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# On a $p(x)$-biharmonic problem with Navier boundary condition

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**Abstract**

In this paper, we study a $p(x)$-biharmonic equation with Navier boundary condition

$$\begin{cases} \Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)}u$ is a $p(x)$-biharmonic operator with $p(x) \in C(\overline{\Omega})$, $p(x) > 1$. $\lambda, \mu \in \mathbb{R}$, $a \in L^\infty(\Omega)$ such that $\inf_{x\in\Omega} a(x) = a^- > 0$. By variational methods, we establish the results of existence and non-existence of solutions.

**MSC:** 35J60; 35B30; 35B40

**Keywords:** $p(x)$-biharmonic; Critical points theory; Variational methods

## 1 Introduction

In recent years, the study on variational problems with variable exponent is an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics (see [1–3]). We also refer to [4–14] for an overview.

Fourth-order equations have various applications in areas of applied mathematics and physics such as micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, thin plate theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells (see [15–17]). In addition, this type of equation can describe the static from change of beam or the sport of rigid body [18]. Many authors study the existence of multiple nontrivial solutions for some fourth order problems [18, 19]. The interplay between the fourth-order equation and the variable exponent equation goes to the $p(x)$-biharmonic problems. The $p(x)$-biharmonic operator possesses more complicated nonlinearities than the $p$-biharmonic one, for example, it is inhomogeneous.

For a $p(x)$-biharmonic problem, there are three common boundary conditions:

(1) Navier boundary condition, i.e.,

$$u = \Delta u = 0 \quad \text{on } \partial\Omega.$$

(2) Neumann type boundary condition, i.e.,

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}\bigl(|\Delta u|^{p(x)-2}\Delta u\bigr) = 0 \quad \text{on } \partial\Omega.$$

(3) No-flux boundary condition, i.e.,

$$\begin{cases} u = \text{constant}, \qquad \Delta u = 0 \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u)\, dS = 0. \end{cases}$$

Study on $p(x)$-biharmonic problems with Navier boundary condition was started probably in 2009, readers may refer to [20–23]. In the paper [20], El Amrouss et al. studied the $p(x)$-biharmonic problem both with Navier boundary condition and Neumann type boundary condition. The no-flux boundary condition represents the situations when the surfaces are impermeable to some contaminants. This condition was extended to the $p(x)$-biharmonic case by Boureanu et al. [24] from the work [9], where the no-flux boundary condition in $p(x)$-Laplacian equations is given by

$$\begin{cases} u = \text{constant} \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu}\, dS = 0. \end{cases}$$

Recently, Afrouzi et al. [23] studied the following $p(x)$-biharmonic problem with Navier boundary condition:

$$\begin{cases} \Delta^2_{p(x)} u + |u|^{p(x)-2}u = \lambda |u|^{q(x)-2}u + \mu |u|^{\gamma(x)-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q(x), \gamma(x) \in C_+(\overline{\Omega})$ and $\gamma^+ < p^- < p^+ < q^-$. Using variational methods, they established some existence and non-existence results of solutions for this problem.

Equations with two parameters involving $p$ or $p(x)$-Laplacian have been studied extensively. For example, in [25], Ricceri showed a further three critical points theorem which gives also information on the localization to the interval of the parameters, and gave an example as an application, i.e.,

$$\begin{cases} \Delta_p u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Subsequently, Ji generalized the equation above to a $p(x)$-Laplace equation both in Dirichlet and Neumann boundary conditions [26], i.e.,

$$\begin{cases} \Delta_{p(x)} u + |u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ Bu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In this paper, we consider the $p(x)$-biharmonic equation with Navier boundary condition:

$$\begin{cases} \Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the operator of fourth order called the $p(x)$-biharmonic operator with $p(x) \in C(\overline{\Omega})$, $p(x) > 1$. $\lambda, \mu \in \mathbb{R}$, $a \in L^\infty(\Omega)$ such that $\inf_{x\in\Omega} a(x) = a^- > 0$.

To apply the theorem in [25], as in papers [25–27] we denote by $\aleph$ the class of all functions $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfying the subcritical growth conditions:

$$\left|f(x,t)\right| \leq \tilde{c}\left(1 + |t|^{q(x)-1}\right) \quad \text{for all } (x,t) \in (\overline{\Omega} \times \mathbb{R}),$$

where $q \in C(\overline{\Omega})$ and $1 \leq q(x) \ll p_2^*(x)$ for all $x \in \overline{\Omega}$, $q_1 \ll q_2$ denotes $\operatorname{essinf}_{x\in\overline{\Omega}}(q_2(x) - q_1(x)) > 0$, and

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & \text{if } 2p(x) < N, \\ +\infty, & \text{if } 2p(x) \geq N. \end{cases}$$

Applying Ricceri's theorem [25], we get our first result as follows.

**Theorem 1.1** *Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with smooth boundary, assume $f \in \aleph$,*

$$\max\left\{\limsup_{\xi\to 0} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{p^+}}, \limsup_{\xi\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{p^-}}\right\} \leq 0, \tag{1.4}$$

*and*

$$\sup_{u\in X} \int_\Omega F\left(x,u(x)\right) > 0.$$

*Set*

$$\theta = \inf\left\{\frac{\int_\Omega \frac{|\Delta u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}\,dx}{\int_\Omega F(x,u(x))\,dx} : u \in X, \int_\Omega F\left(x,u(x)\right) > 0\right\}.$$

*Then, for each compact interval $[a,b] \subset (\theta,+\infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and every $g \in \aleph$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation* (P) *has at least three solutions whose norms are less than $r$.*

Notice that the work space $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is a separable, reflexive Banach space, we apply the fountain theorem and the dual fountain theorem to obtain infinitely many solutions. In this part, we suppose $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary, and assume

($H_0$) $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

$$\left|f(x,t)\right|\le c_1|t|^{q(x)-1},\qquad \left|g(x,t)\right|\le c_2|t|^{\gamma(x)-1},\quad \forall(x,t)\in\overline{\Omega}\times\mathbb{R},$$

where $c_1,c_2>0$, $q(x),\gamma(x)\in C(\overline{\Omega})$ and $1<q(x),\gamma(x)<p_2^*(x)$, $\forall x\in\overline{\Omega}$;

($H_1$) $\exists l>0$, $\mu>p^+$, $0<\mu F(x,s)\le f(x,s)s$ for $|s|\ge l$ and $x\in\Omega$;

($H_2$) $f(x,-s)=-f(x,s)$, $g(x,-s)=-g(x,s)$;

($H_3$) $\liminf_{t\to 0}\frac{G(x,t)}{|t|^\alpha}\ge 0$, $0<\alpha<p^-$;

($H_4$) $\limsup_{t\to\infty}\frac{F(x,t)}{|t|^{p^-}}=0$;

($H_5$) $f(x,t)\cdot t>0$, $g(x,t)\cdot t>0$ for all $(x,t)\in(\Omega\times\mathbb{R})$.

Then we establish our second result as follows.

**Theorem 1.2** *Assume that* ($H_0$) *holds, and* $p^+<q^-\le q(x)<p_2^*(x)$, $\gamma^+<p^-$, *then*

(i) *For every* $\lambda>0$, $\mu\in\mathbb{R}$, *with* ($H_1$), ($H_2$) *satisfied*, (P) *has a sequence of weak solutions* $(\pm u_k)$ *such that* $I_{\lambda,\mu}(\pm u_k)\to+\infty$ *as* $k\to+\infty$.

(ii) *For every* $\lambda>0$, $\mu>0$, *with* ($H_2$), ($H_3$) *satisfied*, (P) *has a sequence of weak solutions* $(\pm u_k)$ *such that* $I_{\lambda,\mu}(\pm u_k)<0$ *and* $I_{\lambda,\mu}(\pm u_k)\to 0$ *as* $k\to+\infty$.

(iii) *For every* $\lambda<0$, $\mu>0$, *with* ($H_3$), ($H_4$) *satisfied*, (P) *has at least one nontrivial weak solution.*

(iv) *For every* $\lambda<0$, $\mu<0$, *with* ($H_5$) *satisfied*, (P) *has no nontrivial weak solution.*

Assumption ($H_1$) is used to get the (P.S.) condition in part (i). ($H_3$) is used to verify condition ($B_2$) in the dual fountain theorem. ($H_4$) is used in (iii) to get the coerciveness of the functional. Actually, (1.1) comes completely as a special case of this work, and problems (1.2), (1.3) are generalized.

The paper consists of four sections. In Sect. 2, we start with some preliminary basic results for the variable exponent Lebesgue–Sobolev spaces. In Sect. 3, we prove Theorem 1.1. In Sect. 4, we prove Theorem 1.2.

## 2 Preliminaries

We start with some preliminary basic results for the variable exponent Lebesgue–Sobolev spaces. Set

$$C_+(\bar{\Omega})=\left\{h\in C(\bar{\Omega}):h(x)>1\ \forall x\in\bar{\Omega}\right\},$$

and

$$h^-=\inf_{x\in\bar{\Omega}}h(x),\qquad h^+=\sup_{x\in\bar{\Omega}}h(x)\quad\text{for any } h\in C(\bar{\Omega}).$$

For any $p(x)\in C_+(\bar{\Omega})$, denote

$$p_k^*(x)=\begin{cases}\frac{Np(x)}{N-kp(x)}, & \text{if } kp(x)<N,\\ +\infty, & \text{if } kp(x)\ge N.\end{cases}$$

The Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\Omega):=\left\{u:\Omega\to\mathbb{R}\text{ measurable and }\int_\Omega\left|u(x)\right|^{p(x)}dx<\infty\right\},$$

equipped with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

and it is a separable and reflexive Banach space.

**Proposition 2.1** ([10]) *For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega u(x)v(x)\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(\cdot)}|v|_{q(\cdot)} \le 2|u|_{p(\cdot)}|v|_{q(\cdot)},$$

*where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$.*

**Proposition 2.2** ([14]) *Let $\rho(u) = \int_\Omega |u|^{p(x)}\,dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < (=; >); 1 \Leftrightarrow \rho(u) < (=; >)\ 1$;
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$;
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho(u) \le |u|_{p(\cdot)}^{p^-}$;
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$;
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

Define the variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ by

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) | D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le m\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} u$ with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(\cdot)} := \sum_{|\alpha| \le m} |D^\alpha u|_{p(\cdot)},$$

becomes a separable, reflexive, and uniformly convex Banach space (see [11]).

**Proposition 2.3** ([10]) *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

*If we replace $\le$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. The weak solutions of problem (P) are considered in the Banach space

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

equipped with the norm

$$\|u\|_{W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)} = \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W_0^{1,p(x)}(\Omega)},$$

which is equivalent to the norm $\|\nabla u\|_{L^{p(x)}(\Omega)}$ (see [28]). Taking into account the particularity of problem (P), the following representation of the norm might be best:

$$\|u\|_a = \inf\left\{\theta > 0 : \int_\Omega \left( \left| \frac{\Delta u}{\theta} \right|^{p(x)} + a(x) \left| \frac{u}{\theta} \right|^{p(x)} \right) dx \le 1 \right\} \tag{2.1}$$

for all $u \in W^{2,p(x)}(\Omega)$ or $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. According to [20, 24], the three norms above are equivalent. For convenience, we choose $\|u\|_a$ as the norm on $X$ in the following.

**Proposition 2.4** ([20]) *Set* $\rho_a(u) = \int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)})\,dx$. *For* $u, u_n \in W^{2,p(\cdot)}(\Omega)$, *we have*

(1) $\|u\|_a < (=;>)\ 1 \Leftrightarrow \rho_a(u) < (=;>)\ 1$;
(2) $\|u\|_a > 1 \Rightarrow \|u\|_a^{p^+} \le \rho_a(u) \le \|u\|_a^{p^-}$;
(3) $\|u\|_a > 1 \Rightarrow \|u\|_a^{p^-} \le \rho_a(u) \ge \|u\|_a^{p^+}$;
(4) $\|u\|_a \to 0 \Leftrightarrow \rho_a(u_n) \to 0$;
(5) $\|u_n\|_a \to \infty \Leftrightarrow \rho_a(u_n) \to \infty$.

**Definition 2.1** We say that $u \in X$ is a weak solution of the boundary value problem (P) iff, for all $v \in X$,

$$\begin{aligned} &\int_\Omega |\Delta u|^{p(x)-2} \Delta u \cdot \Delta v\,dx + \int_\Omega a(x)|u|^{p(x)-2} uv\,dx \\ &\quad - \lambda \int_\Omega f(x,u) v\,dx - \mu \int_\Omega g(x,u) v\,dx = 0. \end{aligned}$$

We define $I_{\lambda,\mu} = \Phi - \lambda J - \mu \Psi$, $\lambda, \mu \in \mathbb{R}$, where

$$\begin{aligned} &\Phi(u) = \int_\Omega \frac{1}{p(x)} \left( |\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx, \\ &J(u) = \int_\Omega F(x,u)\,dx, \qquad \Psi(u) = \int_\Omega G(x,u)\,dx, \end{aligned}$$

and $F(x,t) = \int_0^t f(x,s)\,ds$, $G(x,t) = \int_0^t g(x,s)\,ds$.

**Proposition 2.5** ([20, 24]) *Let* $I_0$ *be the functional defined above, then*

(i) $\Phi \in C^1(X,\mathbb{R})$, *with the Gâteaux derivative defined by*

$$\langle \Phi'(u), v \rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\,dx + \int_\Omega a(x)|u|^{p(x)-2} uv\,dx.$$

(ii) $\Phi$ *is sequentially weakly lower semicontinuous, that is, for any* $u \in X$ *and any subsequence* $(u_n)_n \subset X$ *such that* $u_n \rightharpoonup u$ *weakly in* $X$, *there holds* $I\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$.
(iii) *the mapping* $\Phi' : X \to X'$ *is strictly monotone, bounded homeomorphism and is of type* $S_+$, *namely* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty} \Phi'(u_n)(u_n - u) \le 0$ *implies that* $u_n \to u$.

Similar to [4, 29], we have the following.

**Proposition 2.6** *If h satisfies assumption* ($\mathrm{H}_0$), *then* $\phi$ *is a continuously Gâteaux differentiable functional with*

$$\phi'(u)(v) = \int_\Omega h\big(x, u(x)\big)v(x)\,dx$$

*for each* $u, v \in W^{2,p(x)}(\Omega)$ *and* $\phi'$ *is a compact operator.*

*Remark* 2.1 (See Remark 2.1 in [9]) Noting that a sum of a mapping of type $(S_+)$ and a weakly-strongly continuous mapping is still a mapping of type $(S_+)$, then $I'_{\lambda,\mu} = \Phi' - \lambda J' - \mu\Psi'$ is a mapping of type $(S_+)$. Hence, any bounded $(P.S.)$ sequence of $I_{\lambda,\mu}$ has a convergent subsequence.

Due to the properties fulfilled by $f$ and $g$, we can deduce that $I_{\lambda,\mu}$ is of class $C^1$ with the Gâteaux derivative

$$\begin{aligned}\big\langle I'_{\lambda,\mu}(u), v\big\rangle &= \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,dx + \int_\Omega a(x)|u|^{p(x)-2}uv\,dx \\ &\quad - \lambda\int_\Omega f(x,u)v\,dx - \mu\int_\Omega g(x,u)v\,dx.\end{aligned}$$

So any critical point of $I_{\lambda,\mu}$ is a weak solution to (P).

## 3 Proof of Theorem 1.1

Firstly, we define a class of functions needed in this section as follows:

If $X$ is a real Banach space, we denote by $\mathscr{W}_X$ the functional $\Phi : X \to \mathbb{R}$ possessing the following property: if $\{u_n\}$ is a sequence in $X$ converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to $u$.

**Lemma 3.1** ([25]) *Let X be a separable and reflexive and real Banach space*;

$\Phi : X \to \mathbb{R}$ *be a coercive, sequentially weakly lower semicontinuous* $C^1$ *functional belonging to* $\mathscr{W}_X$, *bounded on each bounded subset of X and whose derivative admits a continuous inverse on* $X^*$;

$J : X \to \mathbb{R}$ *be a* $C^1$ *functional with compact derivative. Assume that* $\Phi$ *has a strict local minimum* $x_0$ *with* $\Phi(x_0) = J(x_0) = 0$.

*Finally, setting*

$$\alpha = \max\left\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)}, \lim_{x\to x_0} \frac{J(x)}{\Phi(x)}\right\}, \qquad \beta = \sup_{x\in\Phi^{-1}(0,+\infty)} \frac{J(x)}{\Phi(x)},$$

*assume that* $\alpha < \beta$.

*Then, for each compact interval* $[a,b] \subset (\frac{1}{\beta}, \frac{1}{\alpha})$ (*with the conventions* $\frac{1}{0} = +\infty$, $\frac{1}{\infty} = 0$), *there exists* $r > 0$ *with the following property*: *for every* $\lambda \in [a,b]$ *and every* $C^1$ *functional* $\Psi : X \to \mathbb{R}$ *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the equation*

$$\Phi'(x) = \lambda J'(x) + \mu\Psi'(x)$$

*has at least three solutions whose norms are less than r.*

*Proof of Theorem* 1.1 Fix $\epsilon > 0$, in view of (P), there exist $\rho_1, \rho_2$ with $0 < \rho_1 < 1 < \rho_2$ such that

$$F(x,\xi) \le \epsilon|\xi|^{p^+} \quad \text{for all } (x,\xi) \in \Omega \times [-\rho_1, \rho_1], \tag{3.1}$$

and

$$F(x,\xi) \le \epsilon|\xi|^{p^-} \quad \text{for all } (x,\xi) \in \Omega \times \bigl(\mathbb{R} \setminus [-\rho_2, \rho_2]\bigr). \tag{3.2}$$

It is clear that

$$F(x,\xi) \le \epsilon|\xi|^{p^+} \quad \text{for all } (x,\xi) \in \Omega \times \bigl(\mathbb{R} \setminus \bigl([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2]\bigr)\bigr). \tag{3.3}$$

Since $F$ is bounded on each bounded subset of $\Omega \times \mathbb{R}$, we can choose $c_3 > 0$ and $q_1'$ with $p^+ < q_1' \ll p_2^*(x)$ such that

$$F(x,\xi) \le \epsilon|\xi|^{p^+} + c_3|\xi|^{q_1'} \quad \text{for all } (x,\xi) \in (\Omega \times \mathbb{R}). \tag{3.4}$$

By the embedding theorem, there exist $c_4, c_5 > 0$, we have if $\|u\|_a < 1$,

$$J(u) \le c_4^{p^+}\epsilon\|u\|_a^{p^+} + c_5\|u\|_a^{q_1'}.$$

So we get that

$$\limsup_{u\to 0} \frac{J(u)}{\Phi(u)} \le p^+ c_4^{p^+}\epsilon. \tag{3.5}$$

If $\|u\|_a > 1$, by (3.2), we have

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int_\Omega F(x,u(x))\,dx}{\int_\Omega \frac{|\Delta u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}\,dx} \\
&\le \frac{p^+ \int_{\Omega(|u|\le\rho_2)} F(x,u)\,dx}{\|u\|_a^{p^-}} + \frac{p^+ \int_{\Omega(|u|>\rho_2)} F(x,u)\,dx}{\|u\|_a^{p^-}}.
\end{aligned}$$

Then we have

$$\limsup_{\|u\|_a\to+\infty} \frac{J(u)}{\Phi(u)} \le p^+ c_6^{p^-}\epsilon. \tag{3.6}$$

Since $\epsilon$ is arbitrary, combining (3.5) and (3.6) we get

$$\max\left\{\limsup_{u\to 0}\frac{J(u)}{\Phi(u)}, \limsup_{\|u\|_a\to+\infty}\frac{J(u)}{\Phi(u)}\right\} \le 0.$$

Then by Lemma 3.1, $\alpha = 0$, and $\beta > 0$ by assumption. Thus all the hypotheses of Lemma 3.1 are satisfied, the proof is complete. □

## 4 Proof of Theorem 1.2

In this section, we study the existence and non-existence of weak solutions for (P). In the following, the letter $c_i > 0$ is constant.

**Lemma 4.1** *Let $X$ be a separable and reflexive Banach space, then there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that*

$$X = \overline{\operatorname{span}}\{e_j : j = 1, 2, \ldots\}, \qquad X^* = \overline{\operatorname{span}}\{e_j^* : j = 1, 2, \ldots\},$$

*with*

$$\langle e_j, e_j^* \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Define

$$X_j = \operatorname{span}\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{4.1}$$

Similar to [21], we have the following.

**Proposition 4.1** *If $q(x), \gamma(x) \in C_+(\overline{\Omega})$, $q(x), \gamma(x) < p_2^*(x)$ for $x \in \overline{\Omega}$, let*

$$\beta_k = \sup\{|u|_{q(x)} : \|u\|_a = 1, u \in Z_k\},$$
$$\theta_k = \sup\{|u|_{\gamma(x)} : \|u\|_a = 1, u \in Z_k\},$$

*then* $\lim_{k\to\infty} \beta_k = 0$, $\lim_{k\to\infty} \theta_k = 0$.

**Lemma 4.2** (Fountain theorem [30]) *Let*

(A$_1$) *$I \in C^1(X, \mathbb{R})$ be an even functional, where $(X, \|\cdot\|)$ is a separable and reflexive Banach space, the subspaces $X_k$, $Y_k$, and $Z_k$ are defined by (4.1).*

*If for each $k \in \mathbb{R}$, there exist $\rho_k > r_k > 0$ such that*

(A$_2$) $\inf\{I(u) : u \in Z_k, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$;

(A$_3$) $\max\{I(u) : u \in Y_k, \|u\| = \rho_k\} \leq 0$;

(A$_4$) *$I$ satisfies the (P.S.) condition for every $c > 0$.*

*Then $I$ has an unbounded sequence of critical points.*

**Lemma 4.3** (Dual fountain theorem [30]) *Assume that* (A$_1$) *is satisfied and there is $k_0 > 0$ such that, for each $k > k_0$, there exist $\rho_k > r_k > 0$ such that*

(B$_1$) $\inf\{I(u) : u \in Z_k, \|u\| = \rho_k\} \geq 0$;

(B$_2$) $\max\{I(u) : u \in Y_k, \|u\| = r_k\} < 0$;

(B$_3$) $\inf\{I(u) : u \in Z_k, \|u\| \leq \rho_k\} \to 0$ *as* $k \to +\infty$;

(B$_4$) *$I$ satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.*

*Then $I$ has a sequence of negative critical values converging to 0.*

**Definition 4.1** We say that $I_{\lambda,\mu}$ satisfies the $(PS)_c^*$ condition (with respect to $(Y_n)$) if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I_{\lambda,\mu}(u_{n_j}) \to m$, and $(I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of $I_{\lambda,\mu}$.

**Proposition 4.2** *$I_{\lambda,\mu}$ is weakly lower semicontinuous on $X$.*

*Proof* By Proposition 2.5, we know that $I_0$ is weakly lower semicontinuous. Assume $u_n \rightharpoonup u$ in $X$, the compact embedding by Proposition 2.3 gives us

$$u_n \to u \quad \text{in } L^{p(x)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^1(\Omega). \tag{4.2}$$

Using the mean value theorem, there exists $v$ which takes values strictly between $u$ and $u_n$ such that

$$\int_\Omega \bigl|F(x,u_n) - F(x,u)\bigr|\,dx \le \int_\Omega |u_n - u| \sup_{x\in\Omega}\bigl|f(x,v)\bigr|\,dx,$$

hence by assumption $(\mathrm{H}_0)$ and (4.2) the functional $J(u) = \int_\Omega F(x,u)\,dx$ is weakly continuous, and so is $\Psi(u) = \int_\Omega G(x,u)\,dx$. Consequently, the functional $I_{\lambda,\mu}$ is weakly lower semicontinuous. □

Consequently, we come to the following.

*Proof of Theorem* 1.2 (i) First we verify that $I_{\lambda,\mu}$ satisfies the (P.S.) condition. Suppose that $(u_n) \subset X$ is a (P.S.) sequence, i.e.,

$$I_{\lambda,\mu}(u_n) \le m, \qquad I'_{\lambda,\mu}(u_n) \to 0 \quad \text{as } n \to \infty.$$

By Remark 2.1, it is sufficient to verify that $(u_n)$ is bounded. Assume that $(\mathrm{H}_1)$ holds and $\|u_n\|_a > 1$ for $n$ large enough, we have

$$\begin{aligned}
&I'_{\lambda,\mu}(u_n) - \frac{1}{\mu}\bigl\langle I'_{\lambda,\mu}(u_n), u_n\bigr\rangle \\
&\quad\ge \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_n\|_a^{p^-} + \lambda \int_\Omega \left(\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\qquad - \mu \int_\Omega \left(\frac{1}{\mu} G(x,u_n) - g(x,u_n)u_n\right) dx \\
&\quad= \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_n\|_a^{p^-} + \lambda \int_{\Omega\cap\{|u_n|\le l\}} \left(\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\qquad + \lambda \int_{\Omega\cap\{|u_n|> l\}} \left(\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\qquad - \mu \int_\Omega \left(\frac{1}{\mu} G(x,u_n) - g(x,u_n)u_n\right) dx \\
&\quad\ge \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_n\|_a^{p^-} + c_7 - \mu \int_\Omega \left(\frac{1}{\mu} G(x,u_n) - g(x,u_n)u_n\right) dx \\
&\quad\ge \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_n\|_a^{p^-} + c_7 - c_8\mu\|u_n\|_a^{\gamma^+}.
\end{aligned} \tag{4.3}$$

Since $p^- > \gamma^+$, we know that $\{u_n\}$ is bounded in $X$. In the following we will prove that $(\mathrm{A}_2)$ holds.

For any $u \in Z_k$, $\|u\|_a = r_k > 1$ ($r_k$ will be given below), we have

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda\int_\Omega F(x,u)\,dx - \mu\int_\Omega G(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda\int_\Omega F(x,u)\,dx - \mu c_9\|u\|_a^{\gamma^+}. \end{aligned} \tag{4.4}$$

By condition ($H_1$) and Proposition 2.2, we have

$$\int_\Omega F(x,u)\,dx \leq \int_\Omega c_1|u|^{q(x)}\,dx \leq \begin{cases} c_{10}, & \text{if } |u|_{q(x)} \leq 1, \\ c_{11}|u|_{q(x)}^{q^+}, & \text{if } |u|_{q(x)} > 1. \end{cases} \tag{4.5}$$

However, if $|u|_{q(x)} > 1$, we have

$$|u|_{q(x)}^{q^+} = \left|\frac{u}{\|u\|_a}\|u\|_a\right|_{q(x)}^{q^+} \leq \left(\beta_k\|u\|_a\right)^{q^+}. \tag{4.6}$$

Combining (4.4) and (4.6), we conclude that

$$I_{\lambda,\mu}(u) \geq \frac{1}{2p^+}\|u\|_a^{p^-} - \lambda c_{12}\left(\beta_k\|u\|_a\right)^{q^+} - c_{13}.$$

Choose $r_k = (2\lambda q^+ c_{12}\beta_k^{q^+})^{\frac{1}{p^- - q^+}}$, notice that $p^- < p^+ < q^+$, by Proposition 4.1, we deduce that $r_k \to +\infty$ as $k \to \infty$, hence

$$I_{\lambda,\mu}(u) \geq \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{q^+}\right)r_k^{p^-} - c_{13} \to \infty.$$

For ($A_3$), let $u \in Y_k$ such that $\|u\|_a = \rho_k > r_k > 1$, then

$$\begin{aligned} I_{\lambda,\mu}(u) &\leq \frac{1}{p^-}\|u\|_a^{p^+} - \lambda\int_\Omega F(x,u)\,dx - \mu\int_\Omega G(x,u)\,dx \\ &\leq \frac{1}{p^-}\|u\|_a^{p^+} - \lambda\int_{\Omega\cap\{|u|\leq l\}} F(x,u)\,dx - \lambda\int_{\Omega\cap\{|u|>l\}} F(x,u)\,dx \\ &\quad + |\mu|c_2\int_\Omega |u|^{\gamma(x)}\,dx. \end{aligned}$$

Since $\dim Y_k < \infty$, all norms are equivalent in $Y_k$, we get that

$$I_{\lambda,\mu}(u) \leq \frac{1}{p^-}\|u\|_a^{p^+} - c_{14} - c_{15}\lambda\|u\|_a^{\mu} + c_{16}|\mu|\|u\|_a^{\gamma^+}.$$

We get that $I_{\lambda,\mu}(u) \to -\infty$ as $\|u\|_a \to +\infty$ since $\gamma^+ < p^+ < \mu$, so ($A_3$) holds. Obviously $I_{\lambda,\mu}(u)$ is even, by Lemma 4.2 then (i) is verified.

(ii) We will use Lemma 4.3 to prove conclusion (ii). For ($B_1$), for any $u \in Z_k$, we have

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{1}{p^+}\|u\|_a^{p^+} - \lambda\int_\Omega F(x,u)\,dx - \mu\int_\Omega G(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^+} - \lambda c_{17}\|u\|_a^{q^-} - c_2\mu\int_\Omega |u|^{\gamma(x)}\,dx. \end{aligned}$$

Notice that $q^- > p^+$, there exists $\rho_0 > 0$ small enough such that $\lambda c_{17}\|u\|_a^{q^-} \le \frac{1}{2p^+}\|u\|_a^{p^+}$ as $0 < \rho = \|u\|_a \le \rho_0$. Then, by Proposition 2.4, we have

$$I_{\lambda,\mu}(u) \ge \begin{cases} \frac{1}{2p^+}\|u\|_a^{p^+} - \mu c_{18}(\theta_k\|u\|_a)^{\gamma^-}, & \text{if } |u|_{\gamma(x)} \le 1, \\ \frac{1}{2p^+}\|u\|_a^{p^+} - \mu c_{19}(\theta_k\|u\|_a)^{\gamma^+}, & \text{if } |u|_{\gamma(x)} > 1. \end{cases} \tag{4.7}$$

Choosing

$$\rho_k = \max\Bigl\{\bigl(2p^+c_{18}\mu\theta_k^{\gamma^-}\bigr)^{\frac{1}{p^+-\gamma^-}}, \bigl(2p^+c_{19}\mu\theta_k^{\gamma^+}\bigr)^{\frac{1}{p^+-\gamma^+}}\Bigr\},$$

notice that $p^+ > \gamma^+$, from Proposition 4.1 we deduce that $\rho_k \to 0$ as $k \to \infty$, hence $I_{\lambda,\mu}(u) \ge 0$.

($B_2$) For $u \in Y_k$ with $\|u\|_a \le 1$. Assumption ($H_3$) is equivalent to

$$\exists\delta > 0, \quad G(x,t) \ge c_{20}t^\alpha, \quad \alpha < p^-, \forall t \in (0,\delta). \tag{4.8}$$

Then we have

$$\begin{aligned} I_{\lambda,\mu}(u) &\le \frac{1}{p^-}\|u\|_a^{p^-} + \lambda\int_\Omega c_1|u|^{q(x)}\,dx - \mu c_{20}\int_\Omega |u|^\alpha\,dx \\ &\le \frac{1}{p^-}\|u\|_a^{p^-} + \lambda c_1\|u\|_a^{q^-} - \mu c_{20}\|u\|_a^\alpha. \end{aligned} \tag{4.9}$$

Since $\alpha < p^- < q^-$, there exists $r_k \in (0,\rho_k)$ such that $I_{\lambda,\mu}(u) < 0$ when $\|u\| = r_k$.

($B_3$) Notice that $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf_{u\in Z_k,\|u\|_a\le\rho_k} I_{\lambda,\mu}(u) \le b_k = \max_{u\in Y_k,\|u\|_a=r_k} \inf I_{\lambda,\mu}(u) < 0.$$

For $u \in Z_k$, $\|u\|_a \le \rho_k$ small enough. From (4.7), we have

$$I_{\lambda,\mu}(u) \ge \frac{1}{2p^+}\|u\|_a^{p^+} - \mu c_{21}\theta_k^{\gamma^+}\|u\|_a^{\gamma^+} \ge -\mu c_{21}\theta_k^{\gamma^+}\|u\|_a^{\gamma^+}.$$

Since $\theta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, then ($B_3$) holds.

In the following, we verify the $(PS)_c^*$ condition. Suppose $\{u_{n_j}\} \subset X$ such that

$$u_{n_j} \in Y_{n_j}, \quad I_{\lambda,\mu}(u_{n_j}) \to m, \qquad (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) \to 0 \quad \text{as } n_j \to +\infty.$$

Similar to ($A_1$), we can get the boundedness of $\|u_{n_j}\|_a$. Hence, there exists $u \in X$ such that $u_{n_j} \rightharpoonup u$ weakly in $X = \overline{\bigcup_{n_j} Y_{n_j}}$. Then we can find $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \rightharpoonup u$. We have

$$\bigl\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\bigr\rangle = \bigl\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - v_{n_j}\bigr\rangle + \bigl\langle I'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\bigr\rangle.$$

Notice that $u_{n_j} - v_{n_j} \in Y_{n_j}$, it yields

$$\begin{aligned} \bigl\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\bigr\rangle &= \bigl\langle (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\bigr\rangle + \bigl\langle I'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\bigr\rangle \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.10}$$

Since $I'_{\lambda,\mu}$ is of $(S_+)$ type, we deduce that $u_{n_j} \to u$ in $X$; furthermore $I'_{\lambda,\mu}(u_{n_j}) \to I'_{\lambda,\mu}(u)$.

Now we claim that $u$ is a critical point of $I_{\lambda,\mu}$. Taking $\omega_k \in Y_k$, when $n_j \geq k$ we have

$$\begin{aligned}\langle I'_{\lambda,\mu}(u), \omega_k\rangle &= \langle I'_{\lambda,\mu}(u) - I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle + \langle I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle \\ &= \langle I'_{\lambda,\mu}(u) - I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle\rangle + \langle (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), \omega_k\rangle.\end{aligned}$$

Taking $n_j \to \infty$, we obtain

$$\langle I'_{\lambda,\mu}(u), \omega_k\rangle = 0, \quad \forall \omega_k \in Y_k.$$

So $I'_{\lambda,\mu}(u) = 0$, this verifies that $I_{\lambda,\mu}$ satisfies the $(PS)^*_c$ condition.

(iii) By Proposition 4.2 we know that $I_{\lambda,\mu}$ is weakly lower semi-continuous, next we will prove the coerciveness of $I_{\lambda,\mu}$. By adding $(\mathrm{H}_4)$, for any $\epsilon > 0$ small, there exists $M > 0$ such that

$$|F(x,t)| \leq \epsilon |t|^{p^-} \quad \text{for } |t| > M.$$

Therefore, when $\lambda \leq 0$, $\mu > 0$, we get that

$$\begin{aligned}I_{\lambda,\mu}(u) &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda\int_\Omega F(x,u)\,dx - \mu\int_\Omega G(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} + \lambda\epsilon\|u\|_a^{p^-} - \mu c_{22}\|u\|_a^{\gamma^+} \to +\infty,\end{aligned} \tag{4.11}$$

as $\|u\|_a \to \infty$ since $\gamma^+ < p^-$. By the Weierstrass theorem, we know that there exists a global minimizer $u_0$ to $I_{\lambda,\mu}(u)$ in $X$. Next we show that $u_0$ is nontrivial.

Choose $v_0 \in C_0^\infty(\Omega)$ such that $0 < v_0 \leq \delta$, by $(\mathrm{H}_0)$ and (4.8), we have

$$\begin{aligned}I_{\lambda,\mu}(tv_0) &\leq t^{p^-}\|v_0\|_a^{p^-} - \lambda\int_\Omega F(x,tv_0)\,dx - \mu\int_\Omega G(x,tv_0)\,dx \\ &\leq t^{p^-}\|v_0\|_a^{p^-} + |\lambda|c_1\int_\Omega t^{p(x)}|v_0|^{p(x)}\,dx - \mu c_{20}\int_\Omega t^\alpha|v_0|^\alpha\,dx \\ &\leq t^{p^-}\|v_0\|_a^{p^-} + |\lambda|c_1 t^{p^-}\int_\Omega |v_0|^{p(x)}\,dx - \mu c_{20}t^\alpha\int_\Omega |v_0|^\alpha\,dx.\end{aligned} \tag{4.12}$$

Notice that $\alpha < p^-$, we can find $t_0 \in (0,1)$ such that $I_{\lambda,\mu}(tv_0) < 0$, so $u_0$ is nontrivial.

(iv) When $\lambda < 0$, $\mu < 0$, we argue by contradiction that $u \in X \setminus \{0\}$ is a weak solution of (1.1). Multiplying (P) by $u$ and integrating by part, we have

$$\int_\Omega |\Delta u|^{p(x)}\,dx + \int_\Omega a(x)|u|^{p(x)}\,dx = \lambda\int_\Omega f(x,u)u\,dx + \mu\int_\Omega g(x,u)u\,dx.$$

It is contrary to condition $(\mathrm{H}_5)$, the proof is complete. □

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