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Periodic solution for second-order impulsive differential inclusions with relativistic operator

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Abstract

In this paper, the boundary value problem of a second-order impulsive differential inclusion involving a relativistic operator is studied. First, the singular problem is reduced to an equivalent non-singular problem in order to better apply the variational methods. Then the existence of a periodic solution is obtained by nonsmooth critical point theory. Moreover, the boundedness and nonnegativity of solutions are obtained by restricting the discontinuous nonlinear term.

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1 Introduction

By physical experiments, a change of the mass of an object occurs when the velocity of the object is comparable with the speed of light. So in the range of high speeds, the mass of an object is no longer a constant. In the special theory of relativity, Newton's second law $F = \frac{d}{dt}(mv)$ still holds. Just the mass m is no longer a constant, but a function of the rate of the object's movement. So the relativistic dynamics fundamental equation $F = \frac{d}{dt}[\frac{m_0}{\sqrt{1-(\frac{v}{c})^2}}v]$ can explain the problem of the "Bell slow contraction effect" better, where F is the force of an object in motion, m_0 is the mass of an object at rest, v is the velocity and $c = 3.0 \times 10^8$ m/s is the velocity of light. For the theory and application of special relativity, we refer the reader to [1–7].

Based on the understanding of the relativistic dynamics fundamental equation, mathematicians put forward relativistic operators which have a close relationship with physics. In recent years, the research of relativistic operators has attracted widespread attention of mathematical scholars due to the physical significance and applicability. In 2010, Brezis and Mawhin [8] showed that the solution of the following problem:

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + a\sin u = e(t), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{1.1}$$

namely $u \in C^1[0,T]$ satisfying $\|u\|_\infty < 1$, using $\frac{u'}{\sqrt{1-u'^2}}$, is absolutely continuous. Moreover, the solution of associated solutions of (1.1) can be associated to the critical points of the energy functional

$$R(u) := \int_0^T \left[1 - \sqrt{1 - |u'|^2} + a\cos u + eu\right] dt,$$

which is defined on the closed convex set

$$K = \left\{u \in W^{1,\infty}\big([0,T]\big) : u(0) = u(T), \|u\|_\infty < 1\right\}.$$

In 2011, Bereanu, Jebelean and Mawhin [9] studied the multiple solutions for Neumann and periodic problems involving a singular ϕ-Laplacian. Furthermore, the authors creatively extended the energy functional $R(u)$ to $S(u) = \Phi(u) + g(u)$ in the space $C_T = \{u \in C([0,T]) : u(0) = u(T)\}$, where

$$\Phi(u) = \begin{cases} \int_0^T [1 - \sqrt{1 - u'^2}]\, dt & \text{if } u \in W^{1,\infty}([0,T]), \\ +\infty & \text{if } u \in C([0,T]) \backslash W^{1,\infty}([0,T]), \end{cases}$$

$$g(u) = \int_0^T [a\cos u + eu]\, dt.$$

In 2015, Jebelean and Mawhin [10] discussed some existence results of the problem

$$-\big(\varphi(u')\big)' = \nabla_u F(t,u), \qquad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where

$$\varphi(y) = \frac{y}{\sqrt{1 - |y|^2}}, \quad y \in B(1). \tag{1.2}$$

Here $B(1) \subset \mathbb{R}^N$ denotes the open ball of center 0 and radius 1. The potential $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfied the L^1 Carathéodory conditions. There exists $\alpha(t) \in L^1$ such that $|\nabla_x F(t,x)| \leq \alpha(t)$. The authors reduced the singular problem to the non-singular problem

$$\big(-\psi(u')\big)' = \nabla_u F(t,u), \qquad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1.3}$$

and they defined $\psi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\psi(y) = \begin{cases} \frac{y}{\sqrt{1-|y|^2}}, & |y| \leq R, \\ \frac{y}{\sqrt{1-R^2}}, & |y| > R, \end{cases} \tag{1.4}$$

where $R \in (0,1)$. Furthermore, they proved the problem (1.3) has at least $N + 1$ geometrically distinct periodic solutions and the system with oscillating potential has infinitely many solutions.

In 2016, Mawhin [11] gave recent results on the multiplicity of T-periodic solutions of differential systems of the form

$$\left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)' + \nabla_u F(t,u) = e(t),$$

based on several techniques of critical point theory. Besides, [12–15] studied the equations describing relativistic pendulum well.

In 2016, Jebelean, Mawhin and Serban [16] studied the existence and multiplicity of periodic solutions for the differential inclusion

$$-\big(\varphi(u')\big)' \in \partial F(t,u), \qquad u(0)-u(T)=0=u'(0)-u'(T), \tag{1.5}$$

where φ is defined as (1.2). The problem (1.5) has at least one solution and infinitely many solutions by restricting different assumptions to nonlinear terms.

It was observed that the above literature did not consider the impulsive effect. The impulsive effect [17] can describe the processes which undergo the sudden changes or discontinuous jumps in the real world. Thus impulsive effects caused great concern in many fields such as industrial robotics, population dynamics, control theory, physics and so on. Many mathematicians have conducted detailed and in-depth research on the impulsive differential equations [18–30]. Besides, nonlinear boundary value problems [8, 31–36] play an important role in solving mathematical physics problems.

However, to the best of our knowledge, there are few papers concerned with impulsive differential inclusion involving a relativistic operator. Motivated by [10–12, 16, 37–40], we study the existence of solutions for the following problem:

$$\begin{cases} -(\varphi(u'(t)))' + \varphi(u(t)) \in \lambda\partial F(t,u), & t\in[0,T]\backslash\{t_1,t_2,\dots,t_m\}, \\ -\Delta\varphi(u'(t_i)) = I_i(u(t_i)), & i=1,2,\dots,m, \\ u(0)=u(T), \qquad u'(0)=u'(T), \end{cases} \tag{1.6}$$

where φ is defined as (1.2), $\Delta\varphi(u'(t_i)) = \varphi(u'(t_i^+)) - \varphi(u'(t_i^-))$. $\partial F(t,u)$ is the generalized Clarke gradient of $F(t,\cdot)$ at $u\in\mathbb{R}^N$, $I_i\in C(\mathbb{R}^N;\mathbb{R}^N)$, λ is a positive parameter.

In order to treat the problem (1.6), we firstly consider the following non-singular problem:

$$\begin{cases} -(\psi(u'(t)))' + \psi(u(t)) \in \lambda\partial F(t,u), & t\in[0,T]\backslash\{t_1,t_2,\dots,t_m\}, \\ -\Delta\psi(u'(t_i)) = I_i(u(t_i)), & i=1,2,\dots,m, \\ u(0)=u(T), \qquad u'(0)=u'(T), \end{cases} \tag{1.7}$$

where ψ is defined as (1.4).

We shall apply the critical point theorem [38] to obtain the existence and the property of weak solution for (1.7). Under some assumptions, the solution of non-singular problem (1.7) is the solution of problem (1.6). Moreover, we will prove problem (1.6) has at least one nonnegative solution by variational approach. With the impulsive effects and differential inclusion taken into consideration, difficulties such as how to change the problem (1.7) into problem (1.6), how to deal with the non-differentiablity of the energy functional and how to prove the critical point of energy functional is classical solution of (1.7) have to be overcome. We obtained the existence of periodic solution by critical point theorem. Moreover, the nonnegativity and boundedness of the solutions are presented.

This paper is organized as follows. In Sect. 2, we recall some definitions and lemmas which are critical to main results. In Sect. 3, the existence results of solutions are given.

In Sect. 4, we give conclusions about the differences between the research results and the reference [12]. In Sect. 5, an example is presented to verify Theorem 3.2. In the Appendix, the derivation of complex inequalities is given.

2 Preliminaries

In order to better understand the main contents of this article, we introduce some nonsmooth theory in this section.

Let the space

$$X = \{u \in H^1([0,T];\mathbb{R}^N) : u(0) = u(T)\},$$

with the norm

$$\|u\|_X = \left(\int_0^T |u|^2 + |u'|^2\,dt\right)^{\frac{1}{2}}.$$

Clearly, $(X, \|\cdot\|_X)$ is a reflexive real Banach space and its topological dual is $(X^*, \|\cdot\|_{X^*})$. $I : X \to \mathbb{R}$ is locally Lipschitz if for each $u \in X$, there exist a neighborhood Ω of u and a real number k satisfying

$$|I(x) - I(y)| \le k\|x-y\|_X, \quad \forall x, y \in \Omega.$$

For a locally Lipschitz functional I and a function $u \in X$, the generalized directional derivative at u along the direction $v \in X$ is defined by

$$I^0(u;v) = \lim_{y\to u}\sup_{h\to 0^+}\frac{I(y+hv) - I(y)}{h}.$$

The generalized Clarke gradient of I at u is

$$\partial I(u) = \{u^* \in X^* : \langle u^*, v\rangle \le I^0(u;v), \forall v \in X\},$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X and X^*.

Definition 2.1 ([37]) The function $u \in X$ is a critical point of locally Lipschitz functional I if $0 \in \partial I(u)$, i.e., $I^\circ(u;v) \ge 0$, $\forall v \in X$.

Lemma 2.1 ([39]) *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and*

$$\varphi^0(u;v) = \langle\varphi'(u), v\rangle \quad \text{for all } u, v \in X;$$
$$\partial\varphi(u) = \{\varphi'(u)\} \quad \text{for all } u \in X.$$

Lemma 2.2 ([16]) *Assume that $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H_F) and let I_F be defined by $I_F := \hat{F}|H_T = \hat{F} \circ i$ and $u \in H_T$. If $l \in \partial I_F$, then there is some $u_l \in L^1$ such that $|u_l(t)| \le \alpha(t)$, $u_l \in \partial F(t,u(t))$ for a.e. $t \in [0,T]$, $\langle l, v\rangle = -\int_0^T (u_l|v)\,dt$, $\forall v \in H_T$.*

Lemma 2.3 ([38]) *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. Suppose that there exist two numbers $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < r_2$ such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2),$$

where

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1},$$

for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ the function $I_\lambda = \Phi - \lambda \Psi$ fulfills the $^{[r_1]}(PS)^{[r_2]}$-condition (PS: Palais–Smale).

Then, for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there exists $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ with $u_{0,\lambda}$ being a critical point of I_λ.

Next we consider the non-singular system (1.7). Let the functional $I : X \to \mathbb{R}$ defined by

$$I(u) = \int_0^T \Psi\left(u'\right) dt + \int_0^T \Psi(u)\, dt - \lambda \int_0^T F(t,u)\, dt \circ i - \sum_{i=1}^m \int_0^{u(t_i)} I_i(s)\, ds,$$

where $\Psi : \mathbb{R}^N \to \mathbb{R}$,

$$\Psi(y) = \begin{cases} 1 - \sqrt{1 - |y|^2}, & |y| \le R, \\ 1 - \sqrt{1 - |R|^2} + \frac{|y|^2 - R^2}{2\sqrt{1-R^2}}, & |y| > R, \end{cases}$$

and $R < 1$. i is an embedded map satisfying $H_T \overset{i}{\hookrightarrow} C$.

In the next discussion, we let $I(u) = \Phi(u) - \lambda \tilde{\Psi}(u)$, where $\Phi(u) = \int_0^T \Psi(u')\, dt + \int_0^T \Psi(u)\, dt - \sum_{i=1}^m \int_0^{u(t_i)} I_i(s)\, ds$, $\tilde{\Psi}(u) = \hat{F}(u)|_{H_T} = \hat{F}(u) \circ i$, where $\hat{F}(u) = \int_0^T F(t,u)\, dt$.

Definition 2.2 A function u is a classical solution of problem (1.7), if $u : [0,T] \backslash \{t_1, t_2, \dots, t_m\} \to \mathbb{R}^N$ is of class C^1 with $\psi(u')$ absolutely continuous, satisfying

$$\begin{cases} -(\psi(u'(t)))' + \psi(u(t)) = u^*, & t \in [0,T] \backslash \{t_1, t_2, \dots, t_m\}, \\ -\Delta \psi(u'(t_i)) = I_i(u(t_i)), & i = 1, 2, \dots, m, \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{cases} \tag{2.1}$$

where $u^* \in \lambda \partial F(t,u)$.

Lemma 2.4 ([10]) *The functional $\Psi(y)$ satisfies the inequality*

$$\frac{1}{2}|y|^2 \le \Psi(y) \le \frac{1}{\sqrt{1-R^2}}|y|^2 \quad \textit{for all } y \in \mathbb{R}^N.$$

Lemma 2.5 ([41]) *Let $u, v \in L^1(0,T;\mathbb{R}^N)$. If for every $f \in C_T^\infty$, $\int_0^T (u(t), f'(t))\,dt = -\int_0^T (v(t), f(t))\,dt$, then $\int_0^T v(s)\,ds = 0$ and there exists $c \in \mathbb{R}^N$ such that $u(t) = \int_0^t v(s)\,ds + c$ a.e. on $[0,T]$.*

3 Main results

Proposition 3.1 *If function $u \in X$ is a critical point of I, u is a solution of problem (1.7).*

Proof Assume that u is a critical point of I, then we have

$$\int_0^T \big(\psi\big(u'\big)|v'\big)\,dt + \int_0^T \big(\psi(u)|v\big)\,dt - \lambda\tilde{\Psi}^\circ(u;v) - \sum_{i=1}^m \big(I_i\big(u(t_i)\big)|v(t_i)\big) \geq 0, \tag{3.1}$$

for any $v \in H_T$. Since $C_0^\infty([0,T];\mathbb{R}^N) \subset H_T$, clearly,

$$\int_0^T \big(\psi\big(u'\big)|v'\big)\,dt + \int_0^T \big(\psi(u)|v\big)\,dt - \lambda\tilde{\Psi}^\circ(u;v) - \sum_{i=1}^m \big(I_i\big(u(t_i)\big)|v(t_i)\big) \geq 0$$

holds for any $v \in C_0^\infty([0,T];\mathbb{R}^N)$. Considering the subadditivity and positive homogeneity of the function $\tilde{\Psi}^\circ(u;\cdot)$ and applying the Hahn–Banach theorem, there exists a linear functional $h: H_T \to \mathbb{R}$ satisfying

$$-\lambda\tilde{\Psi}^\circ(u;v) \geq \langle h, v\rangle, \quad \text{for any } v \in H_T, \tag{3.2}$$

and

$$-\left(\int_0^T \big(\psi\big(u'\big)|v'\big)\,dt + \int_0^T \big(\psi(u)|v\big)\,dt - \sum_{i=1}^m \big(I_i\big(u(t_i)\big)|v(t_i)\big)\right) = \langle h, v\rangle,$$
$$\text{for any } v \in C_0^\infty\big([0,T];\mathbb{R}^N\big). \tag{3.3}$$

Since F is Lipschitz, there exists a positive constant B with $-\lambda\tilde{\Psi}^\circ(u;v) \leq B\|v\|$ for any $v \in H_T$. This combining (3.2) implies $|\langle h, v\rangle| \leq B\|v\|$ for any $v \in H_T$. Furthermore, we obtain $h \in (H_T)^*$. From the definition of the generalized directional derivative, we have $h \in \lambda\partial(-\tilde{\Psi}(u))$. Taking Lemma 2.2 into consideration, there exists some $u_h \in L^1$ such that

$$u_h \in \lambda\partial F\big(t, u(t)\big), \quad \text{for a.e. } t \in [0,T], \tag{3.4}$$

and

$$\langle h, v\rangle = -\int_0^T (u_h|v)\,dt, \quad \forall v \in H_T. \tag{3.5}$$

By (3.3), (3.4), (3.5), we obtain $\int_0^T (\psi(u')|v')\,dt + \int_0^T (\psi(u)|v)\,dt - \sum_{i=1}^m (I_i(u(t_i))|v(t_i)) = \int_0^T (u_h|v)\,dt$ for any $v \in C_0^\infty([0,T];\mathbb{R}^N)$, where $u_h \in \lambda\partial F(t,u(t))$ for a.e. $t \in [0,T]$. We take $v(t) = \prod_{i=0}^{m+1}(t-t_i)(1,1,\dots,1)^{\mathrm{T}} \in C_0^\infty([0,T];\mathbb{R}^N)$, then $\int_0^T (\psi(u')|v')\,dt + \int_0^T (\psi(u)|v)\,dt = \int_0^T (u_h|v)\,dt$ for a.e. $t \in [0,T]\backslash\{t_1,t_2,\dots,t_m\}$. By Lemma 2.5, we have $\psi(u') = \int_0^t \psi(u) -$

$u_h\,dt + c$. Since $\int_0^t \psi(u) - u_h\,dt$ is absolutely continuous, the gradient of $\int_0^t \psi(u) - u_h\,dt$ exists almost everywhere for $t \in [0,T]$, i.e.,

$$-\big(\psi\big(u'\big)\big)' + \psi(u) = u_h(t), \tag{3.6}$$

for a.e. $t \in [0,T]\backslash\{t_1, t_2, \dots, t_m\}$.

In the following we will prove the boundary conditions and impulsive conditions.

By $u_h \in \lambda\partial F(t,u(t))$, we obtain

$$\big(u_h(t)|v(t)\big) \le \lambda F^\circ\big(t,u(t);v(t)\big), \tag{3.7}$$

for a.e. $t \in [0,T]$, any $v \in H_T$. Multiplying (3.6) by $v \in H_T$, integrating over $[0,T]$ and using the integration by parts formula, one has

$$\begin{aligned}&\big(\psi\big(u'(0)\big) - \psi\big(u'(T)\big)\big)v(T) + \int_0^T \big(\psi\big(u'\big)|v'\big)\,dt + \int_0^T \big(\psi(u)|v\big)\,dt \\ &\quad - \sum_{i=1}^m \big(-\triangle\psi\big(u'(t_i)\big)|v(t_i)\big) = \int_0^T (u_h|v)\,dt.\end{aligned}$$

By (3.7), we obtain

$$\begin{aligned}&\int_0^T \big(\psi\big(u'\big)|v'\big)\,dt + \int_0^T \big(\psi(u)|v\big)\,dt \\ &\quad \le \int_0^T \lambda F^\circ(t,u;v)\,dt + \big(\psi\big(u'(T)\big) - \psi\big(u'(0)\big)|v(T)\big) + \sum_{i=1}^m -\big(\triangle\psi\big(u'(t_i)\big)|v(t_i)\big).\end{aligned} \tag{3.8}$$

By the fact that $-\lambda\tilde{\Psi}^\circ(u;v) \le -\lambda\hat{F}^\circ(u;v) \le \int_0^T \lambda(-F)^\circ(t,u;v)\,dt$ ([42]) and (3.1),

$$\begin{aligned}&\int_0^T \big(\psi\big(u'\big)|v'\big)\,dt + \int_0^T \big(\psi(u)|v\big)\,dt + \int_0^T \lambda(-F)^\circ(t,u;v)\,dt \\ &\quad - \sum_{i=1}^m \big(I_i\big(u(t_i)\big)|v(t_i)\big) \ge 0,\end{aligned} \tag{3.9}$$

holds. By (3.8) and (3.9) the inequality

$$\begin{aligned}&\int_0^T \lambda(-F)^\circ(t,u;v)\,dt + \int_0^T \lambda F^\circ(t,u;v)\,dt \\ &\quad \ge \big(\psi\big(u'(0)\big) - \psi\big(u'(T)\big)|v(T)\big) + \sum_{i=1}^m \big(\triangle\psi\big(u'(t_i)\big) + I_i\big(u(t_i)\big)|v(t_i)\big)\end{aligned} \tag{3.10}$$

holds. Let $v_n \in H_T$ be defined by

$$v_n = \begin{cases}(1-nt)y, & 0 \le t \le \frac{1}{n}, \\ 0, & \frac{1}{n} < t \le T - \frac{1}{n}, \\ (n(t-T)+1)y, & T - \frac{1}{n} < t \le T,\end{cases}$$

for any $y \in \mathbb{R}^N$.

By [37], we obtain

$$\int_0^T F^\circ(t,u;v_n)\,dt \to 0 \quad \text{as } n\to\infty \tag{3.11}$$

and

$$\int_0^T (-F)^\circ(t,u;v_n)\,dt \to 0 \quad \text{as } n\to\infty, \tag{3.12}$$

when F is Lipschitz. Then we take $v = v_n$ in (3.10), letting $n \to \infty$, by (3.11), (3.12), a straightforward computation shows that $(\psi(u'(0)) - \psi(u'(T))|y) + \sum_{i=1}^m (\triangle\psi(u'(t_i)) + I_i(u(t_i))|v(t_i)) \le 0$. As y is arbitrarily chosen in $\mathbb{R}^N$, one has

$$\bigl(\psi\bigl(u'(0)\bigr) - \psi\bigl(u'(T)\bigr)|y\bigr) + \sum_{i=1}^m \bigl(\triangle\psi\bigl(u'(t_i)\bigr) + I_i\bigl(u(t_i)\bigr)|v(t_i)\bigr) = 0. \tag{3.13}$$

Apparently, $v(t_i) = 0$, $i = 1,2,\ldots,m$. (3.13) is changed into $(\psi(u'(0)) - \psi(u'(T))|y) = 0$. Taking $y = e_i$, $i = 1,2,\ldots,N$, we get $\psi(u'(0)) - \psi(u'(T)) = 0$. Since ψ is a homeomorphism, we have $u'(0) = u'(T)$. Equation (3.13) is changed into

$$\sum_{i=1}^m \bigl(\triangle\psi\bigl(u'(t_i)\bigr) + I_i\bigl(u(t_i)\bigr)|v(t_i)\bigr) = 0. \tag{3.14}$$

Let

$$v_n = \begin{cases} nty, & 0 \le t \le \frac{1}{n}, \\ y, & \frac{1}{n} < t \le T - \frac{1}{n}, \\ -(n(t-T))y, & T - \frac{1}{n} < t \le T. \end{cases}$$

We take $v = v_n$ in (3.14), letting $n \to \infty$, then $v(t_i) = y$. Taking $i \ne j$, $v(t_i) = y = \prod_{j\ne i}^m (t_i - t_j)e$, $v(t_j) = y = \prod_{h=1}^m (t_j - t_h)e$, $e = e_1, e_2, \ldots, e_N$, we obtain $-\triangle\psi(u'(t_1)) = I_1(u(t_1))$ when $i = 1$, $-\triangle\psi(u'(t_2)) = I_2(u(t_2))$ when $i = 2$, …, $-\triangle\psi(u'(t_m)) = I_m(u(t_m))$ when $i = m$. So $-\triangle\psi(u'(t_i)) = I_i(u(t_i))$ when $i = 1,2,\ldots,m$. Besides $u \in X$, it is clear $u(0) = u(T)$. □

To better illustrate the main results, we give the following assumptions:

(A_1) $\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u) \ge F(t,\mathbf{0}) = 0$, where $\mathbf{0} = (0,0,0,\ldots,0)^{\mathrm{T}}$ and $\int_0^{\frac{1}{2}} F(t,(0,0,\ldots,d_1t,\ldots,0)^{\mathrm{T}})\,dt \ge 0$, $\int_{\frac{3}{4}}^T F(t,(0,0,\ldots,\frac{2d_1(t-T)}{3-4T},\ldots,0)^{\mathrm{T}})\,dt \ge 0$;

(A_2) $[(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2})^2 - (2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|)^2]^{\frac{1}{2}} < \frac{F(t_j,\hat{u}_0)}{4d_1^2}$, where $F(t_j,\hat{u}_0) = \min_{t\in[\frac{1}{2},\frac{3}{4}]} F(t,\hat{u}_0)$;

(A_3) $\frac{1}{2}\|u_0\|_X^2 > |\sum_{i=1}^m \int_0^{u_0(t_i)} I_i(s)\,ds|$;

(A_4) $\partial_{u_i} F(t,\mathbf{0}) \ge 0$;

(A_5) $\frac{4d_1^2(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})}{F(t_j,\hat{u}_0)} < b\frac{c_1^2(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})}{\sqrt{1-\bar{z}}\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}$, where $0 < b \le 1$;

(A_6) $|I_i(u)| < D_{1i} + D_{2i}|u|^\gamma$, where D_{1i}, D_{2i} are positive constants, $D_1 = \max\{D_{1i}\}$, $D_2 = \max\{D_{2i}\}$ $\gamma \le 1$ and $I_i(u)u < 0$ for all $u \in X$, $i = 1,2,\ldots,m$;

(A_7) $[(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2})^2 - (2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|)^2]^{\frac{-1}{2}} \max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)| \, b(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3}) + m(D_1 + D_2R^\gamma) + 1 - \sqrt{1 - |\frac{2r_2}{T}|} \leq R$, where $R < 1$, r_2 and $\bar{z}$ appear in the proof of Theorem 3.2;

(A_8) $\sum_{i=1}^m (I_i(u_n(t_i))|(u(t_i) - u_n(t_i))) \geq \alpha\|u_n\|_X^2$, where $u_n \rightharpoonup u$ in X, $\alpha \geq \frac{\tilde{d}-\tilde{c}}{2}$, where $\tilde{d}$, $\tilde{c}$ are given in the Appendix;

(A_9) $\sqrt{\frac{2r_2}{T}} \leq R$, $\forall u \in X$;

(B_0) $I_i(u) > \mathbf{0}$ as $u < \mathbf{0}$;

(B_1) $(\partial F(t,u)|u) \leq 0$ as $u_i < 0$.

Theorem 3.2 *Assume there exist positive constants* c_1, d_1 *with* $d_1 < c_1$ *such that* (A_1)–(A_9), (B_0) *and* (B_1) *hold, then for* $\lambda \in \left]\frac{4d_1^2(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})}{F(t_j,\hat{u}_0)}, b\frac{c_1^2(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})}{\sqrt{1-\bar{z}}\max_{[0,c_1]} F}\right[$, *problem* (1.7) *admits at least one nonnegative solution* u_* *such that* $|u_*| < \sqrt{\frac{2r_2}{T}} \leq R$ *and*

$$\begin{aligned} |u_*'| \leq {}& \left[\left(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2}\right)^2 - \left(2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|\right)^2\right]^{-\frac{1}{2}} \\ & \times \max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)| \\ & \times b\left(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2}\right)\left(\frac{2T+13}{24} + \frac{1}{4T-3}\right) + |m(D_1 + D_2R^\gamma)| \\ & + 1 - \sqrt{1 - \left|\frac{2r_2}{T}\right|} \leq R. \end{aligned}$$

Proof We will apply Lemma 2.3 and Ref. [37] to complete this section in four steps.

Step 1. We prove $\Phi(u)$, $\tilde{\Psi}(u)$ are locally Lipschitz.

Clearly, $\Phi(u) \in C^1(X)$. From Lemma 2.1, $\Phi(u)$ is locally Lipschitz. By calculation we can see $\tilde{\Psi}(u) - \tilde{\Psi}(v) = \hat{F} \circ i(u) - \hat{F} \circ i(v) = \hat{F} \circ i(u-v) \leq L|i(u-v)| \leq L|\bar{u} - \bar{v}|_C \leq LM\|u - v\|_{H_T}$. So $\tilde{\Psi}(u)$ is locally Lipschitz on X.

Step 2. We show that there exist $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that $\beta(r_1, r_2) < \rho(r_1, r_2)$.

Firstly, we prove $\rho(r_1, r_2) \geq \frac{\frac{1}{4}F(t_j,\hat{u}_0)}{(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})d_1^2}$.

By $u \in H_T$, we have $u \in L^1$, i.e., for every $t \in [0,T]$, there exists c_1 such that $|u(t)| \leq c_1$. Let $L = \frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2}$, $M = \max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|$, $\bar{z} = 4\frac{M^2}{L^2}$. Apparently, we have

$$\begin{aligned} \sqrt{1-\bar{z}} = {}& \left[\left(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2}\right)^2 - \left(2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|\right)^2\right]^{\frac{1}{2}} \\ & \times \frac{c_1^2}{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} & \left[\left(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2}\right)^2 - \left(2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|\right)^2\right]^{-\frac{1}{2}} \\ & \quad = \frac{1}{\sqrt{1-\bar{z}}} \frac{c_1^2}{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}. \end{aligned} \tag{3.15}$$

By (A_2), (3.15), we obtain

$$\frac{1}{4}F(t_j,\hat{u}_0) > \sqrt{1-\bar{z}} \max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)\frac{d_1^2}{c_1^2}. \tag{3.16}$$

We claim that

$$d_1 \leq \frac{c_1}{2\sqrt[4]{1-\bar{z}}}. \tag{3.17}$$

We assume (3.17) is not established then we have $d_1^2 > \frac{c_1^2}{4\sqrt{1-\bar{z}}}$ i.e. $\sqrt{1-\bar{z}}d_1^2 > \frac{1}{4}c_1^2$. Combining (3.16), we obtain

$$\frac{\frac{1}{4}F(t_j,\hat{u}_0)}{\sqrt{1-\bar{z}}d_1^2} < \frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2} < \frac{\frac{1}{4}F(t_j,\hat{u}_0)}{\sqrt{1-\bar{z}}d_1^2}.$$

This is a contradiction. So (3.17) holds.

From (3.16), there exists $\bar{\varepsilon} \in]0,c[$ such that for every $\varepsilon \in]0,\bar{\varepsilon}[$

$$\sqrt{1-\bar{z}} \max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)\frac{d_1^2}{c_1^2}\frac{c}{c-\varepsilon} < \frac{1}{4}F(t_j,\hat{u}_0). \tag{3.18}$$

By Lemma 2.4, we have

$$\frac{1}{2}\|u\|_X^2 \leq \int_0^T \Psi\left(u'\right)dt + \int_0^T \Psi(u)\,dt \leq \frac{1}{\sqrt{1-R^2}}\|u\|_X^2. \tag{3.19}$$

In order to apply Lemma 2.3, we take $u_0(t) \in X$, where

$$u_{0i} = \begin{cases} d_1 t, & 0 \leq t \leq \frac{1}{2}, \\ \frac{d_1}{2}, & \frac{1}{2} < t \leq \frac{3}{4}, \\ \frac{2d_1(t-T)}{3-4T}, & \frac{3}{4} < t \leq T. \end{cases}$$

Let $u_0(t) = (0,\dots,u_{0i},0,\dots,0)^{\mathrm{T}}$. Specially, $u_0(t) = \hat{u}_0 = (0,0,\dots,\frac{d_1}{2},\dots,0)$ when $\frac{1}{2} < t \leq \frac{3}{4}$. So there exists t_j satisfying $F(t_j,\hat{u}_0) = \min_{t\in[\frac{1}{2},\frac{3}{4}]} F(t,\hat{u}_0)$. By computing, we obtain $\|u_0\|_X^2 = (\frac{2T+13}{24} + \frac{1}{4T-3})d_1^2$.

Put $r_1 = 0$, $r_2 = (\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3})T^0\frac{c_1^2}{\sqrt{1-\bar{z}}}$, where

$$T^0 = \begin{cases} 1, & T < 1, \\ T, & T > 1. \end{cases}$$

Clearly, let $T_1 = \frac{-46+\sqrt{2596}}{16} < \frac{5}{16}$. A straightforward computation shows that $r_2 > 0$ when $\{0 < T < T_1\} \cup \{T > \frac{3}{4}\}$. By (A_3), (3.17) and (3.19), we obtain

$$r_1 = 0 < \Phi(u_0) \leq \left(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2}\right)\left(\frac{2T+13}{24} + \frac{1}{4T-3}\right)d_1^2 < r_2 \tag{3.20}$$

and $\Phi^{-1}(]-\infty,0[) = \{0\}$. Clearly, $\tilde{\Psi}(0) = 0$ is established.

By (A_1), one has

$$\tilde{\Psi}(u_0) = \int_0^T F(t,u_0)\,dt \geq \int_{\frac{1}{2}}^{\frac{3}{4}} F(t,\hat{u}_0)\,dt \geq \frac{1}{4}F(t_j,\hat{u}_0). \tag{3.21}$$

By (3.20), (3.21), we have

$$\begin{aligned}\rho(r_1,r_2) &= \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\tilde{\Psi}(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\tilde{\Psi}(u)}{\Phi(v)-r_1} > \frac{\tilde{\Psi}(u_0)}{\Phi(u_0)} \\ &\geq \frac{\frac{1}{4}F(t_j,\hat{u}_0)}{(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})d_1^2}.\end{aligned}$$

In the following, we will show $\beta(r_1,r_2) < \frac{\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)}{(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})\frac{c_1^2}{\sqrt{1-\bar{z}}}}$.

To better deal with $\beta(r_1,r_2)$, we give some notations.

$G(t,s) = \int_0^s g(t,u)\,du$, where

$$g(t,u) = \begin{cases} \partial F(t,u), & u_i \geq 0, i = 1,2,\ldots,N, \\ \partial F(t,0), & u_i < 0, i = 1,2,\ldots,N, \\ 0 & u_i < 0, \text{ for some } i \text{ and } u_i > 0, \end{cases}$$

for some i and $s = (s_1,s_2,\ldots,s_N)$.

Indeed,

$$\max_{t\in[0,T],s_i\in[-c_1',c_1']} G(t,s) = \max\Big\{\max_{t\in[0,T],s_i\in[0,c_1']} G(t,s), \max_{t\in[0,T],s_i\in[-c_1',0]} G(t,s)\Big\}.$$

By (A_4), we obtain

$$\begin{aligned}&\max_{t\in[0,T],s_i\in[0,c_1']} G(t,s) = \max_{t\in[0,T],s_i\in[0,c_1']} F(t,s) \geq F(t,\mathbf{0}) = 0, \\ &\max_{t\in[0,T],s_i\in[-c_1',0]} G(t,s) = \max_{t\in[0,T],s_i\in[-c_1',0]} \int_0^s g(t,u)\,du = \max_{t\in[0,T],s_i\in[-c_1',0]} \big(\partial F(t,\mathbf{0})|s\big) \leq 0,\end{aligned}$$

where c_1' is a positive constant. So

$$\max_{t\in[0,T],s_i\in[-c_1',c_1']} G(t,s) = \max_{t\in[0,T],s_i\in[0,c_1']} F(t,s). \tag{3.22}$$

According to the property of the supremum and infimum, we have

$$\begin{aligned}\beta(r_1,r_2) &= \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\tilde{\Psi}(u) - \tilde{\Psi}(v)}{r_2-\Phi(v)} \\ &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\tilde{\Psi}(u) - \tilde{\Psi}(u_0)}{r_2-\Phi(u_0)}.\end{aligned}$$

By (3.22) and (3.21), we obtain

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\tilde{\Psi}(u)-\tilde{\Psi}(u_0)}{r_2-\Phi(u_0)} \le \frac{T^0\max_{t\in[0,T],|s_i|\le c_1'}G(t,s)-\frac{1}{4}F(t_j,\hat{u}_0)}{r_2-\Phi(u_0)} = \frac{T^0\max_{t\in[0,T],|u_i|\in[0,c_1']}F(t,u)-\frac{1}{4}F(t_j,\hat{u}_0)}{r_2-\Phi(u_0)}.$$

From (3.20), we have

$$\frac{T^0\max_{t\in[0,T],|u_i|\in[0,c_1']}F(t,u)-\frac{1}{4}F(t_j,\hat{u}_0)}{r_2-\Phi(u_0)} \le \frac{T^0\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)-\frac{1}{4}F(t_j,\hat{u}_0)}{r_2-(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})d_1^2}. \tag{3.23}$$

Considering (3.16) and the expression of r_2, one has

the right of inequality (3.23)

$$\le \frac{T^0\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)-\sqrt{1-\bar{z}}\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)\frac{d_1^2}{c_1^2}}{r_2(1-\frac{d_1^2\sqrt{1-\bar{z}}}{c_1^2T^0})}$$
$$= \frac{T^0\max_{t\in[0,T],|u_i|\in[0,c_1]}F(t,u)(1-\frac{d_1^2\sqrt{1-\bar{z}}}{c_1^2T^0})}{r_2(1-\frac{d_1^2\sqrt{1-\bar{z}}}{c_1^2T^0})}$$
$$= \frac{\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)}{(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})\frac{c_1^2}{\sqrt{1-\bar{z}}}}.$$

Thus we have

$$\beta(r_1,r_2) < \frac{\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)}{(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})\frac{c_1^2}{\sqrt{1-\bar{z}}}}. \tag{3.24}$$

By (3.16) and (3.24), it is clear $\beta(r_1,r_2)<\rho(r_1,r_2)$.

Step 3. We show that $I=\Phi-\lambda\tilde{\Psi}$ fulfills the ${}^{[r_1]}(PS)^{[r_2]}$-condition.

By the mean value theorem and the Hölder inequality, we have $\|u\|_\infty=\max_{t\in[0,T]}|u(t)|\le\bar{N}\|u\|_X$, where $\bar{N}=\max\{\sqrt{T},\frac{1}{\sqrt{T}}\}$.

To this end, let $\{u_n\}\subseteq X$ be a sequence such that

(a_1) $(\Phi-\lambda\tilde{\Psi})(u_n)$ is bounded;

(a_2) there exists a sequence $\{\epsilon_n\}\subset\mathbb{R}_+$, $\epsilon_n\to0^+$ such that $(\Phi-\lambda\tilde{\Psi})^\circ(u_n;v)\ge-\epsilon_n\|v\|_X$ for all $v\in X$;

(a_3) $\Phi(u_n)<r_2$ for all $n\in N$.

By Lemma 2.4, (A_6) and (a_3), we have $\frac{1}{2}\|u_n\|_X^2-mD_1\bar{N}\|u_n\|_X-\frac{mD_2\bar{N}}{\gamma+1}\|u_n\|_X^{\gamma+1}\le\Phi(u_n)<r_2$. Clearly, $\{u_n\}$ is a bounded sequence in X as $\gamma<1$. We have the same conclusion as $\gamma=1$, $mD_2<1$. Since X is a reflexive Banach space and we have the compact embedding $X\hookrightarrow L^2([a,b])$, we may assume that

$$u_n\rightharpoonup u \quad \text{in } X, \qquad u_n\to u \quad \text{in } L^2([0,T]). \tag{3.25}$$

Letting $v = u - u_n$ in (a_2), we have

$$\bigl(\Phi'(u_n)|(u-u_n)\bigr) + (-\lambda\tilde{\Psi})^\circ(u_n; u-u_n) \geq -\epsilon_n\|u-u_n\|_X. \tag{3.26}$$

By calculating, we get

$$\begin{aligned}\bigl(\Phi'(u_n)|(u-u_n)\bigr) &= \int_0^T \bigl(\psi(u_n')|(u'-u_n')\bigr)\,dt + \int_0^T \bigl(\psi(u_n)|(u-u_n)\bigr)\,dt \\ &\quad - \sum_{i=1}^m \bigl(I_i(u_n(t_i))|(u(t_i)-u_n(t_i))\bigr) \\ &= \int_0^T \bigl(\psi(u_n')|u'\bigr) - \bigl(\psi(u_n')|u_n'\bigr)\,dt + \int_0^T \bigl(\psi(u_n)|u\bigr) - \bigl(\psi(u_n)|u_n\bigr)\,dt \\ &\quad - \sum_{i=1}^m \bigl(I_i(u_n(t_i))|(u(t_i)-u_n(t_i))\bigr).\end{aligned}$$

By the definition of ψ, the inequality $ab \leq \frac{a^2+b^2}{2}$ and (A_8), we obtain

$$\bigl(\Phi'(u_n)|(u-u_n)\bigr) \leq \frac{1}{2}\tilde{d}\|u\|_X^2 - \left(\frac{1}{2}\tilde{c}+\alpha\right)\|u_n\|_X^2, \tag{3.27}$$

where $\tilde{d} = \max\{\frac{1}{\sqrt{1-|u_n'|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\}$, $\tilde{c} = \min\{\frac{1}{\sqrt{1-|u_n'|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\}$, as $|u_n'| \leq R$, $|u_n| \leq R$; $\tilde{d} = \max\{\frac{1}{\sqrt{1-|u_n'|^2}}, \frac{1}{\sqrt{1-R^2}}\} = \frac{1}{\sqrt{1-R^2}}$, $\tilde{c} = \min\{\frac{1}{\sqrt{1-|u_n'|^2}}, \frac{1}{\sqrt{1-R^2}}\} = \frac{1}{\sqrt{1-|u_n'|^2}}$, as $|u_n'| \leq R$, $|u_n| > R$; $\tilde{d} = \max\{\frac{1}{\sqrt{1-|u_n|^2}}, \frac{1}{\sqrt{1-R^2}}\} = \frac{1}{\sqrt{1-R^2}}$, $\tilde{c} = \min\{\frac{1}{\sqrt{1-|u_n|^2}}, \frac{1}{\sqrt{1-R^2}}\} = \frac{1}{\sqrt{1-|u_n|^2}}$, as $|u_n'| > R$, $|u_n| \leq R$; $\tilde{d} = \frac{1}{\sqrt{1-R^2}}$, $\tilde{c} = \frac{1}{\sqrt{1-R^2}}$, as $|u_n'| > R$, $|u_n| > R$. One can see the Appendix for more details. Equation (3.26) on combining with (3.27) gives the following inequality:

$$-\epsilon_n\|u-u_n\|_X + \left(\frac{1}{2}\tilde{c}+\alpha\right)\|u_n\|_X^2 \leq \frac{1}{2}\tilde{d}\|u\|_X^2 + (-\lambda\tilde{\Psi})^\circ(u_n; u-u_n). \tag{3.28}$$

Note that $\tilde{\Psi}$ is well defined and locally Lipschitz on $L^2([0,T])$. Taking $(-\lambda\tilde{\Psi}|_X)^\circ \leq (-\lambda\tilde{\Psi}^\circ)|_X$ for all $u, v \in X$ into consideration, the upper semicontinuity of $(-\lambda\tilde{\Psi})^\circ$, in the strong topology of $L^2([0,T]) \times L^2([0,T])$, implies that

$$\lim_{n\to\infty}\sup(-\lambda\tilde{\Psi})^\circ(u_n; u-u_n) \leq (-\lambda\tilde{\Psi})^\circ\Bigl(\lim_{n\to\infty} u_n; \lim_{n\to\infty} u-u_n\Bigr) = 0. \tag{3.29}$$

Considering the upper limit and using (3.29), the inequality (3.28) becomes

$$\lim_{n\to\infty}\sup\left(\frac{1}{2}\tilde{c}+\alpha\right)\|u_n\|_X \leq \frac{1}{2}\tilde{d}\|u\|_X. \tag{3.30}$$

Applying (A_8) to (3.30), we have

$$\lim_{n\to\infty}\sup\|u_n\|_X \leq \|u\|_X. \tag{3.31}$$

We know that $L^2[0,T]$ is uniformly convex and $X \hookrightarrow\hookrightarrow L^2[0,T]$, X is uniformly convex. Combining (3.25), (3.31) with the convexity of X, we obtain $u_n \to u$ in X (see [43]). Hence the functional $\Phi - \lambda\tilde{\Psi}$ satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition.

From Lemma 2.3, there exists $u_* \in X$ being a critical point of I satisfying

$$\begin{aligned}\Phi(u_*) - \lambda\tilde{\Psi}(u_*) &= \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \big(\Phi(v) - \lambda\tilde{\Psi}(v)\big) \le \Phi(u_0) - \lambda\tilde{\Psi}(u_0) \\ &\le \left(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2}\right)\left(\frac{2T+13}{24} + \frac{1}{4T-3}\right)d_1^2 - \frac{1}{4}\lambda F(t_j,\hat{u}_0).\end{aligned}$$

Clearly, $0 < \Phi(u_*) < r_2$ is established. So we have $\int_0^T \Psi(u_*')\,dt + \int_0^T \Psi(u_*)\,dt - \sum_{i=1}^m \int_0^{u_*} I_i(s)\,ds < r_2$. By Lemma 2.4, (A_6), we can see $\frac{T|u_*|^2}{2} < r_2$. According to (A_9), $|u_*| \le R$ is valid.

Let $v(\cdot) = \varphi(u'(\cdot)) = \psi(u'(\cdot))$ be continuously differentiable, so $u' = \varphi^{-1}(v) \in C^0([0,T])$, i.e. $u_* \in C^1([0,T])$. Because $u_*(0) = u_*(T)$, there exists t_0 such that $u_*'(t_0) = 0$. According to Lemma 2.3, (A_5), for any $\lambda \in \left]\frac{4d_1^2}{F(t_j,\hat{u}_0)}(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3}), b\frac{c_1^2(\frac{1}{\sqrt{1-R^2}}+\frac{1}{2})(\frac{2T+13}{24}+\frac{1}{4T-3})}{\sqrt{1-\bar{z}}\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)}\right[$, then there exists $\varepsilon \in]0,\bar{\varepsilon}[$ and a positive constant c such that

$$\lambda\frac{c}{c-\varepsilon}T^0 < \frac{bc_1^2(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3})}{\sqrt{1-\bar{z}}\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)}. \tag{3.32}$$

Let

$$\Psi^0(y) = \begin{cases} \frac{1}{\sqrt{1-|y|^2}}, & |y| \le R, \\ \frac{1}{\sqrt{1-|R|^2}}, & |y| > R. \end{cases}$$

By the integral mean value theorem and (3.32), (3.15), (A_6) we obtain

$$\begin{aligned}\left|u_*'(t)\right| &\le \Psi^0\big(u_*'(t)\big)\left|u_*'(t)\right| \\ &= \left|\int_{t_0}^t \lambda\partial F\big(u_*(s)\big)\,ds\right| + \left|\sum_{i=1}^m I_i\big(u(t_i)\big)\right| + \int_{t_0}^t \psi\big(u_*(s)\big)\,ds \\ &\le \frac{\lambda T^0 c}{c-\varepsilon}\left|\max_{t\in[0,T],|u|\in[0,c_1]}\partial F(t,u)\right| + \left|\sum_{i=1}^m I_i\big(u(t_i)\big)\right| + \int_0^T \psi\big(u_*(s)\big)\,ds \\ &< \frac{c_1^2\max_{t\in[0,T],|u|\in[0,c_1]}|\partial F(t,u)|b(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3})}{\sqrt{1-\bar{z}}\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)} \\ &\quad + \left|\sum_{i=1}^m I_i\big(u(t_i)\big)\right| + 1 - \sqrt{1-|u_*|^2} \\ &< \left[\left(\frac{\max_{t\in[0,T],|u|\in[0,c_1]}F(t,u)}{c_1^2}\right)^2 - \left(2\max_{t\in[0,T],|u|\in[0,c_1]}\left|\partial F(t,u)\right|\right)^2\right]^{\frac{-1}{2}} \\ &\quad \times b\left(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2}\right)\left(\frac{2T+13}{24} + \frac{1}{4T-3}\right)\max_{t\in[0,T],|u|\in[0,c_1]}\left|\partial F(t,u)\right| \\ &\quad + \left|m\big(D_1 + D_2R^\gamma\big)\right| + 1 - \sqrt{1 - \left|\frac{2r_2}{T}\right|}.\end{aligned}$$

By (A_7), we have $|u_*'(t)| \le R$.

Step 4. We prove each solution of (1.7) is nonnegative.

Assume that u is the solution of (1.7). Multiplying u^- on both sides of differential inclusion and integrating from 0 to T, we have

$$\int_0^T \left(-\left(\psi\left(u'\right)\right)'|u^-\right) dt + \int_0^T \left(\left(\psi(u)\right)|u^-\right) dt = \int_0^T \left(\tilde{g}(t,u)|u^-\right) dt,$$

where $\tilde{g}(t,u) \in \lambda\partial F(t,u)$ and $\tilde{g}(t,u) \in L^\delta$, $\delta > 1$.

We define $u^- = \mathbf{0}$ when $u_i(t) > 0$ and $u_m(t) < 0$ for $i, m \in \{1,2,\dots,N\}$. $u^- = -u$ when $u_i(t) < 0$ for any $i \in \{1,2,\dots,N\}$. $u^- = \mathbf{0}$ when $u_i(t) > 0$ for any $i \in \{1,2,\dots,N\}$.

$$\begin{aligned}
&\int_0^T \left(-\left(\psi\left(u'\right)\right)'|u^-\right) dt \\
&\quad = \left(-\psi\left(u'(T)\right)|u^-(T)\right) + \left(\psi\left(u'(0)\right)|u^-(0)\right) - \sum_{i=1}^m \left(I_i\left(u(t_i)\right)|u^-(t_i)\right) \\
&\qquad + \int_0^T \left(\psi\left(u'\right)|\left(u^-\right)'\right) dt \\
&\quad = \int_0^T \left(\psi\left(u'\right)|\left(u^-\right)'\right) dt - \sum_{i=1}^m \left(I_i\left(u(t_i)\right)|u^-(t_i)\right) \\
&\quad = \int_{\{t:u_i(t)<0\}} \left(\psi\left(u'\right)|\left(u^-\right)'\right) dt + \int_{\{t:u_i(t)\geq 0\}} \left(\psi\left(u'\right)|\left(u^-\right)'\right) dt \\
&\qquad - \sum_{i=1}^m \left(I_i\left(u(t_i)\right)|u^-(t_i)\right) \\
&\quad = \int_{\{t:u_i(t)<0\}} \left(\psi\left(u'\right)|\left(u^-\right)'\right) dt - \sum_{i=1}^m \left(I_i\left(u(t_{ij})\right)|u^-(t_{ij})\right) \\
&\qquad \left(t_{ij} \text{ such that } u(t_{ij}) < 0\right),
\end{aligned}$$

$\int_0^T (\tilde{g}(t,u)|u^-)\, dt = \int_{\{t:u_i(t)<0\}} (\tilde{g}(t,u)|u^-)\, dt + \int_{\{t:u_i(t)\geq 0\}} (\tilde{g}(t,u)|u^-)\, dt = -\int_{\{t:u_i(t)<0\}} (\tilde{g}(t,u)|u)\, dt$. By (B_0), (B_1) we have $\int_0^T (\tilde{g}(t,u)|u^-)\, dt \geq 0$ and $-\sum_{i=1}^m I_i(u(t_{ij}))u^-(t_{ij}) < 0$. So

$$\int_{\{t:u_i(t)<0\}} \psi\left(u'\right)u'\, dt + \int_{\{t:u_i(t)<0\}} \psi(u)u\, dt < 0.$$

Hence the measure of $\{t : u_i(t) < 0\}$ is 0. The proof is completed. □

4 Conclusion

We apply the nonsmooth critical theorem of Ref. [38] to a study of the boundary value problem due to the non-differentiability and non-smoothness of energy functionals. We construct a specific function v_n to verify the boundary condition and the impulsive condition. The boundedness of solutions is obtained to convert singular and non-singular problems into each other. These contents are not covered in the literature [12]. Moreover, the proof that the critical point of the energy functional is the solution of the boundary value problem and the proof of the nonnegativity of the solution are obviously different from [12]. In particular, the restrictions of the nonlinear terms are completely different.

Besides, we obtain $|u| < R$, $|u'| < R$ by $u \in \Phi^{-1}(r_1, r_2)$ and related inequalities. Reference [12] has similar results by defining the solution space and embedding maps.

5 Example

Take

$$\partial F = \begin{cases} \frac{u}{|u|}(-|u|^4 - |u| + \sin|u| - \frac{l(t)}{10}), & u < 0, \\ \frac{l(t)}{10}, & u = 0, \\ \frac{u}{|u|}(|u|^{\frac{1}{4}} + |u|^4 + \frac{l(t)}{10}), & u > 0, \end{cases}$$

and

$$F(t,u) = \begin{cases} (-\frac{|u|^5}{5} - \frac{1}{2}|u|^2 - \cos|u| - \frac{l(t)}{10}|u|), & u < 0, \\ \frac{l(t)}{10}u, & u = 0, \\ \frac{4}{5}|u|^{\frac{5}{4}} + \frac{|u|^5}{5} + \frac{l(t)}{10}|u|, & u > 0. \end{cases}$$

where $\max_{t\in[0,T]} l(t) = 1$, $\min_{t\in[0,T]} l(t) > 0$, $\min_{t\in[\frac{1}{2},\frac{3}{4}]} l(t) = 1$. (A_1), (A_4) and (B_1) are satisfied.

Clearly, 0 is a nonnegative solution. There are other nonzero solutions to the equation. Let nonzero solution $u = (0,0,\ldots,\tilde{u},0,\ldots,0)^{\mathrm{T}}$. $\tilde{u} = t^{-0.9}$ for any $t \in [3.8105, T)$; $\tilde{u}(0) = \tilde{u}(T) \le c_1$; $\tilde{u}(t) \le c_1$ for any $t \in (0, 3.8105)$.

By calculation, $\frac{\max_{t\in[0,T],|u|\in[0,c_1]}}{c_1^2} F(t,u) = \frac{4}{5}c_1^{\frac{-3}{4}} + \frac{c_1^3}{5} + \frac{1}{10c_1}$, $2\max_{|u|\in[0,c_1]}|\partial F(t,u)| = 2(c_1^{\frac{1}{4}} + c_1^4 + \frac{1}{10})$ as $c_1 < 1$, $F(t_j,\hat{u}_0) = \frac{4}{5}(\frac{d_1}{2})^{\frac{5}{4}} + \frac{1}{5}(\frac{d_1}{2})^5 + \frac{d_1}{20}$. Let $c_1 = 0.300$, $d_1 = 0.005$, $b = 0.3$, a straightforward computation shows that $\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2} = 2.3123$, $2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)| = 1.6964$, $\frac{\frac{1}{4}F(t_j,\hat{u}_0)}{d_1^2} = 6.9721$, $[(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2})^2 - (2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|)^2]^{\frac{1}{2}} = 1.5713 < \frac{F(t_j,\hat{u}_0)}{4d_1^2} = 6.9721$, and $\frac{4d^2(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3})}{F(t_j,\hat{u}_0)} = 0.6267 < b\frac{c_1^2(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3})}{\sqrt{1-\bar{z}}\max_{t\in[0,T],|u_i|\in[0,c_1]} F} = 0.8343$. So (A_2), (A_5) are satisfied. Moreover, $|u_*| < \sqrt{\frac{2r_2}{T}} = 0.3106 < 1$, (A_9) is established.

Put

$$I_i(u_j) = \begin{cases} \frac{1}{100m}(-u_j)^{\frac{1}{2}}, & u_j < 0, \\ -\frac{1}{100m}(u_j)^{\frac{1}{2}}, & u_j \ge 0, \end{cases}$$

where u_j is the jth component of u. (A_3), (A_6), (A_8) and (B_0) are satisfied.

Let $D_1 = 0$, $D_2 = \frac{1}{100m}$, $\gamma = \frac{1}{2}$, we obtain $|u'_*| = [(\frac{\max_{t\in[0,T],|u|\in[0,c_1]} F(t,u)}{c_1^2})^2 - (2\max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)|)^2]^{\frac{-1}{2}} \max_{t\in[0,T],|u|\in[0,c_1]} |\partial F(t,u)| b(\frac{1}{\sqrt{1-R^2}} + \frac{1}{2})(\frac{2T+13}{24} + \frac{1}{4T-3}) + |m(D_1 + D_2 R^\gamma)| + 1 - \sqrt{1 - |\frac{2r_2}{T}|} = 0.7076 + 0.0095 + 0.0495 = 0.7666 < 1$, taking $0.7666 < R < 1$, (A_7) is established. So problem (1.7) has at least one solution u satisfying $|u'| \le R$, $|u| \le R$, and each solution of (1.7) is nonnegative.

Appendix: The calculation of (3.27)

Let $M = (\psi(u'_n)|u') - (\psi(u'_n)|u'_n) + (\psi(u_n)|u) - (\psi(u_n)|u_n)$, $S = \int_0^T M\,dt$.

Case 1: $|u'_n| \le R$, $|u_n| \le R$.

From (1.4), we have $(\psi(u'_n)|u') = \frac{(u'_n|u')}{\sqrt{1-|u'_n|^2}} \le \frac{1}{2}\frac{|u'_n|^2+|u'|^2}{\sqrt{1-|u'_n|^2}}$, $(\psi(u_n)|u) = \frac{(u_n|u)}{\sqrt{1-|u_n|^2}} \le \frac{1}{2}\frac{|u_n|^2+|u|^2}{\sqrt{1-|u_n|^2}}$, $(\psi(u'_n)|u'_n) = \frac{|u'_n|^2}{\sqrt{1-|u'_n|^2}}$, $(\psi(u_n)|u_n) = \frac{|u_n|^2}{\sqrt{1-|u_n|^2}}$.

By straightforward calculation, one has $M \le -(\frac{1}{2}\frac{|u'_n|^2}{\sqrt{1-|u'_n|^2}} + \frac{1}{2}\frac{|u_n|^2}{\sqrt{1-|u_n|^2}}) + \frac{1}{2}\frac{|u'|^2}{\sqrt{1-|u'_n|^2}} + \frac{1}{2}\frac{|u|^2}{\sqrt{1-|u_n|^2}}$, $S \le -\frac{1}{2}\min\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\}\|u_n\|_X + \frac{1}{2}\max\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\}\|u\|_X$.

By (A_8), we obtain

$$\begin{aligned}\bigl(\Phi'(u_n)|(u-u_n)\bigr) \le{}& \frac{1}{2}\max\left\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\right\}\|u\|_X \\ &- \left(\frac{1}{2}\min\left\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\right\} + \alpha\right)\|u_n\|_X.\end{aligned}$$

Case 2: $|u'_n| \le R$, $|u_n| > R$.

From (1.4), we have $(\psi(u'_n)|u') = \frac{(u'_n|u')}{\sqrt{1-|u'_n|^2}} \le \frac{1}{2}\frac{|u'_n|^2+|u'|^2}{\sqrt{1-|u'_n|^2}}$, $(\psi(u_n)|u) = \frac{(u_n|u)}{\sqrt{1-|R|^2}} \le \frac{1}{2}\frac{|u_n|^2+|u|^2}{\sqrt{1-|R|^2}}$, $(\psi(u'_n)|u'_n) = \frac{|u'_n|^2}{\sqrt{1-|u'_n|^2}}$, $(\psi(u_n)|u_n) = \frac{|u_n|^2}{\sqrt{1-|R|^2}}$.

By straightforward calculation, one has $M \le -(\frac{1}{2}\frac{|u'_n|^2}{\sqrt{1-|u'_n|^2}} + \frac{1}{2}\frac{|u_n|^2}{\sqrt{1-|R|^2}}) + \frac{1}{2}\frac{|u'|^2}{\sqrt{1-|u'_n|^2}} + \frac{1}{2}\frac{|u|^2}{\sqrt{1-|R|^2}}$, $S \le -\frac{1}{2}\min\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|R|^2}}\}\|u_n\|_X + \frac{1}{2}\max\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|R|^2}}\}\|u\|_X$.

By (A_8), we obtain

$$\begin{aligned}\bigl(\Phi'(u_n)|(u-u_n)\bigr) \le{}& \frac{1}{2}\max\left\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|R|^2}}\right\}\|u\|_X \\ &- \left(\frac{1}{2}\min\left\{\frac{1}{\sqrt{1-|u'_n|^2}}, \frac{1}{\sqrt{1-|R|^2}}\right\} + \alpha\right)\|u_n\|_X.\end{aligned}$$

Case 3: $|u'_n| > R$, $|u_n| \le R$.

From (1.4), we have $(\psi(u'_n)|u') = \frac{(u'_n|u')}{\sqrt{1-|R|^2}} \le \frac{1}{2}\frac{|u'_n|^2+|u'|^2}{\sqrt{1-|R|^2}}$, $(\psi(u_n)|u) = \frac{(u_n|u)}{\sqrt{1-|u_n|^2}} \le \frac{1}{2}\frac{|u_n|^2+|u|^2}{\sqrt{1-|u_n|^2}}$, $(\psi(u'_n)|u'_n) = \frac{|u'_n|^2}{\sqrt{1-|R|^2}}$, $(\psi(u_n)|u_n) = \frac{|u_n|^2}{\sqrt{1-|u_n|^2}}$.

By straightforward calculation, one has $M \le -(\frac{1}{2}\frac{|u'_n|^2}{\sqrt{1-|R|^2}} + \frac{1}{2}\frac{|u_n|^2}{\sqrt{1-|u_n|^2}}) + \frac{1}{2}\frac{|u'|^2}{\sqrt{1-|R|^2}} + \frac{1}{2}\frac{|u|^2}{\sqrt{1-|u_n|^2}}$, $S \le -\frac{1}{2}\min\{\frac{1}{\sqrt{1-|R|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\}\|u_n\|_X + \frac{1}{2}\max\{\frac{1}{\sqrt{1-|R|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\}\|u\|_X$.

By (A_8), we obtain

$$\begin{aligned}\bigl(\Phi'(u_n)|(u-u_n)\bigr) \le{}& \frac{1}{2}\max\left\{\frac{1}{\sqrt{1-|R|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\right\}\|u\|_X \\ &- \left(\frac{1}{2}\min\left\{\frac{1}{\sqrt{1-|R|^2}}, \frac{1}{\sqrt{1-|u_n|^2}}\right\} + \alpha\right)\|u_n\|_X.\end{aligned}$$

Case 4: $|u'_n| > R$, $|u_n| > R$.

From (1.4), we have $(\psi(u'_n)|u') = \frac{(u'_n|u')}{\sqrt{1-|R|^2}} \le \frac{1}{2}\frac{|u'_n|^2+|u'|^2}{\sqrt{1-|R|^2}}$, $(\psi(u_n)|u) = \frac{(u_n|u)}{\sqrt{1-|R|^2}} \le \frac{1}{2}\frac{|u_n|^2+|u|^2}{\sqrt{1-|R|^2}}$, $(\psi(u'_n)|u'_n) = \frac{|u'_n|^2}{\sqrt{1-|R|^2}}$, $(\psi(u_n)|u_n) = \frac{|u_n|^2}{\sqrt{1-|R|^2}}$.

By straightforward calculation, one has $M \le -(\frac{1}{2}\frac{1}{\sqrt{1-|R|^2}}(|u'_n|^2 + |u_n|^2) + \frac{1}{2}\frac{1}{\sqrt{1-|R|^2}}(|u'|^2 + |u|^2)$, $S \le -\frac{1}{2}\frac{1}{\sqrt{1-|R|^2}}\|u_n\|_X + \frac{1}{2}\frac{1}{\sqrt{1-|R|^2}}\|u\|_X$.

By (A_8), we obtain $(\Phi'(u_n)|(u-u_n)) \leq \frac{1}{2}\frac{1}{\sqrt{1-|R|^2}}\|u\|_X - (\frac{1}{2}\frac{1}{\sqrt{1-|R|^2}}+\alpha)\|u_n\|_X$.

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