omega G_N(x,y;t-\tau)\bigl(\varepsilon^q - u^q(y,\tau)\bigr)\,dy\,d\tau \\ & + \int_0^t \int_\partial \Omega G_N(x,y;t-\tau) \int_\Omega k(\xi,y,\tau)u_\varepsilon^l(y,\tau)\,dy\,dS_\xi\,d\tau \\ = {} & Lu_\varepsilon(x,t). \end{aligned} \tag{2.17}$$

To show that (2.17) is solvable for small T, we will use the contraction mapping argument. To this end, we define a sequence of functions $u_{\varepsilon,n}(x,t)$, $n = 1, 2, \ldots$ in the following way:

$$u_{\varepsilon,1}(x,t) = \varepsilon, \quad (x,t) \in \overline{Q}_T, \tag{2.18}$$

$$u_{\varepsilon,n+1}(x,t) = Lu_{\varepsilon,n}(x,t), \quad (x,t) \in \overline{Q}_T, n = 1, 2, \ldots. \tag{2.19}$$

By (2.15)–(2.19) and the properties of $u_{0\varepsilon}(x)$, we get

$$u_{\varepsilon,n}(x,t) \geq \varepsilon, \quad (x,t) \in \overline{Q}_T, n = 1, 2, \ldots. \tag{2.20}$$

Set

$$M_{0\varepsilon} = \sup_{x\in\Omega} u_{0\varepsilon}(x).$$

Using the method of mathematical induction, we prove that the inequalities

$$\sup_{Q_{T_1}} u_{\varepsilon,n}(x,t) \le M, \quad n = 1,2,\ldots \tag{2.21}$$

hold for some constants $T_1 > 0$ and $M > \max\{\varepsilon, M_{0\varepsilon}\}$. For $n = 1$, it is established obviously. Supposing that (2.19) is true for $n = m$, we shall prove it for $n = m + 1$. Indeed, by (2.12)–(2.14) and (2.19), we have

$$\begin{aligned} u_{\varepsilon,m+1}(x,t) &= \int_\Omega G_N(x,y;t)u_{0\varepsilon}(y)\,dy \\ &\quad + \int_0^t \int_\Omega G_N(x,y;t-\tau)au_{\varepsilon,m}^p \int_\Omega u_{\varepsilon,m}^q(x,\tau)\,dx\,dy\,d\tau \\ &\quad + b\int_0^t \int_\Omega G_N(x,y;t-\tau)\bigl(\varepsilon^q - u\varepsilon, m^q(y,\tau)\bigr)\,dy\,d\tau \\ &\quad + \int_0^t \int_\partial \Omega G_N(x,y;t-\tau)\int_\Omega k(\xi,y,\tau)u_{\varepsilon,m}^l(y,\tau)\,dy\,dS_\xi\,d\tau \\ &\le M_{0\varepsilon} + M^{p+q}\gamma(t) + M^l\beta(t), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} \gamma(t) &= \int_0^t \int_\Omega a|\Omega|G_N(x,y;t-\tau)\,dy\,d\tau, \\ \beta(t) &= \sup_{x\in\Omega} \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau)\int_\Omega k(\xi,y,\tau)\,dy\,dS_\xi\,d\tau. \end{aligned}$$

We note that [15] there exist positive constants δ_1 and a_1 such that

$$\beta(t) \le a_1\sqrt{t}, \quad t \le \delta_1. \tag{2.23}$$

Due to (2.15) and (2.16), we have

$$\gamma(t) = a|\Omega|t \le a_2 t, \quad t \le \delta_2, \tag{2.24}$$

where δ_2 and a_2 are some positive constants. We choose $0 < T_1 < \min\{\delta_1, \delta_2\}$ such that

$$\sup_{0<t<T_1} \bigl(M^{(p+q)}\gamma(t) + M^l\beta(t)\bigr) \le M - M_{0\varepsilon}. \tag{2.25}$$

Because of (2.22) and (2.25), we have (2.21) with $n = m + 1$. Using mean value theorem, we obtain for $n = 2,3,\ldots$

$$\begin{aligned} &\sup_{Q_{T_1}} |u_{\varepsilon,n+1} - u_{\varepsilon,n}| \\ &\quad = \sup_{Q_{T_1}} \left| a\int_0^t \int_\Omega G_N(x,y;t-\tau)\Bigl[u_{\varepsilon,n}^p \int_\Omega u_{\varepsilon,n}^q(x,\tau)\,dx \right. \end{aligned}$$

$$
\begin{aligned}
& - u^p_{\varepsilon,n-1} \int_\Omega u^q_{\varepsilon,n-1}(x,\tau)\,dx \biggr]\,d\xi\,d\tau \\
& - b \int_0^t \int_\Omega G_N(x,\xi;t-\tau)\bigl(u^m_{\varepsilon,n}(\xi,\tau) - u^m_{\varepsilon,n-1}(\xi,\tau)\bigr)\,d\xi\,d\tau \\
& + \int_0^t \int_\partial \Omega G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau)\bigl(u^l_{\varepsilon,n}(y,\tau) - u^l_{\varepsilon,n-1}(y,\tau)\bigr)\,dy\,dS_\xi\,d\tau \biggr| \\
= {} & \sup_{Q_{T_1}} \biggl| a \int_0^t \int_\Omega G_N(x,y;t-\tau) \biggl[(u_{\varepsilon,n} - u_{\varepsilon,n-1}) p {\theta_{1,n}}^{p-1} \int_\Omega u^q_{\varepsilon,n}(x,\tau)\,dx \\
& + u^p_{\varepsilon,n-1} \int_\Omega (u_{\varepsilon,n} - u_{\varepsilon,n-1}) q {\theta_{2,n}}^{q-1}(x,\tau)\,dx \biggr]\,d\xi\,d\tau \\
& - b \int_0^t \int_\Omega G_N(x,\xi;t-\tau) m (u_{\varepsilon,n} - u_{\varepsilon,n-1}) {\theta_{3,n}}^{m-1}(\xi,\tau)\,d\xi\,d\tau \\
& + \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau) l \bigl(u_{\varepsilon,n}(y,\tau) - u_{\varepsilon,n-1}(y,\tau)\bigr) \theta^{l-1}_{4,n}\,dy\,dS_\xi\,d\tau \biggr|,
\end{aligned}
$$

where $\theta_{i,n}(x,t)$ $(i = 1,2,3,4)$ are continuous functions in $\overline{Q}_{T_1}$ such that $\alpha_1 \le \theta_{i,n}(x,t) \le M_1$ for $(x,t) \in \overline{Q}_{T_1}$. Thus,

$$
\begin{aligned}
& \sup_{Q_{T_1}} |u_{\varepsilon,n+1} - u_{\varepsilon,n}| \\
& \quad \le \sup_{(0,T_1)} \rho(t) \sup_{Q_{T_1}} \bigl| u_{\varepsilon,n}(x,t) - u_{\varepsilon,n-1}(x,t) \bigr| \\
& \quad \le (M+\varepsilon) \Bigl(\sup_{(0,T_1)} \rho(t) \Bigr)^{n-1},
\end{aligned}
$$

where

$$
\begin{aligned}
\rho(t) = {} & \bigl(p \bigl({\alpha_1}^{p-1} + {M_1}^{p-1} \bigr) M^q + M^p q \bigl({\alpha_1}^{q-1} + {M_1}^{q-1} \bigr) \bigr) \gamma(t) \\
& - \frac{b}{a|\Omega|} m \bigl({\alpha_1}^{m-1} + {M_1}^{m-1} \bigr) \gamma(t) + l \bigl({\alpha_1}^{l-1} + {M_1}^{l-1} \bigr) \beta(t)
\end{aligned}
$$

for $t \in [0,T_1]$. We note that positive constants α_1 and M_1 do not depend on n. By (2.23) and (2.24) there exists a constant $T \in (0,T_1)$ such that

$$
\sup_{(0,T)} \rho(t) < 1.
$$

Hence, the sequence $u_{\varepsilon,n}(x,t)$ converges uniformly in $\overline{Q}_T$ as $n \to \infty$. We denote

$$
u_\varepsilon(x,t) = \lim_{n\to\infty} u_{\varepsilon,n}(x,t). \tag{2.26}
$$

By virtue of (2.20), (2.21) we have

$$
\varepsilon \le u_\varepsilon(x,t) \le M, \quad (x,t) \in \overline{Q}_T. \tag{2.27}
$$

Passing to the limit as $n \to \infty$ in (2.19), by dominated convergence theorem, we obtain that the function $u_\varepsilon(x,t)$ satisfies (2.17). Thus, $u_\varepsilon(x,t)$ is the solution of problem (2.12)–(2.14) in Q_T. By contradiction we shall prove uniqueness of the solution of (2.12)–(2.14) in Q_T for small values of T. Let problem (2.12)–(2.14) have at least two solutions $u_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$ in Q_T. Arguing as above, we can get

$$\begin{aligned}
&\sup_{Q_T}\left|u_\varepsilon(x,t)-v_\varepsilon(x,t)\right| \\
&\quad=\sup_{Q_T}\left|a\int_0^t\int_\Omega G_N(x,y;t-\tau)\left[u_\varepsilon^p\int_\Omega u_\varepsilon^q(x,\tau)\,dx-v_\varepsilon^p\int_\Omega v_\varepsilon^q(x,\tau)\,dx\right]d\xi\,d\tau\right.\\
&\qquad-b\int_0^t\int_\Omega G_N(x,\xi;t-\tau)\bigl(u_\varepsilon^m(\xi,\tau)-v_\varepsilon^m(\xi,\tau)\bigr)\,d\xi\,d\tau\\
&\qquad\left.+\int_0^t\int_\partial \Omega G_N(x,\xi;t-\tau)\int_\Omega k(\xi,y,\tau)\bigl(u_\varepsilon^l(y,\tau)-v_\varepsilon^l(y,\tau)\bigr)\,dy\,dS_\xi\,d\tau\right|\\
&\quad=\sup_{Q_T}\left|a\int_0^t\int_\Omega G_N(x,y;t-\tau)\left[(u_\varepsilon-v_\varepsilon)p{\theta_{1,n}}^{p-1}\int_\Omega u_\varepsilon^q(x,\tau)\,dx\right.\right.\\
&\qquad\left.+v_\varepsilon^p\int_\Omega(u_\varepsilon-v_\varepsilon)q{\theta_{2,n}}^{q-1}(x,\tau)\,dx\right]d\xi\,d\tau\\
&\qquad-b\int_0^t\int_\Omega G_N(x,\xi;t-\tau)m(u_\varepsilon-v_\varepsilon){\theta_{3,n}}^{m-1}(\xi,\tau)\,d\xi\,d\tau\\
&\qquad\left.+\int_0^t\int_{\partial\Omega} G_N(x,\xi;t-\tau)\int_\Omega k(\xi,y,\tau)l\bigl(u_\varepsilon(y,\tau)-v_\varepsilon(y,\tau)\bigr)\theta_{4,n}^{l-1}\,dy\,dS_\xi\,d\tau\right|\\
&\quad\le\sup_{Q_T}\left(\left(p{\theta_1}^{p-1}M^q+M^pq{\theta_2}^{q-1}-\frac{b}{a|\Omega|}m{\theta_3}^{m-1}\right)\gamma(t)\right.\\
&\qquad\left.+l{\theta_4}^{l-1}\beta(t)\right)\sup_{Q_T}\left|u_\varepsilon(x,t)-v_\varepsilon(x,t)\right|\\
&\quad\le\alpha\sup_{Q_T}\left|u_\varepsilon(x,t)-v_\varepsilon(x,t)\right|,
\end{aligned}$$

where $\theta_i(x,t)$ $(i=1,2,3,4)$ are some positive continuous functions between $u_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$ in $\overline{Q}_T$ and $\alpha<1$ for small values of T. Obviously, $u_\varepsilon(x,t)=v_\varepsilon(x,t)$ in Q_T. □

Theorem 2.2 *For some values of T, problem (1.1)–(1.3) has a maximal solution in Q_T.*

Proof Let u_ε be a solution of (2.12)–(2.14). It is easy to see that u_ε is a supersolution of (1.1)–(1.3). By Lemma 2.1, for $\varepsilon_1\le\varepsilon_2$, we can obtain $u_{\varepsilon_1}\le qu_{\varepsilon_2}$. According to the Dini theorem (see [16]) for some $T>0$, the sequence $u_\varepsilon(x,t)$ converges as $\varepsilon\to0$ uniformly in $\overline{Q}_T$ to some function $u(x,t)$. Passing to the limit as $\varepsilon\to0$ in (2.17) and using dominated convergence theorem, we have that the function $u(x,t)$ satisfies in Q_T the following equation:

$$\begin{aligned}
u(x,t)={}&\int_\Omega G_N(x,y;t-\tau)u_0(y)\,dy\\
&+\int_0^t\int_\Omega G_N(x,y;t-\tau)au^p\int_\Omega u^q(x,\tau)\,dx\,dy\,d\tau\\
&-b\int_0^t\int_\Omega G_N(x,y;t-\tau)u^q(y,\tau)\,dy\,d\tau\\
&+\int_0^t\int_{\partial\Omega} G_N(x,y;t-\tau)\int_\Omega k(\xi,y,\tau)u^l(y,\tau)\,dy\,dS_\xi\,d\tau.
\end{aligned}$$

Thus, $u(x,t)$ solves problem (1.1)–(1.3) in Q_T. It is easy to prove that $u(x,t)$ is a maximal solution of (1.1)–(1.3) in Q_T. □

3 Global existence and blow-up in finite time

In order to state the following results, let us introduce some useful symbols. Throughout this paper, let λ and $\varphi(x)$ be the first eigenvalue and the corresponding normalized eigenfunction of the following problem:

$$\begin{cases} -\Delta\varphi(x) = \lambda\varphi, & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $\lambda > 0$, $\max_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} < 0$, then $\varphi(x) > 0$ and $\int_\Omega \varphi(x)\,dx = 1$.

Denote $L = \sup_{\overline{\Omega}} \varphi(x)$, $M_1 = \inf_{\partial\Omega\times\overline{\Omega}} k(x,y)$, $M_2 = \sup_{\partial\Omega\times\overline{\Omega}} k(x,y)$.

Theorem 3.1 *Assume that $p+q<m$, $l \le 1$. Then the solutions of problem* (1.1)–(1.3) *exist globally for any $k(x,y)$ and any nonnegative initial data.*

Proof Let λ and $\varphi(x)$ satisfy (3.1), then for some $0<\varepsilon<1$ we choose δ to satisfy that

$$\delta \ge \max_{\partial\Omega}\left(-\frac{\partial\varphi}{\partial\upsilon}\right)^{-1} M_2 \int_\Omega \frac{1}{(\delta\varphi(x)+\varepsilon)^l}\,dx.$$

Let

$$\overline{u}(x,t) = \frac{ce^{rt}}{\delta\varphi(x)+\varepsilon},$$

where

$$r \ge \lambda + \sup_{\overline{\Omega}} \frac{2\delta^2|\nabla\varphi|^2}{(\delta\varphi(x)+\varepsilon)^2},$$

$$c = \max\left\{\sup_{\overline{\Omega}}\bigl(u_0(x)+1\bigr)\bigl(\delta\varphi(x)+\varepsilon\bigr), 1, \sup_{\overline{\Omega}}\left[\frac{(\delta\varphi(x)+\varepsilon)^{m-p}}{b}\int_\Omega \frac{1}{(\delta\varphi(x)+\varepsilon)^q}\,dx\right]^{\frac{1}{m-p-q}}\right\}.$$

Then, when $(x,t)\in Q_T$, we have

$$\begin{aligned} &\overline{u}_t - \Delta\overline{u} - a\overline{u}^p \int_\Omega \overline{u}^q(y,t)\,dy + b\overline{u}^m \\ &\quad = r\overline{u} - \overline{u}\left(\frac{\lambda\varphi\delta}{\delta\varphi(x)+\varepsilon} + \frac{2\delta^2|\nabla\varphi|^2}{(\delta\varphi(x)+\varepsilon)^2}\right) \\ &\qquad - a\frac{c^{p+q}e^{(p+q)rt}}{(\delta\varphi(x)+\varepsilon)^p}\int_\Omega \frac{1}{(\delta\varphi(x)+\varepsilon)^q}\,dx + b\frac{c^m e^{mrt}}{(\delta\varphi(x)+\varepsilon)^m} \\ &\quad \ge 0. \end{aligned} \tag{3.2}$$

On the other hand, when $(x,t)\in S_T$, we obtain

$$\begin{aligned}
&\frac{\partial\overline{u}}{\partial\upsilon}-\int_\Omega k(x,y)\overline{u}^l(y,t)\,dy\\
&\quad=\frac{ce^r t\delta}{\varepsilon^2}\left(-\frac{\partial\varphi}{\partial\upsilon}\right)-c^l e^l rt\int_\Omega k(x,y)\frac{1}{(\delta\varphi(x)+\varepsilon)^l}\,dy\\
&\quad\geq ce^r t\left[\delta\left(-\frac{\partial\varphi}{\partial\upsilon}\right)-M_2\int_\Omega\frac{1}{(\delta\varphi(x)+\varepsilon)^l}\,dx\right]\\
&\quad\geq 0.
\end{aligned}\tag{3.3}$$

Since $c>\sup_{\overline{\Omega}}(u_0(x)+1)(\delta\varphi(x)+\varepsilon)$, we get

$$u(x,0)=\frac{c}{\delta\varphi(x)+\varepsilon}\geq\frac{\sup_{\overline{\Omega}}(u_0(x)+1)(\delta\varphi(x)+\varepsilon)}{\delta\varphi(x)+\varepsilon}\geq u_0(x).\tag{3.4}$$

Combining (3.2)–(3.4), it is clear that $\overline{u}(x,t)$ is a supersolution of problem (1.1)–(1.3) in Q_t. By the comparison principle, the solution of problem (1.1)–(1.3) exists globally. □

Theorem 3.2 *Assume that $p+q>m>1$, $l>0$, $\frac{a}{L}>b$. Then, for any $k(x,y)>0$, the solution of problem* (1.1)–(1.3) *blows up in finite time if the initial data $u_0(x)$ satisfies $\int_\Omega u_0(x)\varphi(x)\,dx>1$.*

Proof Let $u(x,t)$ be the solution of problem (1.1)–(1.3), we define the following auxiliary function:

$$J(t)=\int_\Omega\varphi(x)u(x,t)\,dx,$$

where $\varphi(x)$ satisfies (3.1).

Multiplying both sides of the equation of (1.1) by $\varphi(x)$ and integrating over Ω, we could obtain

$$\begin{aligned}
J'(t)&=\int_\Omega\varphi\left(\Delta u+au^p\int_\Omega u^q(y,t)\,dy-bu^m\right)dx\\
&=\int_{\partial\Omega}\varphi\frac{\partial u}{\partial\upsilon}\,dS-\int_{\partial\Omega}u\frac{\partial\varphi}{\partial\upsilon}\,dS+\int_\Omega u\Delta\varphi\,dx\\
&\quad+\int_\Omega\varphi(x)au^p\int_\Omega u^q(y,t)\,dy\,dx-b\int_\Omega\varphi u^m\,dx\\
&=-\lambda\int_\Omega u\varphi\,dx+a\int_\Omega\varphi(x)u^p\int_\Omega u^q(y,t)\,dy\,dx-b\int_\Omega\varphi u^m\,dx-\int_{\partial\Omega}u\frac{\partial\varphi}{\partial\nu}\,dS.
\end{aligned}$$

Using the equality $\int_{\partial\Omega}\frac{\partial\varphi}{\partial\upsilon}\,dS=-\lambda\int_\Omega\varphi\,dx=-\lambda$ and $u(x,t)\geq 0$, we have

$$\begin{aligned}
J'(t)&\geq-\lambda\int_\Omega u\varphi\,dx+a\int_\Omega\varphi(x)u^p\int_\Omega u^q(y,t)\,dy\,dx-b\int_\Omega\varphi u^m\,dx+\lambda\inf_{S_T}u(x,t)\\
&\geq\int_\Omega\left(-\lambda u+au^p\int_\Omega u^q(y,t)\,dy-bu^m\right)\varphi(x)\,dx.
\end{aligned}\tag{3.5}$$

From (3.5), Jensen's inequality, and $\int_\Omega \varphi(x)\,dx = 1$, we obtain

$$\begin{aligned} J'(t) &\geq -\lambda \int_\Omega u\varphi\,dx + a\int_\Omega u^q(y,t)\,dy \int_\Omega u^p\varphi(x)\,dx - b\int_\Omega u^m\varphi(x)\,dx \\ &\geq -\lambda J + \frac{a}{\sup_{\overline{\Omega}}\varphi(x)}\int_\Omega u^q\varphi(x)\,dx\int_\Omega u^p\varphi(x)\,dx - b\int_\Omega u^m\varphi(x)\,dx \\ &\geq -\lambda J + \frac{a}{L}J^{p+q} - bJ^m \geq -\lambda J + \left(\frac{a}{L} - b\right)J^{p+q} - b. \end{aligned}$$

Since $\frac{a}{L} - b > 0$ and the function $f(J) = J^{p+q}$ is convex, then there exists $\eta > 1$ such that

$$\left(\frac{a}{L} - b\right)J^{p+q} \geq 2(\lambda J + b) \tag{3.6}$$

with the initial data $J(0) = \int_\Omega u_0(x)\varphi(x)\,dx$. It follows easily that if $J(0) > \eta$, then $J(t)$ is increasing on its interval of existence and

$$J'(t) \geq \frac{1}{2}\left(\frac{a}{L} - b\right)J^{p+q}.$$

From above inequality (3.6), we have

$$\lim_{t\to T_0^-} J(t) = +\infty,$$

where

$$T_0 = \frac{2}{(p+q-1)J^{p+q-1}(0)}\left(\frac{a}{L} - b\right).$$

By the comparison principle for ordinary equations (see [11, 17]), it is clear that the solution of (1.1)–(1.3) blows up in finite time. □

Theorem 3.3 *Assume $p + q = m > 1$. Then problem (1.1)–(1.3) has blow-up solutions in finite time as well as global solutions. More precisely,*

(i) *if $a|\Omega| > b$ and $u_0(x)$ is large enough, then for any $k(x,y) \geq 0$, the solution blows up in finite time;*

(ii) *if $l \geq 1$ and $\int_\Omega k(x,y)\,dy < 1$, the solution exists globally when $u_0(x) \leq \rho\zeta(x)$ for some $\rho > 0$, where $\zeta(x)$ satisfies*

$$-\Delta\zeta(x) = \sigma, \quad x \in \Omega, \tag{3.7}$$

$$\frac{\partial\zeta}{\partial\upsilon} = \delta, \quad x \in \partial\Omega. \tag{3.8}$$

Proof (i) Consider the following ODE:

$$\begin{aligned} &\underline{u}'(t) = \big(a|\Omega| - b\big)\underline{u}^m, \\ &\underline{u}(0) = \underline{u}_0, \end{aligned}$$

where $0 < \underline{u}_0 < \min_{\overline{\Omega}} u_0(x)$.

It is clear that $\underline{u}(t)$ is a subsolution of (1.1)–(1.3), and we know $\lim_{t\to T_0^-}\underline{u}(t)=+\infty$, where $T_0=\frac{1}{(m-1)(a|\Omega|-b)\underline{u}_0^{m-1}}$.

By the comparison principle, we could obtain our blow-up result immediately.

(ii) Let $\zeta(x)$ be the unique positive solution of the following elliptic problem: (3.7)–(3.8) and we choose $\sigma>0$ such that $0<\zeta(x)<1$.

Let

$$\overline{u}(x)=\rho\zeta(x),$$

where

$$0<\rho<\min\left\{1,\left(\frac{\sigma}{a\zeta^p(x)\int_\Omega\zeta^q(y)\,dy-b\zeta^m(x)}\right)^{\frac{1}{m-1}}\right\}.$$

Calculating directly, for $x\in\Omega$, we have that

$$\begin{aligned}&\overline{u}_t-\Delta\overline{u}-a\overline{u}^p\int_\Omega\overline{u}^q(y,t)\,dy+b\overline{u}^m\\&\quad=\sigma\rho-a\rho^{p+q}\zeta^p(x)\int_\Omega\zeta^p(y)\,dy+b\rho^m\zeta^m(x)\geq 0.\end{aligned}\tag{3.9}$$

On the other hand, for $x\in\partial\Omega$, we find that

$$\begin{aligned}&\frac{\partial\overline{u}}{\partial\nu}-\int_\Omega k(x,y)\overline{u}^l(y,t)\,dy\\&\quad=\rho\delta-\rho^l\int_\Omega k(x,y)\zeta^l(y)\,dy\\&\quad\geq\rho\delta-\rho^lM_2\int_\Omega\zeta(x)\,dx\geq\rho\Big[\delta-\rho^{l-1}M_2\max_\Omega\zeta|\Omega|\Big]\geq 0,\end{aligned}\tag{3.10}$$

where we choose $\max_\Omega\zeta(x)$ small enough.

Combining now (3.9)–(3.10) and by Lemma 2.1, it follows that $u(x,t)$ exists globally provided that $u_0(x)<\rho\zeta(x)$. The proof of Theorem 3.3 is complete. □

4 Conclusion

In this paper, we considered the properties of solutions for the reaction–diffusion equation with nonlinear absorption and with nonlinear nonlocal Neumann boundary condition and proved that the solution either exists globally or blows up in finite time depending on the initial data, the weighting function on the border, and nonlinear indexes in the equation by using the comparison principle. However, as far as we know, there is little literature on the blow-up properties for problem (1.1) with nonlinear inner absorptions and nonlinear nonlocal Neumann boundary condition. Due to the nonlinear diffusion terms and nonlinear nonlocal Neumann boundary condition, we have some new difficulties to overcome. First we should prove the comparison principle for problem (1.1) which plays an important role in the proof of our main results. Then, by the Neumann eigenvalue and its corresponding eigenfunctions to the eigenvalue problem for the equation, we construct a well-ordered positive supersolution and subsolution. Using the comparison principle, we achieve our purpose and obtain the global existence and blow-up of solutions to the problem. It should be pointed out that our results enrich and extend some previous results.

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