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Topological structure of solution sets for fractional evolution inclusions of Sobolev type

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Abstract

The paper is devoted to establishing the solvability and topological property of solution sets for the fractional evolution inclusions of Sobolev type. We obtain the existence of mild solutions under the weaker conditions that the semigroup generated by $-AE^{-1}$ is noncompact as well as F is weakly upper semicontinuous with respect to the second variable. On the same conditions, the topological structure of the set of all mild solutions is characterized. More specifically, we prove that the set of all mild solutions is compact and the solution operator is u.s.c. Finally, an example is given to illustrate our abstract results.

Keywords: Fractional evolution inclusion; Sobolev type; Weakly upper semi-continuous; Measures of noncompactness

1 Introduction

Fractional calculus has been used successfully to study many complex systems in various fields of science and engineering, which mainly rely on the nonlocal character of the fractional differentiation, we refer the readers to [10, 19]. Fractional differential equations and inclusions have recently been proved to be valuable tools in the mathematical modeling of systems and processes in the fields of physics [8], chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. In recent years, there has been a significant development on differential equations involving fractional derivatives. We refer the reader to [22–24] and the references therein.

It is worth mentioning that evolution equations of Sobolev type have been extensively studied due to their various applications such as in the flow of fluid through fissured rocks, thermodynamics and shear in second order fluids (cf. [3, 6, 12]). The fractional evolution equations of Sobolev type, which arise in the theory of control of dynamical systems when the controlled system or the controller is described by a fractional evolution equation of Sobolev type, provide the mathematical modeling and simulations of controlled systems and processes. For the research of fractional evolution equations of Sobolev type, we refer the readers to [9, 13].

Since a differential inclusion usually has many solutions starting at a given point, new issues, such as the investigation of topological properties of solution sets, selection of solutions with given properties, and evaluation of the reachability sets, appear. An important aspect of the topological structure is the compactness of solution sets. Regarding the topological structure of solution sets for differential equations and inclusions, we may cite, among others [2, 11, 21, 25]. However, to the best of our knowledge, the topological structure of the solution sets for fractional evolution inclusions of Sobolev type has not been explored.

In this paper, we study the fractional evolution inclusions of Sobolev type in Banach spaces X and Y:

$$\begin{cases} {}^cD_t^q(Ex(t)) + Ax(t) \in F(t, x(t)), & t \in J := [0, T], \\ x(0) = x_0 \in D(E) \subset X, \end{cases} \tag{1.1}$$

where ${}^cD_t^q$ is the regularized Caputo derivative of order $0 < q < 1$, the operators $E : D(E) \subset X \to Y$ and $A : D(A) \subset X \to Y$ are two closed linear operators, and $F : J \times X \to 2^Y$ is a multi-valued function with convex, closed values.

In this work, motivated by the above consideration, we are interested in investigating the topological structure of the solution set for (1.1) under rather mild conditions. Our purpose is to prove that the solution set for (1.1) is nonempty and compact with the semigroup generated by $-AE^{-1}$ being noncompact.

By assuming that E^{-1} is compact or the resolvent set $R(\lambda, -AE^{-1})$ of $-AE^{-1}$ is compact for every $\lambda \in \rho(-AE^{-1})$, which guarantees that the semigroup generated by $-AE^{-1}$ is compact, there has been many works devoted to solvability and controllability of fractional (or integer order) evolution equations of Sobolev type such as [1, 7, 9]. After reviewing the previous research on the fractional evolution equations and inclusions of Sobolev type, we find that most of the works assume that the semigroup generated by $-AE^{-1}$ is compact. However, much less is known about the fractional evolution equations and inclusions of Sobolev type with the noncompact semigroup. We will prove that the solution set for (1.1) is nonempty and compact when the semigroup generated by $-AE^{-1}$ is noncompact and the multi-valued function F is weakly upper semicontinuous with respect to the second variable. We also prove that the solution operator is u.s.c.

Let us give a description of our approach. When dealing with the solvability of (1.1), the key point is to find a compact convex subset which is invariant under the operator $\mathfrak{L}$ (defined in the proof of Theorem 3.1). It is noted that when the semigroup is compact, the compactness of the convex subset becomes a direct consequence by a standard argument. In this paper, however, we assume that the semigroup is noncompact. To overcome this difficulty, we suppose that the multi-valued function F satisfies some regular properties expressed by the measure of noncompactness. Hence, by utilizing the measure of noncompactness, multi-valued analysis, and fixed point theory, we obtain the solvability of (1.1). The method of proving the result of the topological structure of the solution set of (1.1) mainly comes from [21].

The paper is organized as follows. In Sect. 2, we recall some concepts and facts which are broadly used for deriving the main results of the paper. In Sect. 3, the nonemptiness of solution set of (1.1) is proved, and then we show that the solution set of (1.1) is a compact set; moreover, we prove that the solution operator is u.s.c. An example is given to illustrate our abstract results in Sect. 4.

2 Preliminaries

As usual, for a Banach space V, 2^V stands for the collection of all nonempty subsets of V. $C(J;V)$ denotes the Banach space of all continuous functions from J to V equipped with its usual norm. For $1 \leq p < \infty$, let $L^p(J;V)$ stand for the Banach space consisting of all Bochner integrable functions $u : J \to V$. $W^{1,p}(J;V)$ is the subspace of $L^p(J;V)$ consisting of functions such that the weak derivative u_t belongs to $L^p(J;V)$. Both spaces $L^p(J;V)$ and $W^{1,p}(J;V)$ are endowed with their standard norms.

We present the criterion of weak compactness in $L^p(J;V)$ for $1 < p < +\infty$, which is more useful further.

Lemma 2.1 ([20, Corollary 1.3.1]) *Let V be reflexive and $1 < p < +\infty$. A subset $K \subset L^p(J;V)$ is weakly relatively sequentially compact in $L^p(J;V)$ if and only if K is bounded in $L^p(J;V)$.*

In order to define the fractional derivative, it is important to recall some facts about the theory of fractional calculus. For more details about fractional calculus, please see, e.g., [15, 22]. Define the function $g_q : \mathbb{R} \to \mathbb{R}$ for $q \geq 0$ by

$$g_q(t) = \begin{cases} \frac{1}{\Gamma(q)} t^{q-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $g_0(t) = 0$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.1 ([21]) For a function $u \in L^1(J;V)$, $q \geq 0$, the Riemann–Liouville fractional integral of order q of u can be expressed by

$$J_t^q u(t) := g_q * u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\,\mathrm{d}s, \quad t \in J, q > 0$$

with $J_t^0 u(t) = u(t)$.

Definition 2.2 ([21]) Let $m \in \mathbb{N}$ and $0 \leq m-1 < q < m$. If $u \in C^{m-1}(J;V)$, $g_{m-q} * u \in W^{m,1}(J;V)$. Then

$${}^cD_t^q u(t) := \frac{\mathrm{d}^m}{\mathrm{d}t^m} J_t^{m-q} \left(u(t) - \sum_{i=0}^{m-1} u^{(i)}(0) g_{i+1}(t) \right)$$

is called the Caputo fractional derivative of order q of the function u.

We make the following assumptions on the operators A and E.

(H_1) E and A are closed linear operators.

(H_2) $D(E) \subset D(A)$ and E is bijective.

(H_3) $E^{-1} : Y \to D(E)$ is continuous.

Remark 2.1 We can deduce from assumptions (H_1) and (H_3) that E^{-1} is bounded.

Assumptions (H_1), (H_2) and a closed graph theorem imply the boundedness of the linear operator $-AE^{-1} : Y \to Y$, and then $-AE^{-1}$ generates a semigroup $\{T(t), t \geq 0\}$ which is continuous for $t > 0$ in the uniform operator topology (see [18, Theorem 1.2]). Throughout this paper, we assume that there exists a constant $M > 0$ such that $\sup\{\|T(t)\|, t \geq 0\} \leq M$.

Consider the fractional evolution equation of Sobolev type in the form

$$\begin{cases} {}^cD_t^q(Ex(t)) + Ax(t) = f(t), & t \in J, \\ x(0) = x_0 \in D(E), \end{cases} \tag{2.1}$$

where $f \in L^p(J;Y)$ with $p > 1$ and $pq > 1$.

We use the following definition of mild solution of (2.1) which comes from [9].

Definition 2.3 A function $x \in C(J;X)$ is called a mild solution of (2.1) if it satisfies

$$x(t) = \mathcal{Q}(t)Ex_0 + \int_0^t (t-s)^{q-1}\mathcal{P}(t-s)f(s)\,\mathrm{d}s, \quad t \in J,$$

where

$$\mathcal{Q}(t) = \int_0^\infty E^{-1}\Psi_q(\theta)T\big(t^q\theta\big)\,\mathrm{d}\theta, \qquad \mathcal{P}(t) = q\int_0^\infty \theta E^{-1}\Psi_q(\theta)T\big(t^q\theta\big)\,\mathrm{d}\theta,$$

$$\Psi_q(s) = \frac{1}{\pi q}\sum_{n=1}^\infty (-s)^{n-1}\frac{\Gamma(1+qn)}{n!}\sin(n\pi q), \quad s \in (0,\infty)$$

with Ψ_q the probability density function defined on $(0,\infty)$, that is, $\Psi_q(s) \geq 0$, $s \in (0,\infty)$, and $\int_0^\infty \Psi_q(s)\,\mathrm{d}s = 1$.

For each $x_0 \in D(E)$ and $f \in L^p(0,T;Y)$, we denote by $x(\cdot,x_0,f)$ the unique mild solution to (2.1). For given $x_0 \in D(E)$, we also define the map $\mathcal{S} : L^p(0,T;Y) \to C(J;X)$ by setting

$$\mathcal{S}f(t) = x(t,x_0,f), \quad t \in J,$$

where $f \in L^p(0,T;Y)$.

By using the same argument as in the proof of [9, Lemma 3.2], we have some additional properties of the two families $\{\mathcal{Q}(t), t \geq 0\}$ and $\{\mathcal{P}(t), t \geq 0\}$ of operators.

Lemma 2.2 *Assume that* (H_1)–(H_3) *hold. Then*

(i) *for every* $t \geq 0$, $\mathcal{Q}(t)$ *and* $\mathcal{P}(t)$ *are linear and bounded operators on* Y, *more precisely,*

$$\big\|\mathcal{Q}(t)\omega\big\| \leq M\big\|E^{-1}\big\|\|\omega\|, \qquad \big\|\mathcal{P}(t)\omega\big\| \leq \frac{qM\|E^{-1}\|}{\Gamma(1+q)}\|\omega\|, \quad t \geq 0, \omega \in Y;$$

(ii) $\mathcal{Q}(t)$ *and* $\mathcal{P}(t)$, $t > 0$, *are continuous in the uniform operator topology.*

Remark 2.2 When $E = I$, $I : Y \to Y$ is an identity operator, one has

$$\mathcal{Q}(t) = \int_0^\infty \Psi_q(\theta)T\big(t^q\theta\big)\,\mathrm{d}\theta, \qquad \mathcal{P}(t) = q\int_0^\infty \theta\Psi_q(\theta)T\big(t^q\theta\big)\,\mathrm{d}\theta.$$

From the proof of [21, Lemma 3.1], we can derive the following characterization.

Lemma 2.3 *If D is a bounded subset in X and $\mathcal{K}$ is an L^p-integrable bounded subset in $L^p(J;Y)$, that is,*

$$\|f(t)\| \le \rho(t) \quad \text{for all } f \in \mathcal{K} \text{ and a.e. } t \in J,$$

where $\rho \in L^p(J;\mathbb{R}^+)$. Then the set of all mild solutions for (2.1)

$$\{x(\cdot, x_0, f) : x_0 \in D, f \in \mathcal{K}\}$$

is equicontinuous in $C(J;X)$.

Lemma 2.4 *If the two sequences $\{f_n\} \subset L^p(J;Y)$ and $\{x_n\} \subset C(J;X)$, where x_n is a mild solution of the problem*

$$\begin{cases} {}^cD_t^q(Ex_n(t)) + Ax_n(t) = f_n(t), & t \in J, \\ x_n(0) = x_0 \in D(E) \subset X, \end{cases}$$

$\lim_{n\to\infty} f_n = f$ *weakly in $L^p(J;Y)$ and* $\lim_{n\to\infty} x_n = x$ *in $C(J;X)$, then x is a mild solution of the limit problem*

$$\begin{cases} {}^cD_t^q(Ex(t)) + Ax(t) = f(t), & t \in J, \\ x(0) = x_0 \in D(E) \subset X. \end{cases}$$

Proof Define $\mathcal{H} : X \times L^p(J;Y) \to C(J;X)$ by

$$\mathcal{H}(u_0, f) = x(\cdot, x_0, f).$$

It is obvious that $\mathcal{H}$ is linear and continuous. Hence, its graph is sequentially closed in the product space $[X \times L^p(J;Y)] \times C(J;X)$. Moreover, its graph is convex. Thus it is weakly $\times$ strongly sequentially closed, which completes the proof. □

Let U and Z be metric spaces. Denote

$$C(Z) = \{D \in 2^Z : D \text{ is closed}\},$$

$$C_v(Z) = \{D \in C(Z) : D \text{ is convex}\},$$

$$K(Z) = \{D \in C(Z) : D \text{ is compact}\}.$$

Let $\varphi : U \to 2^Z$ be a multi-valued map and $\mathrm{Gra}(\varphi)$ be the graph of φ. Denote by $\varphi^{-1}(D) = \{y \in U : \varphi(y) \cap D \neq \emptyset\}$ the complete preimage of D under φ, where $D \subset Z$.

(i) φ is called closed if $\mathrm{Gra}(\varphi)$ is closed in $U \times Z$;

(ii) φ is called quasi-compact if $\varphi(K)$ is relatively compact for each compact set $K \subset U$;

(iii) φ is called upper semi-continuous (shortly, u.s.c.) if $\varphi^{-1}(D)$ is closed for each closed set $D \subset Z$, and lower semi-continuous (shortly, l.s.c.) if $\varphi^{-1}(D)$ is open for each open set $D \subset Z$.

The following lemma gives a sufficient condition for u.s.c. multi-valued maps.

Lemma 2.5 ([14, Theorem 1.1.12]) *Let $\varphi : U \to K(Z)$ be a closed and quasi-compact multi-valued map. Then φ is u.s.c.*

Furthermore, in the case when U and Z are Banach spaces, a multi-valued map $\varphi : D \subset U \to 2^Z$ is called weakly upper semi-continuous (shortly, weakly u.s.c.) if $\varphi^{-1}(\mathcal{B})$ is closed in D for every closed set $\mathcal{B} \subset Z$.

Lemma 2.6 ([5, Lemma 2.2(ii)]) *Let $\varphi : D \subset U \to 2^Z$ be a multi-valued map with convex weakly compact values. Then φ is weakly u.s.c. if and only if, for each sequence $\{(u_n, z_n)\} \subset D \times Z$ such that $u_n \to u$ in U and $z_n \in \varphi(u_n)$, $n \geq 1$, it follows that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and $z \in \varphi(u)$ such that $z_{n_k} \to z$ weakly in Z.*

We state the following fixed point result which will be used in the proof of the existence result.

Lemma 2.7 ([4, Lemma 1]) *Let D be a nonempty, compact, and convex subset of a Banach space and $\varphi : D \to 2^D$ u.s.c. with contractible values. Then φ has at least one fixed point.*

Now, we recall some facts about the measure of noncompactness (MNC). The definition of MNC can be found in lots of literature works, for example, [14]. Here, we only introduce some specific properties of the Hausdorff MNC.

The Hausdorff MNC enjoys the following properties (see [14]).

Let $\mathcal{E}$ be a Banach space,

(i) for any bounded linear operators T from $\mathcal{E}$ to $\mathcal{E}$ and $\Omega \subset \mathcal{E}$, it follows that

$$\chi(T\Omega) \leq \|T\| \chi(\Omega); \tag{2.2}$$

(ii) for every bounded subset $D \subset \mathcal{E}$ and $\epsilon > 0$, there is a sequence $\{w_n\} \subset D$ such that

$$\chi(D) \leq 2\chi\left(\{w_n\}\right) + \epsilon. \tag{2.3}$$

We need the following statement which provides us with a basic MNC estimate.

Lemma 2.8 ([17]) *Let the sequence of functions $\{g_n\} \subset L^1(J; \mathcal{E})$ be integrably bounded, i.e.,*

$$\|g_n(t)\| \leq \varrho(t) \quad \text{for a.e. } t \in J \text{ and all } n \geq 1,$$

where $\varrho \in L^1(J; \mathbb{R}^+)$. Then the function $\chi(\{g_n(t)\})$ belongs to $L^1(J; \mathbb{R}^+)$ and satisfies that

$$\chi\left(\left\{\int_0^t g_n(s)\,ds\right\}\right) \leq 2\int_0^t \chi\left(\{g_n(s)\}\right) ds$$

for each $t \in J$.

3 Main results

For the multi-valued function $F : J \times X \to C_v(Y)$, we have the following standing assumptions.

(H_4) $F(t,\cdot)$ is weakly u.s.c. for a.e. $t \in J$ and $F(\cdot,v)$ has an L^p-integral selection for each $v \in X$;

(H_5) there exists a function $\eta \in L^p(J;\mathbb{R}^+)$ such that

$$\|F(t,v)\| := \sup\{\|y\| : y \in F(t,v)\} \le \eta(t)(1+\|v\|)$$

for a.e $t \in J$ and each $v \in X$;

(H_6) there exists $\mu \in L^p(J;\mathbb{R}^+)$ such that

$$\chi(F(t,\Omega)) \le \mu(t)\chi(\Omega)$$

for a.e. $t \in J$ and all bounded subsets $\Omega \subset X$.

Define a multi-valued map $\mathrm{Sel}_F : C(J;X) \to 2^{L^p(J;Y)}$ by

$$\mathrm{Sel}_F(x) := \{f \in L^p(J;Y) \text{ and } f(t) \in F(t,x(t)) \text{ for a.e. } t \in J\}.$$

Then the multi-valued map Sel_F satisfies the following.

Lemma 3.1 ([21, Lemma 3.3]) *Assume that* (H_4)–(H_5) *hold and suppose that* Y *is reflexive. Then* Sel_F *is weakly u.s.c with nonempty, convex, and weakly compact values.*

Now, we are ready to give the existence result. Here, $x \in C(J;X)$ is a mild solution of (1.1) if x is a mild solution of (2.1) with $f \in \mathrm{Sel}_F(x)$.

Set

$$\alpha = M\|E^{-1}\|\|E\|, \qquad \beta = \frac{qM\|E^{-1}\|}{\Gamma(1+q)}, \qquad \gamma = T^{q-\frac{1}{p}}\left(\frac{p-1}{pq-1}\right)^{1-\frac{1}{p}}.$$

Theorem 3.1 *Let* $p > 1$, $pq > 1$, *and assumptions* (H_1)–(H_6) *hold. Assume further that* Y *is reflexive, then* (1.1) *has at least one solution.*

Proof Define the multi-valued map $\mathfrak{L} : C(J;X) \to 2^{C(J;X)}$ by

$$\mathfrak{L} = \mathcal{S} \circ \mathrm{Sel}_F.$$

Then we look for a fixed point x of the multi-valued map $\mathfrak{L}$ which in fact is a mild solution of (1.1). To do this, we will find a compact bounded convex set $\mathcal{M} \subset C(J;X)$ such that $\mathfrak{L}(\mathcal{M}) \subset \mathcal{M}$ and prove that $\mathfrak{L}$ is u.s.c. with contractible values due to Lemma 2.7.

Let $\psi \in C(J;\mathbb{R}^+)$ be the unique solution of the integral equation

$$\psi(t) = \alpha\|x_0\| + \beta\gamma\|\eta\|_{L^p(0,T)} + \beta\int_0^t (t-s)^{q-1}\eta(s)\psi(s)\,ds,$$

and set

$$\mathcal{M}_0 = \{x \in C(J;X) : \|x(t)\| \le \psi(t) \text{ for } t \in J\}.$$

It is easy to see that the set $\mathcal{M}_0$ is closed, bounded, and convex. Moreover, it can be easily verified that $\mathfrak{L}$ maps $\mathcal{M}_0$ into itself, that is, $\mathfrak{L}(\mathcal{M}_0) \subset \mathcal{M}_0$. In fact, taking $x \in \mathcal{M}_0$ and $\hat{x} \in \mathfrak{L}(\mathcal{M}_0)$, there exists $f \in \mathrm{Sel}_F(x)$ such that $\hat{x} = \mathcal{S}f$. For each $t \in J$, it follows from (H_5) that

$$\begin{aligned}\|\hat{x}(t)\| &= \left\| \mathcal{Q}(t)Ex_0 + \int_0^t (t-s)^{q-1}\mathcal{P}(t-s)f(s)\,\mathrm{d}s \right\| \\ &\leq \alpha\|x_0\| + \beta\gamma\|\eta\|_{L^p(0,T)} + \beta\int_0^t (t-s)^{q-1}\eta(s)x(s)\,\mathrm{d}s \\ &= \psi(t).\end{aligned}$$

Put

$$\begin{aligned}&\mathcal{M}_{k+1} = \overline{\mathrm{conv}}\,\mathfrak{L}(\mathcal{M}_k), \quad k = 0,1,\ldots, \\ &\mathcal{M} = \bigcap_{k=0}^{\infty}\mathcal{M}_k.\end{aligned}$$

Notice that $\mathcal{M}_k$ is clearly closed, convex and $\mathcal{M}_{k+1} \subset \mathcal{M}_k$ for each $k = 0,1,\ldots$. Then the set $\mathcal{M}$ is nonempty, closed, and convex. Also, by the same argument as above, we get that $\mathfrak{L}(\mathcal{M}) \subset \mathcal{M}$.

In what follows, we focus on the compactness of $\mathcal{M}$. We only need to prove that $\mathcal{M}$ is relatively compact since $\mathcal{M}$ is closed.

For each $k = 0,1,\ldots$, thanks to (H_5), we know that $\mathrm{Sel}_F(\mathcal{M}_k)$ is integrably bounded. Then, applying Lemma 2.3, we get $\mathfrak{L}(\mathcal{M}_k)$ is equicontinuous. From this, we prove that $\mathcal{M}_{k+1}$ is equicontinuous. Therefore, $\mathcal{M}$ is also equicontinuous.

Given $\epsilon > 0$, for each $k = 0,1,\ldots$, from (2.3) we can take a sequence $\{f_n\} \subset \mathrm{Sel}_F(\mathcal{M}_k)$ such that, for each $t \in J$,

$$\chi\bigl(\mathfrak{L}(\mathcal{M}_k)(t)\bigr) \leq 2\chi\bigl(\bigl(\{\mathcal{S}f_n\}\bigr)(t)\bigr) + \epsilon,$$

which yields that

$$\chi\bigl(\mathcal{M}_{k+1}(t)\bigr) \leq 2\chi\bigl(\bigl(\mathcal{S}\{f_n\}\bigr)(t)\bigr) + \epsilon. \tag{3.1}$$

For the sequence $\{f_n\}$, we consider the set $\{(t-\cdot)^{q-1}\mathcal{P}(t-\cdot)f_n(\cdot)\}$. Let us take $x_n \in \mathcal{M}_k$ such that $x_n = \mathcal{S}f_n$. From (H_5) we know that, for $t,s \in J$, and $s < t$,

$$\begin{aligned}\bigl\|(t-s)^{q-1}\mathcal{P}(t-s)f_n(s)\bigr\| &\leq \beta\bigl(1 + \bigl\|x_n(t)\bigr\|\bigr)(t-s)^{q-1}\eta(s) \\ &\leq \beta\bigl(1 + \psi(T)\bigr)(t-s)^{q-1}\eta(s),\end{aligned}$$

which together with the fact $(t-\cdot)^{q-1}\eta(\cdot) \in L^1(J;\mathbb{R}^+)$ implies that the set $\{(t-\cdot)^{q-1}\mathcal{P}(t-\cdot)f_n(\cdot)\}$ is integrably bounded in $L^1(0,t;Y)$. Hence, according to (2.2), (H_6), and Lemma 2.8, we conclude that, for each $t \in J$,

$$\chi\bigl(\bigl(\{\mathcal{S}f_n\}\bigr)(t)\bigr) \leq \chi\left(\left\{\int_0^t (t-s)^{q-1}\mathcal{P}(t-s)f_n(s)\,\mathrm{d}s\right\}\right)$$

$$\leq 2\beta \int_0^t (t-s)^{q-1}\mu(s)\chi\left(\left\{x_n(s)\right\}\right)\,\mathrm{d}s$$
$$\leq 2\beta \int_0^t (t-s)^{q-1}\mu(s)\chi\left(\mathcal{M}_k(s)\right)\,\mathrm{d}s. \tag{3.2}$$

Combining (3.1) with (3.2), we arrive at

$$\chi\left(\mathcal{M}_{k+1}(t)\right) \leq 4\beta \int_0^t (t-s)^{q-1}\mu(s)\chi\left(\mathcal{M}_k(s)\right)\,\mathrm{d}s + \epsilon, \quad t \in J.$$

Since $\epsilon > 0$ is arbitrary, one can see that

$$\lim_{k\to\infty} \chi\left(\mathcal{M}_{k+1}(t)\right) \leq 4\beta \int_0^t (t-s)^{q-1}\mu(s) \lim_{k\to\infty} \chi\left(\mathcal{M}_k(s)\right)\,\mathrm{d}s, \quad t \in J.$$

By using the known Gronwall's inequality, we conclude that $\lim_{k\to\infty} \chi(\mathcal{M}_k(t)) = 0$ for each $t \in J$, which implies that $\chi(\mathcal{M}(t)) = 0$ for each $t \in J$. Thus, the Arzela–Ascoli theorem ensures that $\mathcal{M}$ is relatively compact.

In the sequel, we verify that $\mathfrak{L}$ is u.s.c. on $\mathcal{M}$. Noticing that $\mathfrak{L}(\mathcal{M}) \subset \mathcal{M}$, we have $\mathfrak{L}$ is quasi-compact. Let $\{(v_n, w_n)\}$ be a sequence in $\mathrm{Gra}(\mathfrak{L})$ such that

$$(v_n, w_n) \to (v, w) \quad \text{in } C(J;X) \times C(J;X).$$

Then there exists a sequence $\{f_n\} \subset L^p(J;Y)$ such that $f_n \in \mathrm{Sel}_F(v_n)$ and $w_n = \mathcal{S}f_n$. Observe that Sel_F is weakly u.s.c. with convex, weakly compact values due to Lemma 3.1. It follows from Lemma 2.6 that there exist $f \in \mathrm{Sel}_F(v)$ and a subsequence of $\{f_n\}$, still denoted by $\{f_n\}$, such that $f_n \to f$ weakly in $L^p(J;Y)$. Lemma 2.4 guarantees that $w = \mathcal{S}f$ and thus $w \in \mathfrak{L}(v)$, which implies that $\mathfrak{L}$ is closed. Hence, it yields from Lemma 2.5 that $\mathfrak{L}$ is u.s.c. on $\mathcal{M}$.

Finally, we proceed to proving that $\mathfrak{L}$ has contractible values. To this end, let $x \in \mathcal{M}$ and $f^* \in \mathrm{Sel}_F(x)$ be fixed. Define a function $h : [0,1] \times \mathfrak{L}(x) \to \mathfrak{L}(x)$ by

$$h(\lambda, v)(t) = \begin{cases} v(t), & t \in [0, \lambda T], \\ \widetilde{x}(t;\lambda, v), & t \in (\lambda T, T], \end{cases}$$

in which

$$\widetilde{x}(t;\lambda, v) = \mathcal{Q}(t)Ex_0 + \int_0^{\lambda T} (t-s)^{q-1}\mathcal{P}(t-s)\hat{f}(s)\,\mathrm{d}s + \int_{\lambda T}^t (t-s)^{q-1}\mathcal{P}(t-s)f^*(s)\,\mathrm{d}s$$

with $\hat{f} \in \mathrm{Sel}_F(x)$ and $v = \mathcal{S}\hat{f}$.

A direct calculation yields that, for every $v \in \mathfrak{L}(x)$, $h(0, v) = \mathcal{S}f^*$ and $h(1, v) = v$. Also, it is not difficult to verify that h is continuous. Hence, $\mathfrak{L}$ has contractible values.

Therefore, an application of Lemma 2.7 enables us to conclude that $\mathfrak{L}$ has at least one fixed point in $\mathcal{M}$. The proof is complete. □

We denote the set of all mild solutions of (1.1) by

$$\Phi(x_0) = \left\{x \in C\left([0,T];X\right) : x \text{ is the mild solution of (1.1) for each } x_0 \in D(E)\right\}.$$

Now, we are in a position to give the topological structure of $\Phi(x_0)$.

Theorem 3.2 *Let the hypotheses in Theorem 3.1 be satisfied. For each $x_0 \in D(E)$, $\Phi(x_0)$ is a compact set.*

Proof Given $x_0 \in D(E)$. Let $\{x_n\} \in \Phi(x_0)$ and $f_n \in \mathrm{Sel}_F(x_n)$ such that $x_n = \mathcal{S}f_n$. It follows from (H_5) that $\{f_n\}$ is L^p-integrable bounded, then Lemma 2.3 ensures the equicontinuity of $\{x_n\}$. Moreover, by an argument similar to that in the proof of Theorem 3.1, we obtain that $\{x_n(t)\}$ is relatively compact for each $t \in J$. Applying the Arzela–Ascoli theorem, we obtain that $\{x_n\}$ is relatively compact. Then, thanks to Lemmas 2.6, 3.1, there exist $\{x_{n_k}\}$ of $\{x_n\}$, $\{f_{n_k}\}$ of $\{f_n\}$, $x \in C([0,T];X)$, and $f \in \mathrm{Sel}_F(x)$ such that

$$\lim_{k\to\infty} x_{n_k} = x \quad \text{in } C\big([0,T];X\big),$$

$$\lim_{k\to\infty} f_{n_k} = f \quad \text{weakly in } L^p(0,T;Y).$$

This together with Lemma 2.4 implies that $x = \mathcal{S}f$, and thus $x \in \Phi(x_0)$. Therefore, the compactness of $\Phi(x_0)$ is established. □

Now, we treat the solution operator

$$\Phi : D(E) \to 2^{C([0,T];X)}.$$

It is easy to see that Φ is well defined. Moreover,

Theorem 3.3 *If the hypotheses in Theorem 3.1 are satisfied, then Φ is u.s.c.*

Proof Assume that $K \subset D(E)$ is a compact subset and take $\{x_n\} \subset \Phi(K)$. As in Theorem 3.1, we know $\{x_n\}$ is equicontinuous.

Taking $\{u_n\} \subset K$ and $f_n \in \mathrm{Sel}_F(x_n)$, for each $t \in J$,

$$\begin{aligned}\chi\big(\{x_n(t)\}\big) &\le \chi\big(\mathcal{Q}(t)Eu_n(t)\big) + \chi\left(\left\{\int_0^t (t-s)^{q-1}\mathcal{P}(t-s)g_n(s)\,\mathrm{d}s\right\}\right)\\ &\le 2\beta\int_0^t (t-s)^{q-1}\mu(s)\chi\big(\{x_n(s)\}\big)\,\mathrm{d}s.\end{aligned}$$

By using the known Gronwall's inequality, we conclude that $\chi(\{x_n(t)\}) = 0$ for each $t \in J$. Thus, the Arzela–Ascoli theorem implies that $\{x_n\}$ is relatively compact, which yields Φ is quasi-compact. Performing the same argument as in the later proof of Theorem 3.2, we obtained that Φ is closed. Therefore, Lemma 2.5 ensures that Φ is u.s.c. □

4 An example

Let $X = Y = L^2(0,\pi)$ and denote its norm by $\|\cdot\|$ and the inner product by $(\cdot,\cdot)$. Consider the following system of partial differential inclusion:

$$\begin{cases} {}^cD_t^q(x(t,\xi) - x_{\xi\xi}(t,\xi)) - x_{\xi\xi}(t,\xi) \in F(t,\xi,x(t,\xi)), & (t,\xi) \in [0,T]\times[0,\pi],\\ x(t,0) = x(t,\pi) = 0, & t \in [0,T],\\ x(0,\xi) = x_0(\xi), & \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $\frac{1}{2} < q < 1$, $F(t,\xi,x) = [f_1(t,\xi,x), f_2(t,\xi,x)]$ is a closed interval for each $(t,\xi,x) \in [0,T] \times [0,\pi] \times \mathbb{R}$.

Let the functions

$$f_i : [0,T] \times [0,\pi] \times \mathbb{R} \to \mathbb{R}, \quad i = 1,2,$$

be such that

- f_1 is l.s.c. and f_2 is u.s.c.,
- $f_1(t,\xi,x) \le f_2(t,\xi,x)$ for each $(t,\xi,x) \in [0,T] \times [0,\pi] \times \mathbb{R}$,
- there exist $l_1, l_2 \in L^\infty(0,T;\mathbb{R}^+)$ such that

  $$\big|f_i(t,\xi,x)\big| \le l_1(t)|x| + l_2(t), \quad i = 1,2$$

  for each $(t,\xi,x) \in [0,T] \times [0,\pi] \times \mathbb{R}$.

Then one can verify (see [21]) that the multi-valued function $F : [0,1] \times X \to 2^X$ defined as

$$F(t,x) = \big\{y \in X : y(\xi) \in \big[f_1\big(t,\xi,x(\xi)\big), f_2\big(t,\xi,x(\xi)\big)\big] \text{ a.e. in } [0,\pi]\big\}$$

satisfies assumptions (H_4)–(H_5) (with $\eta(t) = \sqrt{\pi}\max\{l_1(t), l_2(t)\}$ appearing in (H_5)).

Define $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ by

$$\begin{aligned}
&Ax = -x_{\xi\xi}, \qquad Ex = x - x_{\xi\xi}, \\
&D(A) = D(E) \\
&\quad = \big\{x \in X : x, x_\xi \text{ are absolutely continuous}, x_{\xi\xi} \in X, \text{and } x(t,0) = x(t,\pi) = 0\big\}.
\end{aligned}$$

As in [16], A and E can be written as

$$\begin{aligned}
&Ax = \sum_{n=1}^{\infty} n^2 (x,x_n)x_n, \quad x \in D(A), \\
&Ex = \sum_{n=1}^{\infty} \big(1+n^2\big)(x,x_n)x_n, \quad x \in D(E),
\end{aligned}$$

where $x_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$, $n \in \mathbb{N}^+$ is the orthonormal set of eigenfunctions of A. It is clear that A and E are closed linear operators. Moreover, for any $x \in X$,

$$E^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(x,x_n)x_n, \qquad -AE^{-1}x = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(x,x_n)x_n.$$

Evidently, E^{-1} is continuous. Hence, assumptions (H_1)–(H_3) are satisfied.

From the definition of AE^{-1}, we can verify that $-AE^{-1}$ is a bounded linear operator and it generates a semigroup $T(t)$ on X,

$$T(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}(x,x_n)x_n, \quad x \in X,$$

which is continuous for $t > 0$ in the uniform operator topology. It is easy to see that $\|T(t)\| \leq e^{-t} \leq 1$, and hence $M = 1$.

Therefore, hypotheses (H_1)–(H_5) are satisfied. If F satisfies (H_6), then Theorems 3.1, 3.2 enable us to obtain that the set of all mild solutions of system (4.1) is nonempty and it is a compact set.

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Abbreviations
MNC, measure of noncompactness; u.s.c., upper semi-continuous; l.s.c., lower semi-continuous; weakly u.s.c., weakly upper semi-continuous.

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