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Topological properties of C^0-solution set for impulsive evolution inclusions

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Abstract

In this paper, we study the topological properties to a C^0-solution set of impulsive evolution inclusions. The definition of C^0-solutions for impulsive functional evolution inclusions is introduced. The R_δ-property of C^0-solution set is studied for compact as well as noncompact semigroups on compact intervals. Applying the inverse limit method, the R_δ-structure on noncompact intervals is obtained.

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1 Introduction

Impulsive differential equations and inclusions act as excellent tools to model the real world phenomena exhibiting instantaneous change in state variables. Examples include real-time software verification [2], chemical process plants [19], mobile robotics [8], automotive control [7], nerve impulse transmission [24], etc. For some recent works on the topic, we refer the reader to [1, 13, 20, 34] and the references therein.

For the last few decades, many researchers contributed to the development of the subject by producing significant works on initial and boundary value problems of impulsive differential equations and inclusions. The study of topological properties of solution sets of differential inclusions also gained significant importance. Bothe et al. [11] discussed the existence of integral solutions on a compact interval for the differential inclusions involving equicontinuous (not compact) semigroups. Cardinali et al. [12] proved the existence of local and global mild solutions of semilinear evolution differential inclusions and then studied the compactness of the set of all global mild solutions in the case of a noncompact semigroup. In [13], Cardinali et al. focused on the compactness of the set of mild solutions to semilinear impulsive evolution differential inclusions and obtained the existence of mild solutions for semilinear impulsive evolution differential inclusion on noncompact domains in the case of a noncompact semigroup. Gabor et al. [20, 21] showed that the solution set of impulsive functional differential inclusions is an R_δ-set on compact intervals, and then extended their work to the half-line by using the inverse limit method, when the semigroup is noncompact. Chen et al. [15] considered nonlinear delay evolution differential inclusions and studied the R_δ-structure of C^0-solution set on noncompact intervals in the presence of a compact semigroup. For more results on topological properties of solution sets, we refer the reader to the monographs [10, 14, 18, 23, 26, 30, 31, 39] and a series of articles [3, 5, 6, 22, 28, 29, 32, 33, 35–38], and the references cited therein.

A strong motivation for this paper is mainly due to two reasons: there is no definition of C^0-solutions for impulsive nonlinear evolution inclusions in the related literature at the moment. Secondly, the topological structure of C^0-solutions is yet to be developed when the semigroup is noncompact.

Our aim is to investigate topological properties of the C^0-solution set to the impulsive differential inclusions on noncompact intervals. For a preset $\tau > 0$ and a piecewise continuous function $\varphi : [-\tau, 0] \to E$, where E is the Banach space, we study the following problem:

$$\begin{cases} u'(t) \in Au(t) + f(t), & \text{a.a. (almost all) } t \in \mathbb{R}^+, t \neq t_m, m \in \mathbb{N}^+, \\ f(t) \in F(t, u(t), u_t), & \text{a.a. } t \in \mathbb{R}^+, t \neq t_m, m \in \mathbb{N}^+, \\ u(t) = \varphi(t), & t \in [-\tau, 0], \\ u(t_m^+) = u(t_m) + I_m(u_{t_m}), & m \in \mathbb{N}^+, \end{cases} \tag{1.1}$$

where $A : D(A) \subset E \to 2^E$ is an m-dissipative operator, $F : \mathbb{R}^+ \times E \times \mathcal{C}([-\tau, 0]; E) \to P_{cv}(E)$ is a multivalued function ($P_{cv}(\cdot)$ is defined in the next section), $\varphi \in \mathcal{C}([-\tau, 0]; E)$, $u_t \in \mathcal{C}([-\tau, 0]; E)$ is defined by $u_t(s) = u(t + s)$ ($s \in [-\tau, 0]$) for every $u \in \mathcal{PC}([-\tau, \infty); E)$, $I_m : \mathcal{C}([-\tau, 0]; E) \to E$ are impulse functions, $m \in \mathbb{N}$, $u(t^+) = \lim_{s \to t^+} u(s)$, and the time sequence $(t_m)_{m \in \mathbb{N}}$ is an increasing sequence of given points in $[0, \infty)$ without repeated points.

This paper is organized as follows. Section 2 contains some notations, definitions, and preliminary facts from multivalued analysis, while Sect. 3 describes the concept of a C^0-solution for impulsive evolution inclusions. In Sect. 4.1, we prove that the solution set for inclusions (1.1) is a nonempty compact R_δ-set in a compact interval, when the semigroup is compact. Then we proceed to discussing the R_δ-set on a noncompact interval by the inverse limit method. Section 4.2 deals with the solution set for inclusions (1.1) in a compact interval as a nonempty compact R_δ-set in the case when the semigroup is noncompact. Then we switch onto studying the R_δ-structure of the solution set of (1.1) on a noncompact interval.

2 Preliminaries

In this paper, the topological dual of Banach space E is denoted by E^*. For a multivalued operator $A : D(A) \subset E \to 2^E$ with the domain $D(A)$, we write the range of A as $R(A) = \bigcup_{x \in D(A)} Ax$.

Denote by $L([a, b]; E)$ the Banach space consisting of all Bochner integrable functions from $[a, b]$ to E equipped with the norm $\|u\|_{L([a,b];E)} = \int_a^b |u(t)|\, dt$. Denote by $\mathcal{C}([-\tau, 0]; E)$ the space of piecewise continuous functions $x : [-\tau, 0] \to E$ with a finite number of discontinuity points $\{\hat{t}\}$ such that $\hat{t} \neq 0$ and all values $x(\hat{t}^+) = \lim_{\eta \to 0^+} x(\hat{t} + \eta)$ and $x(\hat{t}^-) = \lim_{\eta \to 0^-} x(\hat{t} + \eta)$ are finite. We equip the space $\mathcal{C}([-\tau, 0]; E)$ with the norm $\|x\|_{\mathcal{C}} = \int_{-\tau}^0 |x(t)|\, dt$. Let $\mathcal{PC}([0, b]; E)$ denote the space of piecewise continuous functions $x : [0, b] \to E$ with a finite number of discontinuity points $\{\hat{t}\}$ and x is continuous from left and has right-hand limits at $\{\hat{t}\}$. Note that the space $\mathcal{PC}([0, b]; E)$ is a Banach space equipped with the norm $\|x\|_{\mathcal{PC}} = \sup\{|x(t)| : t \in [0, b]\}$. It is easy to see that $C([0, b]; E)$ is a closed subspace of it. Denote by $\mathcal{PC}([0,\infty);E)$ the space of piecewise continuous functions $x:[0,\infty)\to E$ with an infinite number of discontinuity points $t_1,t_2,\ldots$ such that $\lim_{m\to\infty} t_m=\infty$. x is continuous from left and has right-hand limits at t_i, $i=1,2,\ldots$.

Let Y and Z be metric spaces. $P(Y)$ stands for the collection of all nonempty subsets of Y. As usual, we define $P_{\mathrm{cl}}(Y)=\{\Omega\in P(Y),\Omega \text{ is closed}\}$, $P_{\mathrm{cp}}(Y)=\{\Omega\in P_{\mathrm{cl}}(Y),\Omega \text{ is compact}\}$, $P_{\mathrm{cv}}(Y)=\{\Omega\in P_{\mathrm{cl}}(Y),\Omega \text{ is convex}\}$, $P_{\mathrm{cp,cv}}(Y)=\{\Omega\in P_{\mathrm{cl}}(Y),\Omega \text{ is compact and convex}\}$, $\mathrm{co}(\Omega)$ (resp., $\overline{\mathrm{co}}(\Omega)$) be the convex hull (resp., convex closed hull in Ω) of the subset Ω.

For the multimap $\mathcal{F}:Y\to P(Z)$, we denote the graph of $\mathcal{F}$ as $\mathrm{Gra}(\mathcal{F})$. If Ω is a subset of Z, then $\mathcal{F}^{-1}(\Omega)=\{y\in Y:\mathcal{F}(y)\cap\Omega\neq\emptyset\}$ defines the complete preimage of Ω under $\mathcal{F}$. We call $\mathcal{F}$ to be closed if $\mathrm{Gra}(\mathcal{F})$ is closed in $Y\times Z$; quasi-compact if $\mathcal{F}(\Omega)$ is relatively compact for any compact subset $\Omega\subset Y$; upper semi-continuous (u.s.c.) if $\mathcal{F}^{-1}(\Omega)$ is closed for any closed subset $\Omega\subset Z$; and weakly upper semi-continuous (weakly u.s.c.) if $\mathcal{F}^{-1}(\Omega)$ is closed for any weakly closed subset $\Omega\subset Z$.

Theorem 2.1 ([27, p. 278]) *Let (X,Σ) be a measure space and E be a separable Banach space. Then a function $f:X\to E$ is measurable if and only if, for every $x'\in E^*$, the function $x'\circ f:X\to\mathbb{R}$ is measurable with respect to Σ and the Borel σ-algebra in $\mathbb{R}$.*

Lemma 2.1 ([17]) *Let E be reflexive. A subset $K\subset L([0,b];E)$ is weakly relatively sequentially compact if and only if it is uniformly integrable.*

Lemma 2.2 ([25]) *Let Y and Z be metric spaces and $\mathcal{F}:Y\to P_{\mathrm{cp}}(Z)$ be a closed quasi-compact multimap. Then $\mathcal{F}$ is u.s.c.*

Lemma 2.3 ([11]) *Let $\mathcal{F}:D\subset Y\to P(Z)$ be a multimap with weakly compact and convex values. Then $\mathcal{F}$ is weakly u.s.c. if and only if $\{x_n\}\subset D$ with $x_n\to x_0\in D$ and $y_n\in\mathcal{F}(x_n)$ implies $y_n\rightharpoonup y_0\in\mathcal{F}(x_0)$, up to a subsequence.*

Lemma 2.4 ([23]) *Let $\mathcal{F}:Y\to P_{\mathrm{cp}}(Z)$ be an u.s.c. multimap. If $D\subset Y$ is a compact set, then its image $\mathcal{F}(D)$ is a compact subset of Z.*

Recall that X is an absolute retract (AR) space if, for each metric space Y and each $\Omega\subset P_{\mathrm{cl}}(Y)$, there exists a continuous function $h:\Omega\to X$, which can be extended to a continuous function $\widetilde{h}:Y\to X$. X is an absolute neighborhood retract (ANR) space if, for each metric space Y, each $\Omega\subset P_{\mathrm{cl}}(Y)$, and a continuous function $h:\Omega\to X$, there exist a neighborhood $U\supset\Omega$ and a continuous extension $\widetilde{h}:U\to X$ of h.

Definition 2.1 A nonempty subset Ω of a metric space is said to be contractible if there exist a point $y_0\in\Omega$ and a continuous function $h:[0,1]\times\Omega\to\Omega$ such that $h(0,y)=y_0$ and $h(1,y)=y$ for every $y\in\Omega$.

Definition 2.2 A subset Ω of a metric space is called an R_δ-set if there exists a decreasing sequence $\{\Omega_n\}$ of compact and contractible sets such that

$$\Omega=\bigcap_{n=1}^{\infty}\Omega_n.$$

Define the Hausdorff measure of noncompactness β on each bounded subset Ω of X by

$$\beta(\Omega) = \inf\left\{r > 0 : \Omega \subset \bigcup_{i=1}^{n} B_r(x_i) \text{ where } x_i \in \Omega\right\},$$

where $B_r(x_i)$ is a ball of radius $\leq r$ centered at x_j, $j = 1, 2, \ldots, m$. It is easy to see that the Hausdorff measure of noncompactness is monotone, nonsingular, and regular.

The following β-estimate, which is similar to that of [25, Theorem 4.2.3], will be used in the sequel.

Lemma 2.5 *Assume that E is a separable Banach space. Let $\mathcal{F} : [0, b] \to P(E)$ be an L^p ($p \geq 1$)-integrable bounded multifunction such that*

$$\beta\bigl(\mathcal{F}(t)\bigr) \leq \zeta(t)$$

for a.e. $t \in [0, b]$, where $\zeta(t) \in L^p([0, b]; \mathbb{R}^+)$. Then

$$\beta\left(\int_0^t \mathcal{F}(\tau)\, d\tau\right) \leq \int_0^t \zeta(\tau)\, d\tau$$

for all $t \in [0, b]$. In particular, if the multifunction $\mathcal{F} : [0, b] \to P_{\mathrm{cp}}(E)$ is measurable and L^p-integrably bounded, then the function $\beta(\mathcal{F}(\cdot))$ is integrable and, moreover,

$$\beta\left(\int_0^t \mathcal{F}(\tau)\, d\tau\right) \leq \int_0^t \beta\bigl(\mathcal{F}(\tau)\bigr)\, d\tau$$

for all $t \in [0, b]$.

Now we state the classical Gronwall inequality, which can be found in [16].

Lemma 2.6 *If*

$$u(t) \leq h(t) + \int_{t_0}^t k(s)u(s)\, ds, \quad t \in [t_0, T),$$

where all the functions involved are continuous on $[t_0, T)$, $T \leq \infty$, and $k(t) \geq 0$, then $x(t)$ satisfies

$$u(t) \leq h(t) + \int_{t_0}^t h(s)k(s) \exp\left(\int_s^t k(\theta)\, d\theta\right) u(s)\, ds, \quad t \in [t_0, T).$$

If, in addition, $h(t)$ is nondecreasing, then

$$u(t) \leq h(t) \exp\left(\int_{t_0}^t k(s)\, ds\right), \quad t \in [t_0, T).$$

Theorem 2.2 ([11]) *Let E be a complete metric space, and $\emptyset \neq \Omega \subset E$. Then the following results are equivalent*:

(i) *Ω is an R_δ-set;*

(ii) *Ω is an intersection of a decreasing sequence $\{\Omega_n\}$ of closed contractible spaces with $\beta(\Omega_n) \to 0$;*

(iii) *Ω is compact and absolutely neighborhood contractible, i.e., Ω is contractible in each neighborhood in $Y \in ANR$.*

Definition 2.3 A multimap $\mathcal{F} : X \to P_{\mathrm{cp}}(E)$ is said to be condensing with respect to an MNC β (β-condensing) if, for every bounded set $\Omega \subset E$ that is not relatively compact, we have

$$\beta\bigl(\mathcal{F}(\Omega)\bigr) \ngeq \beta(\Omega).$$

Let $\Omega \subset P_{\mathrm{cv}}(E)$, $U \subset \Omega$ be a bounded (relatively) open set, β be a monotone nonsingular *MNC* in E, and $\mathcal{F} : \bar{U}_\Omega \to P_{\mathrm{cv}}(\Omega)$ be an u.s.c. β-condensing multimap such that $x \notin \mathcal{F}(x)$ for all $x \in \partial_\Omega U$, where $\bar{U}_\Omega$ and $\partial_\Omega U$ denote the relative closure and the relative boundary of the set U.

In the proof of the subsequent results, we shall also use the following fixed point theorem for a multimap.

Theorem 2.3 ([15, Theorem 2.2]) *Let E be a Banach space and $\Omega \subset E$ be a nonempty compact convex subset. If the multimap $\mathcal{F} : \Omega \to P_{\mathrm{cp}}(\Omega)$ is u.s.c. with contractible values, then $\mathcal{F}$ has a fixed point.*

3 Evolution inclusions governed by *m*-dissipative operators

Let $x, y \in E$ and $h \in \mathbb{R} \setminus \{0\}$. Set

$$[x, y]_h = \frac{|x + hy| - |x|}{h}.$$

Notice that the limit of $[x, y]_h$ exists for $h \to 0$. Then we write

$$[x, y]_+ = \lim_{h \to 0^+ [x,y]_h}$$

and

$$[x, y]_- = \lim_{h \to 0^- [x,y]_h}.$$

For any $x, y, z \in E$ and $\lambda > 0$, $[x, y]_+$ and $[x, y]_-$ satisfy the following properties:

(i) $[\lambda x, y]_+ = [x, y]_+$;

(ii) $|[x, y]_+| \le |y|$;

(iii) $[x, y]_+ = -[x, -y]_- = -[-x, y]_-$;

(iv) $[x, y + z]_+ \le [x, y]_+ + [x, z]_+$ and $[x, y + z]_- \ge [x, y]_- + [x, z]_-$.

Definition 3.1 $A : D(A) \subset E \to E$ is m-dissipative if $R(I - \lambda A) = E$ for all $\lambda > 0$ and A is dissipative, that is,

$$[x_1 - x_2, y_2 - y_1]_+ \ge 0 \quad \text{for all } (x_j, y_j) \in G(A), j = 1, 2.$$

Consider the following impulsive functional differential inclusions:

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in [0,T], T \in (t_{N-1}, t_N), t \neq t_i, i = 1,2,\ldots,N-1, \\ u(t) = \varphi(t), & t \in [-\tau, 0], \\ u(t_m^+) = u(t_m) + I_m(u_{t_m}), & m = 1,\ldots,N-1. \end{cases} \tag{3.1}$$

Let us refer to (3.1) by (IFDI;φ,f).

Since A is dissipative, we have $[u(t) - x, y - Au(t)]_+ \geq 0$ for $x \in D(A)$, $y \in Ax$. Then

$$\left[u(t) - x, y - Au(t)\right]_+ = \left[u(t) - x, y - u'(t) + f(t)\right]_+ \geq 0.$$

So

$$\left[u(t) - x, u'(t)\right]_+ \leq \left[u(t) - x, y + f(t)\right]_+.$$

By the property of $[\cdot,\cdot]_+$,

$$\frac{d}{dt}\left|u(t) - x\right| \leq \left[u(t) - x, y + f(t)\right]_+.$$

When $t_{i-1} \leq s \leq t \leq t_i$, integrating over s, t, we have

$$\left|u(t) - x\right| - \left|u(s) - x\right| \leq \int_s^t \left[u(\tau) - x, y + f(\tau)\right]_+ d\tau.$$

When $t_{j-1} \leq s < t \leq t_{j+N}$ ($N \in \mathbb{N}^+$), integrating over s, t, we have

$$\int_s^t \frac{d}{d\tau}\left|u(\tau) - x\right| d\tau \leq \int_s^t \left[u(\tau) - x, y + f(\tau)\right]_+ d\tau.$$

Then

$$\begin{aligned} &\left|yu(t) - x\right| - \left|u\left(t_{j+N-1}^+\right) - x\right| + \left|u(t_{j+N-2}) - x\right| - \left|u\left(t_{i-2}^+\right) - x\right| \\ &\quad + \cdots + \left|u(t_j) - x\right| - \left|u(s) - x\right| \\ &\leq \int_s^t \left[u(\tau) - x, y + f(\tau)\right]_+ d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} \left|u(t) - x\right| - \left|u(s) - x\right| &\leq \int_s^t \left[u(\tau) - x, y + f(\tau)\right]_+ d\tau + \sum_{i=j}^{j+N-1} \left(\left|u\left(t_i^+\right) - x\right| - \left|u(t_i) - x\right|\right) \\ &\leq \int_s^t \left[u(\tau) - x, y + f(\tau)\right]_+ d\tau + \sum_{s<t_i<t} \left(\left|u\left(t_i^+\right) - x\right| - \left|u(t_i) - x\right|\right) \\ &\leq \int_s^t \left[u(\tau) - x, y + f(\tau)\right]_+ d\tau + \sum_{s<t_i<t} \left|I_i(u_{t_i})\right|. \end{aligned}$$

Let $\Delta_j = \{t_{j,i}\}_{i=1}^{k_j}$, $j = 0, 1, 2, \ldots, N$, be partitions of the interval J_j, $j = 0, 1, \ldots, N$, where

$$J_0 = [0, t_1], \qquad J_j = (t_j, t_j + 1], \quad j = 1, \ldots, N-1, \qquad J_N = [t_N, T]$$

and

$$\begin{aligned}
&0 = t_{0,0} < t_{0,1} < \cdots < t_{0,k_0-1} < t_{0,k_0} = t_1, \\
&t_j < t_{j,0} < t_{j,1} < \cdots < t_{j,k_j-1} < t_j^{(k_j)} = t_{j+1}, \quad j = 1, \ldots, N-1, \\
&t_N < t_{N,0} < t_{N,1} < \cdots < t_{N,k_N-1} < t_{N,k_N} = T.
\end{aligned}$$

Let $u_{j,i} \in E$ and $f_{j,i} \in E$, $j = 0, 1, 2, \ldots, N$, $i = 0, 1, 2, \ldots, n_j$, satisfy the difference inclusion

$$\begin{cases} \frac{x_{j,i}-x_{j,i-1}}{t_{j,i}-t_{j,i-1}} \in Ax_{j,i} + f_{j,i}, \\ x_{j,0} = x_{j-1,k_{j-1}} + I(t_{j-1,k_{j-1}}), \end{cases}$$

$j = 0, 1, 2, \ldots, N$, $i = 1, 2, \ldots, k_j$. Let $\lambda > 0$ and suppose further that the following conditions hold:

(i) $\max_{0\le j\le N} \max_{1\le i\le k_j} (t_{j,i} - t_{j,i-1}) \le \lambda$;
(ii) $\|\varphi_0 - \varphi\| \le \lambda$;
(iii) $\sum_{j=0}^{N} \sum_{i=1}^{k_j} \int_{t_{j,i-1}}^{t_{j,i}} |f_{j,i} - f(t)|\, dt \le \lambda$.

Define a function $u_\lambda : [0, T] \to E$ as follows:

$$u_\lambda(t) = \begin{cases} \varphi, & t \in [-\tau, 0], \\ x_{j,i}, & t \in (t_{j,i-1}, t_{j,i}) \cap (t_j, t_{j+1}), j = 1, \ldots, N, i = 1, \ldots, k_j, \end{cases}$$

and call it a λ-approximate solution of problem (IFDI;φ,f) on the interval $[-\tau, T]$.

Definition 3.2 Let A be m-dissipative. If there exists a sequence $\lambda_n > 0$ and a λ_n-approximate solution such that $\lambda_n \to 0$ and $u_n(t) \to u(t)$ uniformly on $[0, T]$ as $n \to \infty$, then u is said to be a limit solution of (3.1).

In the following, we seek the definition of C^0-solution of (3.1) on $[0, T]$. To this aim, let us observe that: whenever u is a strong solution of (3.1) on $[0, T]$, we have

$$\begin{cases} u'(t) - f(t) \in Au(t), & t \in [0, T], T \in (t_{N-1}, t_N), t \neq t_i, i = 1, 2, \ldots, N-1, \\ u(t) = \varphi(t), & t \in [-\tau, 0], \\ u(t_m^+) = u(t_m) + I_m(u_{t_m}), & m = 1, \ldots, N-1. \end{cases}$$

Since A is dissipative, therefore

$$\bigl[u(\tau) - x, u'(\tau) - f(\tau) - y\bigr]_+ \le 0$$

for each $x \in D(A)$, $y \in Ax$ and for $\tau \neq t_i$, $i = 1, 2, \ldots, N-1$. By the property of $[\cdot,\cdot]_+$, we have

$$\bigl[u(\tau) - x, u'(\tau)\bigr]_- \le \bigl[u(\tau) - x, f(\tau) + y\bigr]_+$$

for each $x \in D(A)$, $y \in Ax$ and for $\tau \neq t_i$, $i = 1,2,\ldots,N-1$. Obviously, $\tau \to |u(\tau) - x|$ is piecewise absolutely continuous on $[0,T]$, and hence it is almost everywhere differentiable on $[0,T]$. Then, for each $x \in D(A)$, $y \in Ax$ and for $\tau \neq t_i$, $i = 1,2,\ldots,N-1$, we have

$$\bigl[u(\tau) - x, u'(\tau)\bigr]_- = \frac{d^-}{d\tau}\bigl(\bigl|u(\tau) - x\bigr|\bigr) = \frac{d}{d\tau}\bigl(\bigl|u(\tau) - x\bigr|\bigr).$$

Integrating both sides of the above inequality over $[s,t] \subset [0,T]$, we get

$$\begin{aligned} \bigl|u(t) - x\bigr| &\leq \bigl|u(s) - x\bigr| + \int_s^t \bigl[u(\sigma) - x, f(\sigma) + y\bigr]_+ \, d\sigma \\ &\quad + \sum_{s<t_m<t} \bigl(\bigl|u(t_m^+) - x\bigr| - \bigl|u(t_m) - x\bigr|\bigr). \end{aligned} \tag{3.2}$$

Definition 3.3 Let A be m-dissipative. By a C^0-solution of (3.1) on $[0,T]$, we mean that an element $u \in \mathcal{PC}([0,T];E)$, $u(t) = \varphi(t)$, $t \in [-\tau, 0]$, $\varphi(0) \in D(A)$, $u(t) \in \overline{D(A)}$ for each $t \in [0,T]$, and satisfies (3.2) for any $(x,y) \in G(A)$ and $0 \leq s \leq t \leq T$.

Remark 3.1 In particular, a C^0-solution also satisfies the inequality

$$\bigl|u(t) - x\bigr| \leq \bigl|u(s) - x\bigr| + \int_s^t \bigl[u(\sigma) - x, f(\sigma) + y\bigr]_+ \, d\sigma + \sum_{s<t_m<t} \bigl|I_m(u_{t_m})\bigr| \tag{3.3}$$

for any $(x,y) \in G(A)$ and $0 \leq s \leq t \leq T$.

Now we are in a position to present an important theorem which relates a limit solution and a C^0-solution.

Theorem 3.1 *Let A be dissipative. Let $f \in L([0,T];E)$, $u(t)$ be a limit solution of* (IFDI; φ, f), *and $v(t)$ be a C^0-solution of* (IFDI; ϕ, g). *Then*

$$\begin{aligned} \bigl|u(t) - v(t)\bigr| &\leq \bigl|u(s) - v(s)\bigr| + \int_s^t \bigl[u(\sigma) - v(\sigma), f(\sigma) - g(\sigma)\bigr]_+ \, d\sigma \\ &\quad + \sum_{s<t_m<t} \bigl|I_m(u_{t_m}) - I_m(v_{t_m})\bigr| \end{aligned} \tag{3.4}$$

for any $s,t \in [0,T]$ with $s \leq t$.

Proof Let $u(t)$ be a limit solution of (IFDI; φ, f) and $v(t)$ be a C^0-solution of (IFDI; ϕ, g).

By Definition 3.2, there exist a sequence $\lambda_n > 0$ and a λ_n-approximate solution such that $\lambda_n \to 0$ and $u_n(t) \to u(t)$ uniformly on $[0,T]$ as $n \to \infty$. Let $u_n(t)$ satisfy the difference equation

$$\frac{x_{j,i}^n - x_{j,i-1}^n}{t_{j,i}^n - t_{j,i-1}^n} \in Ax_{j,i}^n + f_{j,i}^n, \quad \text{for } t_{j,i}^n \in (t_j, t_{j+1}], j = 0,1,2,\ldots,N, i = 1,2,\ldots,k_j.$$

We set $\eta_{j,i}^n = t_{j,i}^n - t_{j,i-1}^n$, $j = 0,1,2,\ldots,N$, $i = 1,2,\ldots,k_j$.

Let $0 \le s \le t \le T$. Since $x_{j,i}^n \in D(A)$ and $(\eta_{j,i}^n)^{-1}(x_{j,i}^n - x_{j,i-1}^n) - f_{j,i}^n \in Ax_{j,i}^n$, the definition of C^0-solutions yields

$$\begin{aligned}
&\left|\nu(t) - x_{j,i}^n\right| - \left|\nu(s) - x_{j,i}^n\right| \\
&\quad\le \int_s^t \left[\nu(\sigma) - x_{j,i}^n, g(\sigma) + \left(\eta_{j,i}^n\right)^{-1}\left(x_{j,i}^n - x_{j,i-1}^n\right) - f_{j,i}^n\right]_+ d\sigma \\
&\qquad + \sum_{s<t_m<t}\left(\left|\nu\left(t_m^+\right) - x_{j,i}^n\right| - \left|\nu(t_m) - x_{j,i}^n\right|\right) \\
&\quad\le \int_s^t \left[\nu(\sigma) - x_{j,i}^n, g(\sigma) - f_{j,i}^n\right]_+ d\sigma + \int_s^t \left[\nu(\sigma) - x_{j,i}^n, \left(\eta_{j,i}^n\right)^{-1}\left(x_{j,i}^n - x_{j,i-1}^n\right)\right]_+ d\sigma \\
&\qquad + \sum_{s<t_m<t}\left(\left|\nu\left(t_m^+\right) - x_{j,i}^n\right| - \left|\nu(t_m) - x_{j,i}^n\right|\right).
\end{aligned}$$

By the property of $[\cdot,\cdot]_+$, we have

$$\begin{aligned}
&\left[\nu(\sigma) - x_{j,i}^n, \left(\eta_{j,i}^n\right)^{-1}\left(x_{j,i}^n - x_{j,i-1}^n\right)\right]_+ \\
&\quad\le \left(\eta_{j,i}^n\right)^{-1}\left(\left|\nu(\sigma) - x_{j,i}^n + \left(x_{j,i}^n - x_{j,i-1}^n\right)\right| - \left|\nu(\sigma) - x_{j,i}^n\right|\right) \\
&\quad= \left(\eta_{j,i}^n\right)^{-1}\left(\left|\nu(\sigma) - x_{j,i-1}^n\right| - \left|\nu(\sigma) - x_{j,i}^n\right|\right).
\end{aligned}$$

In consequence, we obtain

$$\begin{aligned}
&\eta_{j,i}^n\left(\left|\nu(t) - x_{j,i}^n\right| - \left|\nu(s) - x_{j,i}^n\right|\right) + \int_s^t \left(\left|\nu(\sigma) - x_{j,i}^n\right| - \left|\nu(\sigma) - x_{j,i-1}^n\right|\right) d\sigma \\
&\quad\le \eta_{j,i}^n \int_s^t \left[\nu(\sigma) - x_{j,i}^n, g(\sigma) - f_{j,i}^n\right]_+ d\sigma + \eta_{j,i}^n \sum_{s<t_m<t}\left(\left|\nu\left(t_m^+\right) - x_{j,i}^n\right| - \left|\nu(t_m) - x_{j,i}^n\right|\right).
\end{aligned}$$

Adding the above inequalities for $j = \kappa, \kappa+1, \dots, \iota$, $i = 1, 2, \dots, k_j$, we get

$$\begin{aligned}
&\sum_{j=\kappa}^{\iota}\sum_{i=1}^{k_j} \eta_{j,i}^n\left(\left|\nu(t) - x_{j,i}^n\right| - \left|\nu(s) - x_{j,i}^n\right|\right) + \int_s^t \left(\left|\nu(\sigma) - x_{\iota,k_\iota}^n\right| - \left|\nu(\sigma) - x_{\kappa,0}^n\right|\right) d\sigma \\
&\qquad - \sum_{t_{\kappa,0}^n<t_m<t_{\iota,k_\iota}^n} \int_s^t \left(\left|\nu(\sigma) - u\left(t_m^+\right)\right| - \left|\nu(\sigma) - u(t_m)\right|\right) d\sigma \\
&\quad\le \sum_{j=\kappa+1}^{\iota}\sum_{i=1}^{k_j} \eta_{j,i}^n \int_s^t \left[\nu(\sigma) - x_{j,i}^n, g(\sigma) - f_{j,i}^n\right]_+ d\sigma \\
&\qquad + \sum_{j=\kappa+1}^{\iota}\sum_{i=1}^{k_j} \eta_{j,i}^n \sum_{s<t_m<t}\left(\left|\nu\left(t_m^+\right) - x_{j,i}^n\right| - \left|\nu(t_m) - x_{j,i}^n\right|\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{t_{\kappa,0}^n}^{t_{\iota,k_\iota}^n} \left(\left|\nu(t) - u_n(\tau)\right| - \left|\nu(s) - u_n(\tau)\right|\right) d\tau \\
&\qquad + \int_s^t \left(\left|\nu(\sigma) - u_n\left(t_i^n\right)\right| - \left|\nu(\sigma) - u_n\left(t_j^n\right)\right|\right) d\sigma
\end{aligned}$$

$$- \sum_{t^n_{\kappa,0}<t_m<t^n_{\iota,k_\iota}} \int_s^t \left(\left|v(\sigma) - u\left(t_m^+\right)\right| - \left|v(\sigma) - u(t_m)\right|\right) d\sigma$$
$$\le \int_{t^n_{\kappa,0}}^{t^n_{\iota,k_\iota}} \int_s^t \left[v(\sigma) - u_n(\tau), g(\sigma) - f_n(\tau)\right]_+ d\sigma\, d\tau$$
$$+ \int_{t^n_{\kappa,0}}^{t^n_{\iota,k_\iota}} \sum_{t^n_{\kappa,0}<t_m<t^n_{\iota,k_\iota}} \left(\left|v\left(t_m^+\right) - u_n(\tau)\right| - \left|v(t_m) - u_n(\tau)\right|\right) d\tau,$$

where $f_n(\cdot)$ is an integrable function defined almost everywhere on $(0,T)$ by setting

$$f_n(\tau) = f^n_{j,i} \quad \text{for } \tau \in \left(t^n_{j,i}, t^n_{j,i+1}\right) \text{ and } j = 0,1,2,\ldots,N, i = 0,1,2,\ldots,k_j - 1.$$

Considering the first part of the right-hand side of the above inequality, we obtain

$$\int_{t^n_{\kappa,0}}^{t^n_{\iota,k_\iota}} \int_s^t \left[v(\sigma) - u_n(\tau), g(\sigma) - f_n(\tau)\right]_+ d\sigma\, d\tau$$
$$\le \int_{t^n_{\kappa,0}}^{t^n_{\iota,k_\iota}} \int_s^t \left[v(\sigma) - u_n(\tau), g(\sigma) - f(\tau)\right]_+ d\sigma\, d\tau + \int_{t^n_{\kappa,0}}^{t^n_{\iota,k_\iota}} \int_s^t \left|f_n(\tau) - f(\tau)\right| d\sigma\, d\tau$$
$$\le \int_{t^n_{\kappa,0}}^{t^n_{\iota,k_\iota}} \int_s^t \left[v(\sigma) - u_n(\tau), g(\sigma) - f(\tau)\right]_+ d\sigma\, d\tau + T \int_0^T \left|f_n(\tau) - f(\tau)\right| d\tau.$$

We suppose that $t^n_{\kappa,0} \to \alpha$ and $t^n_{\iota,k_\iota} \to \beta$ as $n \to \infty$. Then, in the limit $n \to \infty$, we have

$$\int_\beta^\alpha \left(\left|v(t) - u_n(\tau)\right| - \left|v(s) - u_n(\tau)\right|\right) d\tau + \int_s^t \left(\left|v(\sigma) - u_n(\alpha)\right| - \left|v(\sigma) - u_n(\beta)\right|\right) d\sigma$$
$$- \sum_{\beta<t_m<\alpha} \int_s^t \left(\left|v(\sigma) - u\left(t_m^+\right)\right| - \left|v(\sigma) - u(t_m)\right|\right) d\sigma$$
$$\le \int_\beta^\alpha \int_s^t \left[v(\sigma) - u_n(\tau), g(\sigma) - f(\tau)\right]_+ d\sigma\, d\tau$$
$$+ \int_\beta^\alpha \sum_{s<t_m<t} \left(\left|v\left(t_m^+\right) - u_n(\tau)\right| - \left|v(t_m) - u_n(\tau)\right|\right) d\tau,$$

where we have used the Lebesgue dominated convergence theorem and the u.s.c. of the functional $[\cdot,\cdot]_+$.

Define

$$\Phi(\sigma,\tau) = \left|v(\sigma) - u(\tau)\right|, \qquad \Upsilon(\sigma,\tau) = \left[v(\sigma) - u(\tau), g(\sigma) - f(\tau)\right]_+.$$

We now consider the regularizations Φ_n and Υ_n given by

$$\Phi_n(t,s) = \int_0^T \int_0^T \rho_n(t-\sigma, s-\tau)\Phi(\sigma,\tau)\, d\sigma\, d\tau,$$
$$\Upsilon_n(t,s) = \int_0^T \int_0^T \rho_n(t-\sigma, s-\tau)\Upsilon(\sigma,\tau)\, d\sigma\, d\tau,$$

where $\rho(t,s) = n^2\rho(nt)\rho(ns)$, $\rho \in D(\mathbb{R})$, $\rho \geq 0$, $\operatorname{supp}\rho \in [-1,+1]$, $\int \rho(\xi)\,d\xi = 1$, and $\rho(t) = \rho(-t)$ for all t.

Let $\frac{1}{n} \leq \beta \leq \alpha \leq T$ and $\frac{1}{n} \leq s \leq t \leq T$. Then the above inequality implies that

$$\begin{aligned}
&\int_\beta^\alpha \bigl(\Phi_n(t,\tau) - \Phi_n(s,\tau)\bigr)\,d\tau + \int_s^t \bigl(\Phi_n(\sigma,\alpha) - \Phi_n(\sigma,\beta)\bigr)\,d\sigma \\
&\qquad - \sum_{\beta<t_m<\alpha} \int_s^t \bigl(\Phi_n\bigl(\sigma,t_m^+\bigr) - \Phi_n(\sigma,t_m)\bigr)\,d\sigma \\
&\quad \leq \int_\beta^\alpha d\tau \int_s^t \Upsilon_n(\sigma,\tau)\,d\sigma\,d\tau + \int_\beta^\alpha \sum_{s<t_m<t} \bigl(\Phi_n\bigl(t_m^+,\tau\bigr) - \Phi_n(t_m,\tau)\bigr)\,d\tau.
\end{aligned}$$

As

$$\begin{aligned}
&\Phi_n(t,\tau) - \Phi_n(s,\tau) = \int_s^t \frac{\partial \Phi_n}{\partial\sigma}(\sigma,\tau)\,d\sigma + \sum_{s<t_m<t} \bigl(\Phi_n\bigl(t_m^+,\tau\bigr) - \Phi_n(t_m,\tau)\bigr), \\
&\Phi_n(\sigma,\alpha) - \Phi_n(\sigma,\beta) = \int_\beta^\alpha \frac{\partial \Phi_n}{\partial\tau}(\sigma,\tau)\,d\tau + \sum_{\beta<t_m<\alpha} \bigl(\Phi_n\bigl(\sigma,t_m^+\bigr) - \Phi_n(\sigma,t_m)\bigr),
\end{aligned}$$

we have

$$\frac{\partial \Phi_n}{\partial\sigma}(\sigma,\tau) + \frac{\partial \Phi_n}{\partial\tau}(\sigma,\tau) \leq \Upsilon_n(\sigma,\tau), \quad \text{for } \frac{1}{n} \leq \sigma \leq T, \frac{1}{n} \leq \tau \leq T,$$

and

$$\frac{d}{dt}\Phi_n(t,t) \leq \Upsilon_n(t,t).$$

Integrating over s, t, we get

$$\Phi_n(t,t) - \Phi_n(s,s) \leq \int_s^t \Upsilon_n(\sigma,\sigma)\,d\sigma + \sum_{s<t_m<t} \bigl(\Phi_n\bigl(t_m^+,t_m^+\bigr) - \Phi_n(t_m,t_m)\bigr), \quad \frac{1}{n} \leq s \leq t \leq T,$$

which, in the limit $n \to \infty$, yields

$$\Phi(t,t) - \Phi(s,s) \leq \int_s^t \Upsilon(\sigma,\sigma)\,d\sigma + \sum_{s<t_m<t} \bigl(\Phi\bigl(t_m^+,t_m^+\bigr) - \Phi(t_m,t_m)\bigr), \quad 0 \leq s \leq t \leq T.$$

Consequently, we get

$$\begin{aligned}
\bigl|v(t) - u(t)\bigr| &\leq \bigl|v(s) - u(s)\bigr| + \int_s^t \bigl[v(\sigma) - u(\sigma), g(\sigma) - f(\sigma)\bigr]_+\,d\sigma \\
&\quad + \sum_{s<t_m<t} \bigl(\bigl|v\bigl(t_m^+\bigr) - u\bigl(t_m^+\bigr)\bigr| - \bigl|v(t_m) - u(t_m)\bigr|\bigr) \\
&\leq \bigl|v(s) - u(s)\bigr| + \int_s^t \bigl[v(\sigma) - u(\sigma), g(\sigma) - f(\sigma)\bigr]_+\,d\sigma \\
&\quad + \sum_{s<t_m<t} \bigl|\bigl(v\bigl(t_m^+\bigr) - v(t_m)\bigr) - \bigl(u\bigl(t_m^+\bigr) - u(t_m)\bigr)\bigr|
\end{aligned}$$

$$= \left|v(s) - u(s)\right| + \int_s^t \left[v(\sigma) - u(\sigma), g(\sigma) - f(\sigma)\right]_+ d\sigma$$
$$+ \sum_{s<t_m<t} \left|I_m(u_{t_m}) - I_m(v_{t_m})\right|. \qquad \square$$

In view of Theorem 3.1, we have the following type of uniqueness result for C^0-solutions.

Corollary 3.1 *Let A be dissipative. Let $f \in L([0,T];E)$, and $u(t)$ be a limit solution of* (IFDI; φ, f) *on* $[0,T]$. *Then $u(t)$ is the unique C^0-solution of* (IFDI; φ, f) *on* $[0,T]$.

Proof Let $v \in D(A)$ and $\tilde{v} \in Av$. We set $v(t) \equiv v$ and $g(t) \equiv -\tilde{v}$ for $t \in [0,T]$. Then we can easily check that $v(t)$ is a C^0-solution of (IFDI; ϕ, g) since A is dissipative.

Therefore inequality (3.4) with $u(t)$ and $v(t)$ implies that

$$\begin{aligned} \left|u(t) - v\right| - \left|u(s) - v\right| &\le \int_s^t \left[v - u(\sigma), -\tilde{v} - f(\sigma)\right]_+ d\sigma + \sum_{s<t_m<t} \left|I_m(u_{t_m}) - I_m(v_{t_m})\right| \\ &= \int_s^t \left[u(\sigma) - v, f(\sigma) + \tilde{v}\right]_+ d\sigma + \sum_{s<t_m<t} \left|I_m(u_{t_m}) - I_m(v_{t_m})\right| \end{aligned}$$

for $0 \le s \le t \le T$. Thus $u(t)$ is an integral solution of (IFDI; u_0, f).

Let $v(t)$ be a C_0-solution of (IFDI; u_0, f) on $[0,T]$. Then, by (3.4), we have

$$\begin{aligned} \left|u(t) - v(t)\right| &\le \left|u(0) - v(0)\right| + \sum_{0<t_m<t} \left|I_m(u_{t_m}) - I_m(v_{t_m})\right| \\ &= \sum_{0<t_m<t} \left|I_m(u_{t_m}) - I_m(v_{t_m})\right|. \end{aligned}$$

When $0 \le t \le t_1$, it is easy to deduce that $|u(t) - v(t)| = 0$, that is, $u(t) = v(t)$; when $t_1 \le t \le t_2$, we also obtain $u(t) = v(t)$. In view of the foregoing arguments, it follows that $u(t) = v(t)$ for $t \in [0,T]$. This completes the proof of the corollary. $\square$

Let $x \in E_0 := \overline{D(A)}$, $c \in [a,b)$, and $f \in L([a,b];E)$. We denote by $u(\cdot, c, x, f)$ the unique C^0-solution $u : [c,b] \to E_0$ of the equation

$$u'(t) \in Au(t) + f(t)$$

on $[c,b]$ with $u(c) = x$. We also define the unique C^0-solution $u : [-\tau, T] \to E_0$ of (3.1) by $u(\cdot, [0,T], \varphi, f)$ which satisfies $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$. Let

$$S : E_0 \to E_0 \quad \text{with } S(t)x = u(t, 0, x, 0) \text{ for each } t \ge 0, x \in E_0.$$

Then $\{S(t)\}_{t\ge0}$ is a contraction semigroup on E_0 (see Barbu [9]) and is generated by A.

The semigroup $\{S(t)\}_{t\ge0}$ is called compact if $S(t)$ is a compact operator for each $t > 0$.

Definition 3.4 An m-dissipative operator $A : D(A) \subset E \to P(E)$ is called of compact type if for each $a < b$ and each sequence $\{f_n, u_n\}$ in $L([a,b];E) \times C([a,b];E)$ such that u_n is a C^0-solution on $[a,b]$ of the evolution inclusion

$$u_n'(t) \in Au_n(t) + f_n(t), \quad n = 1, 2, \ldots,$$

$f_n \rightharpoonup f$ in $L([a,b];E)$ and $u_n \to u$ in $C([a,b];E)$, then u is a C^0-solution on $[a,b]$ of the limit problem

$$u'(t) \in Au(t) + f(t), \quad n = 1,2,\ldots.$$

Lemma 3.1 (see [31, Corollary 2.3.1]) *Let X^* be uniformly convex and A be an m-dissipative operator generating a compact semigroup. Then A is of compact type.*

The following compactness criterion is an immediate consequence of Theorem 2.3.3 of [31].

Lemma 3.2 *Let A be an m-dissipative operator generating a compact semigroup. In addition, it is assumed that B is a bounded set in E_0 and $\mathscr{F}$ is uniformly integrable in $L([t_{m-1},t_m];E)$. Then, for each $c \in [t_{m-1},t_m]$, the C^0-solution set*

$$\{u(\cdot,t_{m-1},x,f) : x \in B, f \in \mathscr{F}\}$$

is relatively compact in $C([c,t_m];E)$. Moreover, if B is relatively compact, then the C^0-solution set is relatively compact in $C([t_{m-1},t_m];E)$.

Next, for each $\xi \in \mathcal{C}([-\tau,t_{m-1}];E_0)$ and $f \in L([t_{m-1},t_m];E)$, we define the mapping $S_\xi^m : L([t_{m-1},t_m];E) \to \mathcal{PC}([-\tau,t_m];E_0)$ by

$$S_\xi^m(f)(t) = \begin{cases} \xi(t), & t \in [-\tau,t_{m-1}], \\ u(t,t_{m-1}^+,\xi(t_{m-1}) + I_{m-1}(\xi_{t_{m-1}}),f), & t \in [t_{m-1},t_m]. \end{cases}$$

Clearly, $S_\xi^m(f)$ is the unique C^0-solution for the evolution inclusion

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in (t_{m-1},t_m], \\ u(t) = \xi(t), & t \in [-\tau,t_{m-1}], \\ u(t_{m-1}^+) = \xi(t_{m-1}) + I_{m-1}(\xi_{t_{m-1}}). \end{cases}$$

The following result is an immediate consequence of Lemmas 3.1 and 3.2.

Lemma 3.3 *Let E^* be uniformly convex and A be an m-dissipative operator generating a compact semigroup. Then the following results hold.*

(i) *If $\mathscr{F}$ is uniformly integrable in $L([t_{m-1},t_m];E)$ and $\mathcal{B} \in \mathcal{C}([-\tau,t_m];E_0)$ is relatively compact, then $S_{\mathcal{B}}^m(\mathscr{F})$ is relatively compact in $\mathcal{PC}([-\tau,t_m];E)$.*

(ii) *For each sequence $\{(f_n,u_n)\}_{n=1}^\infty$ in $L([t_{m-1},t_m];E) \times \mathcal{PC}([-\tau,t_m];E_0)$ such that $u_n = S_\xi^m(f_n)$, $n \geq 1$, $f_n \rightharpoonup f$ and $u_n \to u$, we have that $u = S_\xi^m(f)$.*

Lemma 3.4 ([11]) *Let E be a real Banach space and A be an m-dissipative operator generating an equicontinuous semigroup. Then the following results are valid.*

(i) *If $\mathscr{F}$ is uniformly integrable in $L([t_{m-1},t_m];E)$, then $S_\xi^m(\mathscr{F})$ is equicontinuous in $\mathcal{PC}([-\tau,t_m];E)$.*

(ii) *If X^* is uniformly convex and for each sequence $\{f_n\} \subset L([t_{m-1}, t_m]; E)$ such that $|f_n(t)| \leq \psi(t)$ a.e. on J for all $n \geq 1$ with some $\psi \in L([t_{m-1}, t_m])$, then*

$$\beta\bigl(\bigl\{S_\xi^m(f_n)(s) : n \geq 1\bigr\}\bigr) \leq \int_{t_{m-1}}^t \beta\bigl(\bigl\{f_n(s) : n \geq 1\bigr\}\bigr)\,ds \quad \text{on } [t_{m-1}, t_m]. \tag{3.5}$$

In addition, if $f_n \rightharpoonup f$ in $L([t_{m-1}, t_m]; E)$ and $S_\xi^m(f_n) \to g$ in $\mathcal{PC}([-\tau, t_m]; E)$, then $g = S_\xi^m(f_n)$.

Lemma 3.5 (see [31, Theorem 2.3.3]) *Let E be a real Banach space and A be an m-dissipative operator. If $\mathscr{F}$ is uniformly integrable in $L([t_{m-1}, t_m]; E)$, then the following conditions are equivalent*:

(i) *The set $S_\xi^m(\mathscr{F})$ is relatively compact in $\mathcal{PC}([-\tau, t_m]; E)$.*

(ii) *There exists a dense subset D in $[t_{m-1}, t_m]$ such that, for $\forall t \in D$, the set $S_\xi^m(\{f_n\})(t)$ is relatively compact in E.*

From Lemmas 3.3 and 3.5, we deduce the following statement.

Lemma 3.6 *Let E^* be uniformly convex and A be an m-dissipative operator generating an equicontinuous semigroup. For each sequence $\{(f_n, u_n)\}_{n=1}^\infty$ in $L([t_{m-1}, t_m]; E) \times \mathcal{PC}([-\tau, t_m]; E_0)$, we have that $u_n = S_\xi^m(f_n)$, $n \geq 1$, $f_n \rightharpoonup f$ and $u_n \to u$. If $\{f_n\}$ is uniformly integrable and there exists a dense subset D in $[t_{m-1}, t_m]$ such that, for each $t \in D$, the set $S_\xi^m(\{f_n\})(t)$ is relatively compact in E, then $u = S_\xi^m(f)$.*

4 Topological properties of solution sets

This section is concerned with the study of existence of C^0-solution and R_δ-structure of solution sets for problem (1.1) on compact intervals, and R_δ-structure of solution sets for problem (1.1) on noncompact intervals.

In the sequel, we need the following assumptions.

(A1) $A : D(A) \subset E \to 2^E$ is an m-dissipative operator with $0 \in A0$. In addition, E_0 is convex and E^* is uniformly convex.

(A2) The semigroup $\{S(t)\}_{t\geq 0}$ is a compact semigroup.

(A3) The semigroup $\{S(t)\}_{t\geq 0}$ is immediately norm continuous, i.e., it is norm continuous for $t > 0$.

(F1) $F(t, \cdot, \cdot) : E \times \mathcal{C}([-\tau, 0]; E) \to P_{\mathrm{cv}}(E)$ is weakly u.s.c. for a.e. $t \in \mathbb{R}^+$ and $F(\cdot, x, v) : \mathbb{R}^+ \to P_{\mathrm{cv}}(E)$ has a strongly measurable selection for each $(x, v) \in E_0 \times \mathcal{C}([-\tau, 0]; E_0)$.

(F2) There exists a function $\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{R}^+)$ such that

$$\bigl|F(t, x, v)\bigr| \leq \alpha(t)\bigl(1 + |x| + \|v\|_{\mathcal{C}}\bigr)$$

for a.e. $t \in \mathbb{R}^+$ and each $(x, v) \in E_0 \times \mathcal{C}([-\tau, 0]; E_0)$.

(F3) For each $\epsilon > 0$ and each bounded set $\mathcal{U} \times D \subset E_0 \times \mathcal{C}([-\tau, 0]; E_0)$, there exist $\delta > 0$ and a function $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{R}^+)$ such that

$$\beta\bigl(F\bigl(t, \mathcal{U}, O_\delta(D)\bigr)\bigr) \leq \mu(t)\Bigl(\beta(\mathcal{U}) + \sup_{-\tau\leq\theta\leq 0} \beta\bigl(O_\epsilon\bigl(D(\theta)\bigr)\bigr)\Bigr) \quad \text{for a.e. } t \geq 0,$$

where $O_\delta(D)$ denotes a δ-neighborhood of D defined as
$O_\delta(D) := \{z \in \mathcal{C}([-\tau,0];E) : \operatorname{dist}(z,D) < \delta\}$.

For a fixed $m > 0$, we first consider the inclusion problem

$$\begin{cases} x'(t) \in Ax(t) + f(t), & \text{a.e. } t \in [0,t_m], t \neq t_k, k < m, \\ f(t) \in F(t,x(t),x_t), & \text{a.e. } t \in [0,t_m], t \neq t_k, k < m, \\ x(t) = \varphi(t), & t \in [-\tau,0], \\ x(t_k^+) = x(t_k) + I_k(x_{t_k}), & k < m \in \mathbb{N}^+. \end{cases} \tag{4.1}$$

For $z \in \mathcal{PC}([0,t_m];E_0)$, we can define $z_i \in C([t_i,t_{i+1}];E_0)$, $i = 0,1,\dots,m-1$, with $z_i(t) = z(t)$ on $(t_i,t_{i+1}]$ and $z_i(t_i) = z(t_i^+)$. For every set $K \subset \mathcal{PC}([0,t_m];E_0)$, we denote by K_i, $i = 0,1,\dots,m-1$, the set $K_i = \{z_i : z \in K\}$. The following result is obvious.

Proposition 4.1 *A set $K \in \mathcal{PC}([0,t_m];E_0)$ is relatively compact in $\mathcal{PC}([0,t_m];E_0)$ if and only if each set K_i, $i = 0,1,\dots,m-1$ is relatively compact in $C([t_i,t_{i+1}];E_0)$.*

For any $z \in C([t_{k-1},t_k];E)$, we define the function $z[\xi] : [-\tau,t_k] \to E$ as

$$z[\xi](t) = \begin{cases} \xi(t), & \text{for } t \in [-\tau,t_{k-1}], \\ z(t), & \text{for } t \in [t_{k-1},t_k], \end{cases} \tag{4.2}$$

where $\xi \in \mathcal{C}([-\tau,t_{k-1}];E)$ is a fixed function. Letting $C([t_{k-1},t_k];E)[\xi] = \{z[\xi] : z \in C([t_{k-1},t_k];E)\}$, we define a multivalued operator

$$P_F^{k,\xi} : C\bigl([t_{k-1},t_k];E\bigr)[\xi] \to P\bigl(L\bigl([t_{k-1},t_k];E\bigr)\bigr)$$

by

$$P_F^{k,\xi}\bigl(z[\xi]\bigr) = \bigl\{f \in L\bigl([t_{k-1},t_k];E\bigr) : f(s) \in F\bigl(s,z[\xi],z[\xi]_s\bigr) \text{ a.e. } s \in [t_{k-1},t_k]\bigr\}.$$

The following lemma will be used later.

Lemma 4.1 ([15, Lemma 3.1]) *Let E be reflexive, and let the multimap F satisfy* (F1) *and* (F2). *Then $P_F^{k,\xi}$ is weakly u.s.c. with nonempty, convex, and weakly compact values.*

4.1 Compact operator case

Lemma 4.2 ([15, Lemma 3.3]) *Let $\mathcal{D} = E_0 \times \mathcal{C}([-\tau,0];E_0)$ and hypotheses* (F2) *and* (F4) *be satisfied. Then there exists a sequence $\{F_n\} : [0,b] \times \mathcal{D} \to P_{cv}(E)$ such that*

(i) *$F(t,u,v) \subset \cdots \subset F_{n+1}(t,u,v) \subset F_n(t,u,v) \subset \cdots \subset \overline{\mathrm{co}}(F(t,B_{3^{1-n}}(u,v)))$, $n \geq 1$, for each $t \in [0,b]$ and $x \in E$;*

(ii) *$|F_n(t,u,v)| \leq \alpha(t)(3 + |u| + \|v\|_{\mathcal{C}})$, $n \geq 1$, for a.e. $t \in [0,t_m]$ and each $(u,v) \in \mathcal{D}$;*

(iii) *there exists $X \subset [0,t_m]$ with $\operatorname{mes}(X) = 0$ such that, for each $x^* \in E^*$, $\epsilon > 0$ and $(u,v) \in \mathcal{D}$, we can find $N > 0$ such that, for all $n \geq N$,*

$$x^*\bigl(F_n(t,u,v)\bigr) \subset x^*\bigl(F(t,u,v)\bigr) + (-\epsilon,\epsilon);$$

(iv) $F_n(t,\cdot,\cdot):\mathcal{D}\to P_{\mathrm{cv}}(E)$ *is continuous for a.e.* $t\in[0,t_m]$ *with respect to Hausdorff metric for each* $n\geq 1$;

(v) *for each* $n\geq 1$, *there exists a selection* $g_n:[0,b]\times\mathcal{D}\to E$ *of* F_n *such that* $g_n(\cdot,u,v)$ *is measurable for each* $(u,v)\in\mathcal{D}$, *and for any compact subset* $\mathcal{D}'\subset\mathcal{D}$, *there exist constants* $C_V>0$ *and* $\delta>0$ *such that the estimate*

$$\bigl|g_n(t,u_1,v_1)-g_n(t,u_2,v_2)\bigr|\leq C_V\alpha(t)\bigl(|u_1-u_2|+\|v_1-v_2\|_{\mathcal{C}}\bigr)$$

holds for a.e. $t\in[0,b]$ *and each* $(u_1,v_1),(u_2,v_2)\in V$ *with* $V:=(\mathcal{D}'+B_\delta(0))\cap\mathcal{D}$;

(vi) F_n *satisfies condition* (F2) *with* F_n *instead of* F *for each* $n\geq 1$ *provided that* E *is reflexive.*

We denote by $\Theta^m(\varphi)$ the set of all C^0-solutions of inclusion (4.1).

Theorem 4.1 *Assume that* (A1), (A2), (F1), *and* (F2) *are satisfied. If* $I_k:E\to E_0$, $k=1,\dots,m-1$, *are continuous, then for every* $\varphi\in\mathcal{C}([-\tau,0];E)$, $\Theta^m(\varphi)$ *is a nonempty and compact subset of* $\mathcal{PC}([-\tau,t_m];E)$. *Moreover, it is an* R_δ-*set.*

Proof We complete the proof in four steps.

Step 1. Let us consider the non-impulsive inclusion problem

$$(P)_{1;\varphi}\qquad\begin{cases}u'(t)\in Au(t)+F(t,u(t),u_t), & \text{a.e. } t\in[0,t_1],\\ u(t)=\varphi(t), & \text{for } t\in[-\tau,0].\end{cases}$$

Set

$$\begin{aligned}K_1=\bigl\{&u\in\mathcal{PC}\bigl([-\tau,t_1];E_0\bigr):u(t)=\varphi(t)\text{ for }t\in[-\tau,0],\\ &\text{and }\bigl|u(t)\bigr|\leq\psi_\varphi(t)\text{ for }t\in[0,t_1]\bigr\},\end{aligned}$$

where $\psi_\varphi\in C([0,t_1];\mathbb{R}^+)$ is the unique solution of the integral equation

$$\psi_\varphi(t)=\|\varphi\|_{\mathcal{C}}+\int_0^t\alpha(s)\bigl(1+2\bigl|\psi_\varphi(s)\bigr|\bigr)\,ds,\qquad t\in[0,t_1].$$

Define a multivalued mapping Γ^1 on K_1 by setting

$$\Gamma^1(u)=S^1_\varphi\bigl(P^{1,\varphi}_F(u)\bigr),\qquad u\in K_1.$$

Observe that $P^{1,\varphi}_F(u)\neq\emptyset$ for every $u\in K_1$ by Lemma 4.1 and hence $\Gamma^1(u)\subset\mathcal{PC}([-\tau,t_1];E_0)$. Also, $\{z|_{[-\tau,0]}:z\in\Gamma^1(u)\}=\{\varphi\}$ for all $u\in K_1$. Moreover, taking $f\in P^{1,\varphi}_F(x)$ with $u\in K_1$, for every $t\in[0,t_1]$, it follows from (F2) that

$$\begin{aligned}\bigl|S^1_\varphi(f)(t)\bigr|&\leq\bigl|\varphi(0)\bigr|+\int_0^t\bigl|f(s)\bigr|\,ds\\ &\leq\bigl|\varphi(0)\bigr|+\int_0^t\alpha(s)\bigl(1+\bigl|u(s)\bigr|+\|u_s\|_{\mathcal{C}}\bigr)\,ds\end{aligned}$$

$$\begin{aligned}&\le \|\varphi\|_{\mathcal{C}} + \int_0^t \alpha(s)\bigl(1+2\psi_\varphi(s)\bigr)\,ds\\&= \psi_\varphi(t),\end{aligned}$$

where we have used the condition $0 \in A0$ and the fact $\|u_t\|_{\mathcal{C}} \le \psi_\varphi(t)$ for every $t \in [0,t_1]$ and $u \in K_1$. Hence we deduce that $\Gamma^1(u) \subset K_1$ for every $u \in K_1$.

Next we proceed to verifying that Γ^1 is u.s.c. on K_1. In view of Lemma 2.2, it suffices to show that Γ^1 is quasi-compact and closed. For all $f \in P_F^{1,\varphi}(K_1)$, using (F2), we obtain

$$\bigl|f(t)\bigr| \le \alpha(t)\bigl(1+2\psi_\varphi(t_1)\bigr) \quad \text{for a.e. } t \in [0,t_1],$$

which implies that $P_F^{1,\varphi}(K_1)$ is integrably bounded and thus uniformly integrable. In consequence, we obtain by Lemma 3.3(i) that $\Gamma^1(K_1)$ $(= S_\varphi^1(P_F^{1,\varphi}(K_1)))$ is relatively compact in $\mathcal{PC}([-\tau,t_1];E)$. This implies that Γ^1 is quasi-compact.

Let $\{(u_n,v_n)\}$ be a sequence in $G(\Gamma^1)$ such that $(u_n,v_n) \to (u,v)$ in $\mathcal{PC}([-\tau,t_1];E) \times \mathcal{PC}([-\tau,t_1];E)$. Since $v_n \in \Gamma^1(u_n)$, there exists a sequence $\{f_n\} \in L([0,t_1];E)$ satisfying $f_n \in P_F^{1,\varphi}(u_n)$ and $v_n = S_\varphi^1(f_n)$. Noticing that $P_F^{1,\varphi}$ is weakly u.s.c. with convex, weakly compact values due to Lemma 4.1, we deduce by Lemma 2.3 that there exist $f \in P_F^{1,\varphi}(u)$ and a subsequence of $\{f_n\}$, also denoted by $\{f_n\}$, such that $f_n \rightharpoonup f$ in $L([0,t_1];E)$. From this and Lemma 3.3(ii), it follows that $v = S_\varphi^1(f)$ and hence $v \in \Gamma^1(u)$. This shows that Γ^1 is closed.

Consider the closed convex hull of $\Gamma^1(K_1)$ given by

$$\mathcal{K}_1 = \overline{\mathrm{co}}\bigl(\Gamma^1(K_1)\bigr).$$

Clearly $\mathcal{K}_1$ is a compact, convex set in $\mathcal{PC}([-\tau,t_1];E)$ and $\Gamma^1(\mathcal{K}_1) \subset \mathcal{K}_1$.

Next, we show that Γ^1 has a fixed point in $\mathcal{K}_1$. By Theorem 2.3, it suffices to show that Γ^1 has compact and contractible values. Given $u \in \mathcal{K}_1$, we can easily get that $\Gamma^1(u)$ is compact in as much as the closedness and quasi-compactness of Γ^1. Set $f^* \in P_F^{1,\varphi}(u)$ and $u^* = S_\varphi^1(f^*)$, and define a function $h : [0,1] \times \Gamma^1(u) \to \Gamma^1(u)$ as follows:

$$h(r,v)(t) = \begin{cases} v(t), & t \in [-\tau, rt_1],\\ u(t,rt_1,v(rt_1),f^*), & t \in (rt_1,t_1],\end{cases}$$

for each $(r,v) \in [0,1] \times \Gamma^1(u)$, where $u(\cdot,rt_1,v(rt_1),f^*)$, as prescribed in Sect. 3, is the unique C^0-solution of the evolution inclusion problem

$$\begin{cases} u'(t) \in Au(t) + f^*(t), & t \in (rt_1,t_1],\\ u(rt_1) = v(rt_1).\end{cases}$$

It is easy to see that h is well defined since $v = S_\varphi^1 f$ for some $f \in P_F^{1,\varphi}(u)$; hence $h(t,v) = \Gamma^1\tilde{f}$ with $\tilde{f} := f\chi_{[0,rt_1]} + f^*\chi_{(rt_1,t_1]} \in P_F^{1,\varphi}(u)$. Also, it is clear that

$$h(0,y) = u^*, \qquad h(1,y) = y, \quad \text{on } \Gamma^1(x).$$

Moreover, it follows readily that h is continuous. Thus, we obtain that $\Gamma^1(x)$ is contractible. An application of Theorem 2.3 yields that Γ^1 has a fixed point.

We claim that the fixed point set of $\varGamma^1$ is compact. We define a mapping $\widehat{\varGamma}^1$ on $\mathcal{PC}([-\tau,t_1];E_0)$ by

$$\widehat{\varGamma}^1(u) = S_\varphi^1\big(P_F^{1,\varphi}(u)\big), \quad u \in \mathcal{PC}\big([-\tau,t_1];E_0\big),$$

which is regarded as an extension of $\varGamma^1$. Let $\widetilde{\varTheta}_\varphi^1 := \operatorname{Fix} \widehat{\varGamma}^1(C([0,t_1];E_0))$. Then it will be sufficient to show that $u \in K_1$ whenever $u \in \widetilde{\varTheta}_\varphi^1$. Taking $u \in \operatorname{Fix} \widehat{\varGamma}^1(C([0,t_1];E_0))$, it follows that there exists $f \in P_F^{1,\varphi}(u)$ such that $u = S_\varphi^1(f)$. Then, by (F2) and the condition $0 \in A0$, and using the earlier arguments, we have

$$\begin{aligned}
\big|u(t)\big| &\le \big|\varphi(0)\big| + \int_0^t \big|f(s)\big|\,ds \\
&\le \big|\varphi(0)\big| + \int_0^t \alpha(s)\big(1 + \big|u(s)\big| + \|u_s\|_{\mathcal{C}}\big)\,ds \\
&\le \|\varphi\|_{\mathcal{C}} + \int_0^t \alpha(s)\big(1 + 2\|u_s\|_{\mathcal{C}}\big)\,ds, \quad t \in [0,t_1],
\end{aligned}$$

which implies that

$$\|u_t\|_{\mathcal{C}} \le \|\varphi\|_{\mathcal{C}} + \int_0^t \alpha(s)\big(1 + 2\|u_s\|_{\mathcal{C}}\big)\,ds, \quad t \in [0,t_1].$$

Then, for each $t \in [0,t_1]$, an application of Lemma 2.6 yields

$$\begin{aligned}
\|u_t\|_{\mathcal{C}} &\le \|\varphi\|_{\mathcal{C}} + \int_0^t \alpha(s)\,ds + 2\int_0^t \alpha(s)\left(\|\varphi\|_{\mathcal{C}} + \int_0^s \alpha(\sigma)\,d\sigma\right)\exp\left(2\int_s^t \alpha(\sigma)\,d\sigma\right)ds \\
&\le \psi_\varphi(t),
\end{aligned}$$

which implies that $u \in K_1$. Based on the foregoing argument, we get $\widetilde{\varTheta}_\varphi^1 = \operatorname{Fix} \varGamma^1$.

Moreover, as in the above proof, $\mathcal{K}_1$ is compact in $\mathcal{PC}([-\tau,t_1];E_0)$ and $\varGamma^1$ is closed. in consequence, we deduce that $\operatorname{Fix} \varGamma^1$ is a compact set in $\mathcal{PC}([-\tau,t_1];E_0)$ and so is $\widetilde{\varTheta}_\varphi^1$ (in fact, $\widetilde{\varTheta}_\varphi^1 = \varTheta_\varphi^1$).

Step 2. Let us fix $z^1 \in \widetilde{\varTheta}_\varphi^1$ and consider the non-impulsive inclusion problem

$$(P)_{2;z^1} \qquad \begin{cases}
u'(t) \in Ax(t) + f(t), & \text{a.e. } t \in (t_1,t_2], \\
f(t) \in F(t,u(t),u_t), & \text{a.e. } t \in (t_1,t_2], \\
u(t) = z^1(t), & \text{for } t \in [-\tau,t_1], \\
u(t_1^+) = z^1(t_1) + I_1(z_{t_1}^1).
\end{cases}$$

Set

$$\begin{aligned}
K_2 = \big\{&u \in \mathcal{PC}\big([-\tau,t_2];E_0\big) : u(t) = z^1(t) \text{ for } t \in [-\tau,t_1] \\
&\text{and } \big|u(t)\big| \le \psi_{z^1}(t) \text{ for } t \in [t_1,t_2]\big\},
\end{aligned}$$

where $\psi_{z^1} \in C([t_1,t_2];\mathbb{R}^+)$ is the unique solution of the integral equation

$$\psi_{z^1}(t) = \big\|z^1\big\|_{\mathcal{C}} + \big|I_1\big(z_{t_1}^1\big)\big| + \int_{t_1}^t \alpha(s)\big(1 + 2\big|\psi_{z^1}(s)\big|\big)\,ds, \quad t \in [t_1,t_2].$$

Define a multivalued mapping Γ^2 on K_2 by setting

$$\Gamma^2(u) = S^2_{z^1}\big(P^{2,z^1}_F(u)\big), \quad u \in K_2.$$

For every $u \in K_2$, observe that $P^{2,z^1}_F(u) \neq \emptyset$ due to Lemma 4.1 and hence $\Gamma^1(u) \subset \mathcal{PC}([-\tau,t_2];E_0)$. Also, $\{z|_{[-\tau,t_1]} : z \in \Gamma^2(u)\} = \{z^1\}$ for all $u \in K_2$. Moreover, taking $f \in P^{2,z^1}_F(u)$ with $u \in K_2$, it follows from (F2) that, for every $t \in [t_1,t_2]$,

$$\begin{aligned}
\big|S^2_{z^1}(f)(t)\big| &\le \big|z^1(t_1) + I_1\big(z^1_{t_1}\big)\big| + \int_{t_1}^t \big|f(s)\big|\,ds \\
&\le \big|z^1(t_1)\big| + \big|I_1\big(z^1_{t_1}\big)\big| + \int_{t_1}^t \alpha(s)\big(1 + \big|u(s)\big| + \|u_s\|_{\mathcal{PC}[-\tau,t_1]}\big)\,ds \\
&\le \big\|z^1\big\|_{\mathcal{PC}[-\tau,t_1]} + \big|I_1\big(z^1_{t_1}\big)\big| + \int_{t_1}^t \alpha(s)\big(1 + 2\psi_{z^1}(s)\big)\,ds \\
&= \psi_{z^1}(t),
\end{aligned}$$

where we have used the condition $0 \in A0$ and the fact that $\|u_t\|_{\mathcal{C}} \le \psi_{z^1}(t)$ for every $t \in [t_1,t_2]$ and $u \in K_2$. Hence $\Gamma^2(u) \subset K_2$ for every $u \in K_2$.

Since I_1 is continuous, proceeding in the same way as in Step 1, we can claim that problem $(P)_{2;z^1}$ has at least one C_0-solution and the solution set is a compact set, say $\widetilde{\Theta}^2_{z^1}$.

Continuing this iterative process till problem $(P)_{m;z^1,\dots,z^{m-1}}$, we obtain that there exist solutions for this problem, which form a compact set, say $\widetilde{\Theta}^m_{z^1,\dots,z_{m-1}}$.

Now, every solution of $(P)_{m;z^1,\dots,z^{m-1}}$ is a solution of (4.1), which implies that the solution set of inclusion problem (4.1) is nonempty.

Step 3. We prove that the set of all solutions of (4.1), that is,

$$\Theta^m(\varphi) = \bigcup\big\{\widetilde{\Theta}^m_{z^1,\dots,z_{m-1}} : z^1 \in \widetilde{\Theta}^1_\varphi;\dots;z^{m-1} \in \widetilde{\Theta}^{m-1}_{z^1,\dots,z_{m-2}}\big\} \tag{4.3}$$

is compact.

First of all, we define the multifunction $H^1 : \widetilde{\Theta}^1_\varphi \to P(C([-\tau,t_2];E_0))$ as

$$H^1\big(z^1\big) = \widetilde{\Theta}^2_{z^1}.$$

From Step 2, we know hat $\widetilde{\Theta}^1_\varphi$ is compact and H^1 has compact values.

By Lemma 2.4, it suffices to prove that H^1 is u.s.c. On the contrary, we assume that there exists $\bar{z}^1 \in \widetilde{\Theta}^1_\varphi$ such that H^1 is not u.s.c. in $\bar{z}^1$. Therefore there exist $\bar{\varepsilon} > 0$ and two sequences $\{z^1_n\}^\infty_{n=1}$, $z^1_n \to \bar{z}^1$ in $\mathcal{PC}([-\tau,t_1];E_0)$, and $\{z^2_n\}^\infty_{n=1}, z^2_n \in \widetilde{\Theta}^2_{z^1_n}$ such that

$$z^2_n \notin O_{\bar{\varepsilon}}\big(\widetilde{\Theta}^2_{\bar{z}^1}\big), \quad n \ge 1. \tag{4.4}$$

Since $\{z^2_n\}^\infty_{n=1}$ is a sequence of solutions, we have

$$z^2_n(t) = S^2_{z^1_n}\big(f^2_n\big)(t), \quad t \in [-\tau,t_2], \tag{4.5}$$

where $f^2_n \in L([t_1,t_2];E_0), f^2_n(s) \in F(s,z^2_n(s),z^2_{ns})$ for almost every $s \in [t_1,t]$.

Using similar arguments as in the proof of Step 1, it can be shown that the set $\{z_n^2\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{PC}([-\tau,t_2];E_0)$. Therefore, without loss of generality, we can assume that there exists $\bar{z}^2 \in \mathcal{PC}([-\tau,t_2];E_0)$ such that $z_n^2 \to \bar{z}^2$ in $\mathcal{PC}([-\tau,t_2];E_0)$.

Now we show that $\bar{z}^2 \in H^1(\bar{z}^1)$. For every $n \geq 1$, we consider $z_n^2 \in \widetilde{\Theta}_{\bar{z}^1}^2$ and the corresponding function f_n^2 from (4.5).

As in Step 1, we show that there exists $\bar{f}^2 \in L([t_1,t_2];E_0)$ such that $f_n^2 \to \bar{f}^2 \in L([t_1,t_2];E_0)$. Now, by using Lemma 2.3, we have

$$\bar{f}^2(t) \in F\big(t,\bar{z}^2(t),\bar{z}_t^2\big), \quad \text{a.e. } t \in [t_1,t_2].$$

From the fact that the function I_1 is continuous, taking the limit of both sides of (4.5), we get

$$\bar{z}^2(t) = S_{\bar{z}^1}^2\big(\bar{f}^2\big)(t), \quad t \in [-\tau,t_2],$$

where $\bar{f}^2 \in L([t_1,t_2];E_0)$, $\bar{f}^2(s) \in F(s,\bar{z}(s),\bar{z}_s^2)$ for almost every $s \in [t_1,t]$, that is, $\bar{z}^2 \in \widetilde{\Theta}_{\bar{z}^1}^2 = H^1(\bar{z}^1)$.

The fact that $z_n^2 \to \bar{z}^2 \in H^1(\bar{z}^1)$ leads to the contradiction of (4.4). Therefore, H^1 is u.s.c.

By iterating this process, we can obtain the compactness of the solution set $\Theta^m(\varphi)$ on $[-\tau,t_m]$.

Step 4. Consider the inclusion problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in [0,t_m], \\ f(t) \in F_n(t,u(t),u_t), & t \in [0,t_m], \\ u(t) = \varphi(t), & t \in [-\tau,0], \\ u(t_k^+) = u(t_k) + I_k(u_{t_k}), & k \in \mathbb{N}^+, \end{cases} \tag{4.6}$$

where $\varphi \in \mathcal{C}([-\tau,0];E_0)$ and multivalued functions $F_n : [0,t_m] \times X \to P_{\mathrm{cv}}(X)$ are established in Lemma 4.2. Let $\Theta_n^m(\varphi)$ denote the set of all C^0-solutions of the inclusion problem (4.6). Then we show that the set $\Theta_n^m(\varphi)$ is an R_δ-set.

From Lemma 4.2(ii) and (vi), it follows that $\{F_n\}$ satisfies conditions (F1) and (F2) for each $n \geq 1$. Then, using the above arguments, we deduce that each set $\Theta_n^m(\varphi)$ is nonempty and compact in $C([-\tau,t_m];E_0)$ for each $n \geq 1$. In view of Lemma 4.2(i), it is easy to verify that $\Theta^m(\varphi) \subset \cdots \subset \Theta_n^m(\varphi) \subset \cdots \subset \Theta_2^m(\varphi) \subset \Theta_1^m(\varphi)$. Using the method similar to the one employed in [15, Theorem 3.2], we can show that $\Theta^m(\varphi) = \bigcap_{n\geq 1} \Theta_n^m(\varphi)$.

Finally, in order to show that $\Theta^m(\varphi)$ is an R_δ-set, it suffices to verify that $\Theta_n^m(\varphi)$ is contractible for each $n \geq 1$.

Fix $\bar{u} \in \Theta_n^m(\varphi)$ and divide the interval $[0,1]$ into m parts: $0 < \frac{1}{m} < \frac{2}{m} < \cdots < \frac{m-1}{m} < 1$. For $r \in (0,\frac{1}{m}]$, consider the following problem:

$$\begin{cases} u'(t) \in Au(t) + g_n(t,u(t),u_t), & \text{for a.e. } t \in [t_m - mr(t_m - t_{m-1}), t_m], \\ u(t) = \bar{u}(t), & \text{for } t \in [-\tau, t_m - mr(t_m - t_{m-1})], \end{cases} \tag{4.7}$$

where g_n is a measurable locally Lipschitz selection of F_n by Lemma 4.2.

Let $\tilde{u}^m_{n,r}$ stand for the unique solution of this problem. Then $u^m_{n,r}$ can be defined as:

$$u^m_{n,r}(t) = \begin{cases} \bar{u}(t), & \text{for } t \in [-\tau, t_m - mr(t_m - t_{m-1})], \\ \tilde{u}^m_{n,r}(t), & \text{for } t \in [t_m - mr(t_m - t_{m-1}), t_m], \end{cases}$$

with $u^m_{n,r} \in \Theta^m_n(\varphi)$.

Next, for $r \in (\frac{1}{m}, \frac{2}{m}]$, we consider the following problem:

$$\begin{cases} u'(t) \in Au(t) + g_n(t, u(t), u_t), & \text{for a.e. } t \in [t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2}), t_m], \\ u(t) = \bar{u}(t), & \text{for } t \in [-\tau, t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2})], \\ u(t_k^+) = u(t_k) + I_k(u_{t_k}), & k = m-1. \end{cases} \tag{4.8}$$

Let $\tilde{u}^{m-1}_{n,r}$ stand for the unique solution of this problem. Then we obtain that $u^{m-1}_{n,r} \in \Theta^m_n(\varphi)$, where

$$u^{m-1}_{n,r}(t) = \begin{cases} \bar{u}(t), & \text{for } t \in [-\tau, t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2})], \\ \tilde{u}^{m-1}_{n,r}(t), & \text{for } t \in [t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2}), t_m]. \end{cases}$$

Now we consider the following problem for $r \in (\frac{m-1}{m}, 1]$:

$$\begin{cases} u'(t) = Au(t) + g_n(t, u(t), u_t), & \text{for a.e. } t \in [t_1 - m(r - \frac{m-1}{m})t_1, t_m], \\ u(t) = \bar{u}(t), & \text{for } t \in [-\tau, t_1 - m(r - \frac{m-1}{m})t_1], \\ u(t_k^+) = u(t_k) + I_k(u_{t_k}), & k = \{2, 3, \ldots, m-1\}. \end{cases} \tag{4.9}$$

Let $\tilde{u}^1_{n,r}$ stand for the unique solution of the above problem. Then $u^1_{n,r}$ can be defined as:

$$u^1_{n,r}(t) = \begin{cases} \bar{u}(t), & \text{for } t \in [-\tau, t_1 - m(r - \frac{m-1}{m})t_1], \\ \tilde{u}^m_{n,r}(t), & \text{for } t \in [t_1 - m(r - \frac{m-1}{m})t_1, t_m], \end{cases}$$

which belongs to $\Theta^m_n(\varphi)$.

Finally, we define $h_n : [0,1] \times \Theta^m_n(\varphi) \to \Theta^m_n(\varphi)$ by

$$h_n(r, \bar{u}) = \begin{cases} \bar{u}(t), & r = 0, \\ u^m_{n,r}, & r \in (0, \frac{1}{m}], \\ u^{m-1}_{n,r}, & r \in (\frac{1}{m}, \frac{2}{m}], \\ \vdots & \\ u^1_{n,r}, & r \in (\frac{m-1}{m}, 1]. \end{cases} \tag{4.10}$$

Here the functions $u^m_{n,r}, u^{m-1}_{n,r}, \ldots, u^1_{n,r}$ are determined by the choice of $\bar{u} \in \Theta^m_n(\varphi)$. We can show that h_n is continuous by applying a standard method, based on the continuous dependence on initial conditions and the fact that the maps I are continuous, we can establish the continuity for $r \in \{\frac{i}{m}; i = 1, \ldots, m\}$. By the above definition, we have that $h_n(0, \bar{u}) = \bar{x}$ and

$h_n(1,\bar{u}) = u^1_{n,1}$, so $\Theta^m_n(\varphi)$ is a contractible set for each $n \in \mathbb{N}$. Consequently, we conclude that $\Theta^m_n(\varphi)$ is an R_δ-set. □

We denote by $\Theta(\varphi)$ the set of all C^0-solutions of inclusion (1.1).

Theorem 4.2 *Assume that* (A1), (A2), (F1), *and* (F2) *are satisfied. If* $I_k : E \to E_0$, $k = 1,\dots,m-1$, *are continuous, then* $\Theta(\varphi)$ *is a nonempty and compact subset of* $\mathcal{PC}([-\tau,\infty);E_0)$ *for every* $\varphi \in \mathcal{C}([-\tau,0],E_0)$. *Moreover, it is an* R_δ-*set.*

Proof We divide the proof in two steps.

Step 1. Besides (1.1), for each $m \in \mathbb{N}^+$, we study problem (4.1) on the compact interval $[0,t_m]$.

For any $u \in \mathcal{PC}([0,t_m];E)$ with $u(0) = \varphi(0)$, define $u[\varphi]^* : [-\tau,t_m] \to E$ as

$$u[\varphi]^*(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\tau,0], \\ u(t), & \text{for } t \in [0,t_m]. \end{cases} \tag{4.11}$$

We denote $\mathcal{PC}_m = \mathcal{PC}([0,t_m];E)[\varphi]^*$ $(= \{u[\varphi]^* : u \in \mathcal{PC}([0,t_m];E)\})$ and consider the multivalued operator

$$\tilde{P}^{m,\varphi}_F : \mathcal{PC}_m \to P\bigl(L\bigl([0,t_m];E\bigr)\bigr)$$

defined by

$$\tilde{P}^{m,\varphi}_F\bigl(u[\varphi]^*\bigr) = \bigl\{f \in L\bigl([0,t_m];E\bigr) : f(s) \in F\bigl(s,u[\varphi]^*,u[\varphi]^*_s\bigr) \text{ a.e. } s \in [0,t_m]\bigr\}.$$

Next, for $\varphi \in \mathcal{C}([-\tau,0];E_0)$ and $f \in L([0,t_m];E)$, we define the mapping $\tilde{S}^m_\varphi : L([0,t_m];E) \to \mathcal{PC}_m$ as

$$\tilde{S}^m_\varphi(f)(t) = u\bigl(t,[0,t_m],\varphi,f\bigr), \quad t \in [-\tau,t_m].$$

Clearly, $\tilde{S}^m_\varphi(f)$ is the unique C^0-solution for the evolution inclusion with time delay of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in [0,t_m], t \neq t_k, k < m, \\ u(t) = \varphi(t), & t \in [-\tau,0], \\ u(t_k^+) = u(t_k) + I_k(u_{t_k}), & k < m. \end{cases}$$

Consider the sequence of multivalued maps $\tilde{\Gamma}^m : \mathcal{PC}_m \to P(\mathcal{PC}_m)$ defined by

$$\tilde{\Gamma}_m(z) = \tilde{S}^m_\varphi\bigl(\tilde{P}^{m,\varphi}_F\bigl(z[\varphi]^*\bigr)\bigr)$$

for $t \in [-\tau,t_m]$. Now we consider the projection $p^{m+1}_m : \mathcal{PC}_{m+1} \to \mathcal{PC}_m$ given by $p^{m+1}_m(z) = z|_{[-\tau,t_m]}$.

Observe that $\tilde{\Gamma}^m p_m^{m+1} = p_m^{m+1}\tilde{\Gamma}^{m+1}$, so $\{\mathrm{id}, \tilde{\Gamma}^m\}$ is the map of the inverse system $\{\mathcal{PC}_m, p_n^m\}$. The map $\{\mathrm{id}, \tilde{\Gamma}^m\}$ induces the limit map $\tilde{\Gamma} : \mathcal{PC}_\infty \to P(\mathcal{PC}_\infty)$ defined by

$$\begin{aligned}\Gamma^\infty(z) = \bigl\{ &\nu \in \mathcal{PC}_\infty : \nu|_{[-\tau,t_m]} = \tilde{S}_\varphi^m(f|_{[0,t_m]}), f \in L_{\mathrm{loc}}\bigl([0,\infty);E\bigr) \\ &\text{and } f(t) \in F\bigl(t,u(t),u_t\bigr) \text{ for a.e. } t \in [0,\infty)\bigr\}\end{aligned}$$

for each $u \in \mathcal{PC}_\infty$, where

$$\mathcal{PC}_\infty = \mathcal{PC}\bigl([0,\infty);E\bigr)[\varphi]^* = \varprojlim\bigl\{\mathcal{PC}_m, p_m^n\bigr\}.$$

Note that $\Theta(\varphi) := \mathrm{Fix}(\Gamma^\infty) = \lim_{\leftarrow} \Theta^m(\varphi)$ is the solution set of problem (1.1). By Theorem 4.1, for every $m \geq 1$, the solution set $\Theta^m(\varphi)$ to (4.1) is a nonempty compact subset of $PC([0,t_m];E_0)$. By [20, Proposition 2.3], it follows that the set $\Theta(\varphi)$ is nonempty and compact.

Step 2. It has been established in Theorem 4.1 that the solution sets on compact intervals are R_δ-sets (that is, for problem (4.1)). Next, we consider an inverse system similar to the one in Step 1. Applying [4, Proposition 4.1], we deduce that the solution set for problem (1.1) is a compact R_δ-set as claimed. □

4.2 Noncompact operator case

Theorem 4.3 *Suppose that assumptions* (A1), (A3), (F1), (F2), *and* (F3) *are satisfied*. *If* $I_k : E \to E_0$, $k = 1, \ldots, m-1$, *are continuous and there exist constants* $r_k > 0$ *such that*

$$\beta\bigl(I_k(D)\bigr) \leq r_k \sup_{-\tau\leq\theta\leq 0} \beta\bigl(D(\theta)\bigr)$$

for every bounded $D \subset E$, *then for every* $\varphi \in \mathcal{C}([-\tau,0];E)$, $\Theta_m(\varphi)$ *is a nonempty and compact subset of* $\mathcal{PC}([-\tau,t_m];E)$. *Moreover, it is an* R_δ-*set.*

Proof We complete the proof in four steps.

Step 1. For the same K_1, as argued in Theorem 4.1, we note that K_1 is a closed and convex subset of $\mathcal{PC}([-\tau,t_1];E_0)$. Let $\mathcal{K}_{n+1} := \overline{\mathrm{co}}\,\Gamma^1(\mathcal{K}_n)$ for all $n \geq 1$, and $\mathcal{K} := \bigcap_{n\geq 1} \mathcal{K}_n$. We show that $\mathcal{K}$ is compact convex when $\mathcal{K}$ is relatively compact.

By Lemma 3.4(i), we know that $\mathcal{K}$ is an equicontinuous subset of $C([-\tau,t_1];E_0)$, hence $\mathcal{K}$ is relatively compact if $\beta(\mathcal{K}(t)) = 0$ on $[0,t_1]$, since $\beta(\mathcal{K}(t)) = \beta(\{\varphi(t)\}) = 0$ for $t \in [-\tau,0]$. Let $\rho_n(t) = \beta(\mathcal{K}_n(t))$ for $n \geq 1$ and $\rho(t) = \beta(\mathcal{K}(t))$. Then $\rho_{n+1}(t) \leq \beta(\{S_\varphi^1(f)(t) : f \in P_F^{1,\varphi}(\mathcal{K}_n)\})$. In order to apply (3.5), suppose that there is a sequence $\{f_n\}_{n=1}^\infty \subset P_F^{1,\varphi}(\mathcal{K}_n)$ for given $\epsilon > 0$ such that

$$\begin{aligned}\beta\bigl(\bigl\{S_\varphi^1(f)(t) : f \in P_F^{1,\varphi}(\mathcal{K}_n)\bigr\}\bigr) &\leq 2\beta\bigl(\bigl\{\Gamma^1(f_n)(t)\bigr\}_{n=1}^\infty\bigr) + \epsilon \\ &\leq 2\int_0^t \beta\bigl(\bigl\{f_n(s)\bigr\}_{n=1}^\infty\bigr)\,ds + \epsilon.\end{aligned}$$

Taking ℓ to be large enough for the above $\epsilon > 0$, by (F3), there exists $\delta(<\epsilon) > 0$ such that

$$\begin{aligned}\beta\bigl(\bigl\{f_n(s)\bigr\}_{n=1}^\infty\bigr) &= \beta\bigl(\bigl\{f_n(s)\bigr\}_{n=\ell}^\infty\bigr) \leq \beta\bigl(F\bigl(\{s\} \times \bigl\{\mathcal{K}_n(s)\bigr\}_{n=\ell}^\infty \times O_\delta\bigl(\bigl\{(\mathcal{K}_n)_s\bigr\}_{n=\ell}^\infty\bigr)\bigr)\bigr) \\ &\leq \mu(s)\Bigl(\beta\bigl(\bigl\{\mathcal{K}_n(s)\bigr\}_{n=\ell}^\infty\bigr) + \sup_{-\tau\leq\theta\leq 0} \beta\bigl(\bigl\{\mathcal{K}_n(s+\theta)\bigr\}_{n=\ell}^\infty\bigr) + \epsilon\Bigr)\end{aligned}$$

$$\leq \mu(s)\Big(2 \sup_{0\leq\theta\leq s} \beta\big(\{\mathcal{K}_n(\theta)\}_{n=\ell}^{\infty}\big) + \epsilon\Big)$$
$$\leq \mu(s)\Big(2 \sup_{0\leq\theta\leq s} \rho_n(\theta) + \epsilon\Big),$$

which implies that

$$\rho_{n+1}(t) \leq 4\int_0^t \mu(s) \sup_{0\leq\theta\leq s} \rho_n(\theta)\,ds + 2\epsilon \int_0^t \mu(s)\,ds + \epsilon.$$

Since this is true for every $\epsilon > 0$ and $\rho_n(t) \searrow \rho(t)$, therefore

$$\rho(t) \leq 4\int_0^t \mu(s) \sup_{0\leq\theta\leq s} \rho(\theta)\,ds, \qquad \rho(0) = 0.$$

Evidently, it implies that $\rho(t) = 0$ on $[0, t_1]$.

Next we show that Γ^1 is u.s.c. on $\mathcal{K}$. By Lemma 2.2, it suffices to show that Γ^1 is quasi-compact and closed. First, we prove that the multimap $\Gamma^1 = S_\varphi^1 P_F^{1,\varphi}$ has a closed graph. Let $\{v_n\} \subset \mathcal{K}$ with $v_n \to v$ and $u_n \in \Gamma^1(v_n)$ with $u_n \to u$. We first prove that $u \in \Gamma^1(v)$. By the definition of Γ^1, there exist $f_n \in P_F^{1,\varphi}(v_n)$ such that

$$u_n(t) = S_\varphi^1(f_n)(t).$$

In view of (F2), $\{f_n\}$ is bounded in $L([0, t_1]; E_0)$, and that $f_n \rightharpoonup f$ in $L([0, t_1]; E_0)$ (see Lemma 2.1). Since $P_F^{1,\varphi}$ is weakly u.s.c. with weakly compact and convex values (see Lemma 4.1), from Lemma 2.3, we have that $f \in P_F^{1,\varphi}(v)$.

Moreover, for any $\epsilon > 0$, by (F3), there exists $\delta\ (< \epsilon) > 0$ such that

$$\beta\big(\{f_n(t)\}_{n=1}^{\infty}\big) \leq \beta(F\big(\{s\} \times \{v_n(s)\}_{n=1}^{\infty} \times O_\delta\big(\{(v_n)_s\}_{n=1}^{\infty}\big)\big)$$
$$\leq \mu(t)\Big(\beta\big(\{v_n(t)\}_{n=1}^{\infty}\big) + \sup_{-\tau\leq\theta\leq 0} \beta\big(\{(v_n)_t\}_{n=1}^{\infty}\big) + \epsilon\Big)$$
$$\leq \mu(t)\Big(2 \sup_{0\leq\theta\leq t} \beta\big(\{v_n(\theta)\}_{n=1}^{\infty}\big) + \epsilon\Big)$$

for almost every $t \in [0, t_1]$. Thus the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact for almost every $t \in [0, t_1]$. By Lemma 3.4, we obtain

$$\beta\big(\{S_\varphi^1(f_n)(s)\}_{n=1}^{\infty}\big) \leq \int_0^t \beta\big(\{f_n(s)\}_{n=1}^{\infty}\big)\,ds = 0. \tag{4.12}$$

Hence the set $\{S_\varphi^1(f_n)(t)\}_{n=1}^{\infty}$ is relatively compact. Using Lemma 3.6, we find that $u = S_\varphi^1(f)$, that is, $u \in S_\varphi^1(P_F^{1,\varphi}(v)) = \Gamma^1(v)$, demonstrating that the multimap Γ^1 is closed.

It is easy to see that Γ^1 is quasi-compact. In fact, $\Gamma^1(\mathcal{K})$ is compact as $\mathcal{K}$ is closed and compact. By Lemma 2.2, it follows that Γ^1 is u.s.c. on $\mathcal{K}$.

By Theorem 2.3, we deduce that Γ^1 has at least one fixed point since $\Gamma^1 : \mathcal{K} \to P(\mathcal{K})$ has contractible values, which follows as in the proof of Theorem 4.1.

The following result is similar to Step 1 in Theorem 4.1 and leads to the fact that the solution set of problem $(P)_{1;\varphi}$ is a nonempty compact set.

Step 2. Following the proof in Steps 2 and 3 of Theorem 4.1, one can easily infer that the solution set of problem (4.1) is a nonempty compact set as argued Step 1.

Step 3. As in the proof of [20], we can obtain that all solutions to (4.1) are pointwise uniformly bounded by some constant $\bar{K}_m$. Let $\tilde{K}_m = \max\{1,\tau\}\bar{K}_m$. Then $|u(t)| \le \tilde{K}_m$ and $\|u_t\|_{\mathcal{C}} \le \tilde{K}_m$ for every solution x to (4.1). We define a mapping $\tilde{F} : [0,t_m] \times E \times \mathcal{C}([-\tau,0];E) \to P_{\mathrm{cp,cv}}(E)$ by

$$\tilde{F}(t,u,v) = \begin{cases} F(t,u,v), & \text{if } t \in [0,t_m] \text{ and } \max\{|u|,\|v\|_{\mathcal{C}}\} \le \tilde{K}_m, \\ F(t, \frac{\tilde{K}_m u}{|u|}, \frac{\tilde{K}_m v}{\|v\|_{\mathcal{C}}}), & \text{if } t \in [0,t_m] \text{ and } \max\{|u|,\|v\|_{\mathcal{C}}\} \ge \tilde{K}_m, \end{cases}$$

which is integrably bounded and generates the same set of solutions on replacing F by $\tilde{F}$ in (4.1). Without any loss of generality, instead of (F2), we assume that

(F2)′ $|F(t,u,v)| \le \omega_m(t)$ for every $t \in [0,t_m]$, where $\omega_m \in L([0,t_m])$.

Then, similar to Step 2 in Theorem 3.5 of [20], we construct a sequence of multivalued maps $\{F_n\} : [0,b] \times \mathcal{D} \to P_{\mathrm{cv}}(E)$ such that

(i) $F(t,u,v) \subset \cdots \subset F_{n+1}(t,u,v) \subset F_n(t,u,v) \subset \cdots \subset \overline{\mathrm{co}}(F(t,B_{3^{1-n}}(u,v)))$, $n \ge 1$, for each $t \in [0,b]$ and $x \in E$;

(ii) $|F_n(t,u,v)| \le \omega_m(t)$, $n \ge 1$, for a.e. $t \in [0,t_m]$ and each $(u,v) \in \mathcal{D}$;

(iii) there exists $X \subset [0,t_m]$ with $\mathrm{mes}(X) = 0$ such that, for each $x^* \in E^*$, $\epsilon > 0$ and $(u,v) \in \mathcal{D}$, there exists $N > 0$ such that, for all $n \ge N$,

$$x^*\bigl(F_n(t,u,v)\bigr) \subset x^*\bigl(F(t,u,v)\bigr) + (-\epsilon,\epsilon);$$

(iv) $F_n(t,\cdot,\cdot) : \mathcal{D} \to P_{\mathrm{cv}}(E)$ is continuous for a.e. $t \in [0,t_m]$ with respect to Hausdorff metric for each $n \ge 1$;

(v) for each $n \ge 1$, there exists a selection $g_n : [0,b] \times \mathcal{D} \to E$ of F_n such that $g_n(\cdot,u,v)$ is measurable for each $(u,v) \in \mathcal{D}$ and $g_n(t,\cdot,\cdot)$ is locally Lipschitz.

We first consider problem (4.6). Let $\Theta_n^m(\varphi)$ denote the set of all C^0-solutions of inclusion (4.6), which is obviously nonempty in $\mathcal{PC}([-\tau,t_m];E_0)$. Moreover, by Lemma 4.2(i), we have

$$\Theta^m(\varphi) \subset \cdots \subset \Theta_n^m(\varphi) \subset \cdots \subset \Theta_2^m(\varphi) \subset \Theta_1^m(\varphi).$$

Next we prove that each sequence $\{u_n\}$ is such that $u_n \in \Theta_n^m(\varphi)$ for all $n \ge 1$ and has a convergent subsequence $u_{n_k} \to u \in \Theta^m(\varphi)$. Notice that $u_n(t) = \varphi(t) = u(t)$ for all $t \in [-\tau,0]$. Now let us consider the sequence u_n on $[0,t_1]$. Then

$$u_n(t) = \tilde{S}_\varphi^m(f_n) = S_\varphi^1(f_n) \quad \text{for } t \in [0,t_1],$$

where $f_n \in L^1([0,t_1];E)$, $f_n(t) \in F_n(t,u_n(t),(u_n)_t)$ for a.e. $t \in [0,t_1]$.

For ℓ to be large enough with $\epsilon > 0$, using (F3), there exists δ $(<\epsilon) > 0$ such that

$$\begin{aligned} \beta\bigl(\{f_n(s)\}_{n\ge1}\bigr) &= \beta\bigl(\{f_n(s)\}_{n\ge\ell}\bigr) \le \beta\bigl(F\bigl(\{s\} \times \{u_n(s)\}_{n\ge\ell} \times O_\delta\bigl(\{(u_n)_s\}_{n\ge\ell}\bigr)\bigr)\bigr) \\ &\le \mu(s)\Bigl(\beta\bigl(\{u_n(s)\}_{n\ge\ell}\bigr) + \sup_{-\tau\le\theta\le0} \beta\bigl(\{u_n(s+\theta)\}_{n\ge\ell}\bigr) + \epsilon\Bigr) \end{aligned}$$

$$\leq \mu(s)\Big(2\sup_{0\leq\theta\leq s}\beta\big(\{u_n(\theta)\}_{n\geq\ell}\big)+\epsilon\Big)$$
$$\leq \mu(s)\big(2\bar{\rho}(s)+\epsilon\big),$$

where $\bar{\rho}(s) := \sup_{0\leq\theta\leq s}\beta(\{u_n(\theta)\}_{n\geq-})$. Since

$$\beta\big(\{u_n(t)\}_{n\geq1}\big) = \beta\big(\{S^1_\varphi(f_n)(t)\}_{n\geq1}\big) \leq \int_0^t \beta\big(\{f_n(s)\}_{n=1}^\infty\big)\,ds,$$

it follows that

$$\bar{\rho}(t) \leq \int_0^t \mu(s)\big(2\bar{\rho}(s)+\epsilon\big)\,ds.$$

Since it holds for every $\epsilon > 0$, we have

$$\bar{\rho}(t) \leq 2\int_0^t \mu(s)\bar{\rho}(s)\,ds.$$

By Gronwall's inequality, we have $\bar{\rho}(t) = 0$ for $t \in [0, t_1]$. Then $\beta(\{u_n(t)\}_{n\geq1}) = 0$ for $t \in [0, t_1]$. Hence $\beta(\{f_n(t)\}_{n\geq1}) = 0$ for $t \in [0, t_1]$.

For $t = t_1$, we have

$$\begin{aligned}\beta\big(\{u_n(t_1)+I_1\big((u_n)_{t_1}\big)\}_{n=1}\big) &= \beta\big(\{u_n(t_1)+I_1\big((u_n)_{t_1}\big)\}_{n=\ell}\big)\\ &\leq \beta\big(\{u_n(t_1)\}_{n=\ell}\big)+\beta\big(\{I_1\big((u_n)_{t_1}\big)\}_{n=\ell}\big)\\ &\leq r_1\max\Big(\sup_{\tau\leq\theta\leq0}\beta\big(\{\varphi(\theta)\}\big), \sup_{0\leq\theta\leq t_1}\beta\big(\{u(\theta)_{n=\ell}\}\big)\Big)\\ &= 0.\end{aligned}$$

Thus every solution u_n has the form

$$u_n(t) = \tilde{S}^m_\varphi(f_n)(t) = S^2_{u_n|_{[-\tau,t_1]}}(f_n)(t) \quad \text{for } t \in [t_1, t_2],$$

where $f_n \in L^1([t_1, t_2]; E)$, $f_n(t) \in F_n(t, u_n(t), (u_n)_t)$ for a.e. $t \in [t_1, t_2]$. Analogously as before, we can show that $\beta(\{u_n(t)\}_{n\geq1}) = 0$ and $\beta(\{f_n(t)\}_{n\geq1}) = 0$ for $t \in [t_1, t_2]$. Repeating this progress, we find that $\beta(\{u_n(t)\}_{n\geq1}) = 0$ and $\beta(\{f_n(t)\}_{n\geq1}) = 0$ for $t \in [t_{m-1}, t_m]$. Hence $\beta(\{u_n(t)\}_{n\geq1}) = 0$ and $\beta(\{f_n(t)\}_{n\geq1}) = 0$ for $t \in [0, t_m]$.

Since $0 \in A0$ and $|F(t, u, v)| \leq \omega_m(t)$ for every $t \in [0, t_m]$, therefore, for $t \leq t'$ in $(0, t_1)$, we have

$$\big|u_n(t') - u_n(t)\big| \leq \big|\big|u_n(t')\big| - \big|u_n(t)\big|\big| \leq \int_t^{t'} \big|f_n(s)\big|\,ds \leq \int_t^{t'} \omega(s)\,ds.$$

Then $\{u_n\}$ is equicontinuous in $(0, t_1)$ as $\omega_m \in L([0, t_m])$. Similarly, it can be shown that $\{u_n\}$ is equicontinuous in (t_k, t_{k+1}), $k = 1, 2, \ldots, m-1$. By a PC-type Arzela–Ascoli theorem, $\{u_n\}$ has a convergent subsequence on $[0, t_m]$, denoted by $\{u_{n_k}\}$, such that $u_{n_k} \to u$.

We know that $\beta(\{f_{n_k}(t)\}_{n\geq 1}) = 0$. Then, from (F2)$'$ and the Dunford–Pettis theorem, it follows that, up to a subsequence, still denoted by $\{f_{n_k}\}$, $f_{n_k} \rightharpoonup f \in L^1([0,t_m];E)$. By the continuity of impulse functions I_j, we have $u(t) = \tilde{S}^m_\varphi(f)(t)$. To prove that $f(t) \in F(t,u(t),u_t)$ for a.e. $t \in [0,t_m]$, it is sufficient to use the closed and convex values of F, a weakly upper semicontinuity of $F(t,\cdot,\cdot)$, and some standard procedures based on the Mazur lemma.

Step 4. From Step 3, it follows that $\sup\{\operatorname{dist}(\nu,\Theta^m) : \nu \in \Theta^m_n\} \to 0$ and that $\sup\{\operatorname{dist}(\nu,\Theta^m) : \nu \in \overline{\Theta^m_n}\} \to 0$. This together with the fact that Θ^m is compact and $\Theta^m_{n+1} \subset \Theta^m_n$, implies that $\beta(\Theta^m_n) = \beta(\overline{\Theta^m_n}) \searrow 0$, as $n \to \infty$ and $\Theta^m = \bigcap_{n=1}^\infty \Theta^m_n$.

Step 5. Using the method employed in Step 4 of the proof to Theorem 4.1, it can be shown that $\Theta^m_n(\varphi)$ is a contractible set for each $n \in \mathbb{N}$. Consequently, we deduce that the solution set of problem (4.1) is an R_δ-set. □

Theorem 4.4 *Assume that conditions* (A1), (A3), (F1), (F2), *and* (F3) *hold. If* $I_k : E \to E_0$, $k = 1,\dots,m-1$, *are continuous and there exist constants* $r_k > 0$ *such that*

$$\beta\bigl(I_k(D)\bigr) \leq r_k \sup_{-\tau\leq\theta\leq 0} \beta\bigl(D(\theta)\bigr)$$

for every bounded $D \subset E$, *then* $\Theta(\varphi)$ *is a nonempty and compact subset of* $\mathcal{PC}([-\tau,\infty);E)$ *for every* $\varphi \in \mathcal{C}([-\tau,0];E)$. *Moreover, it is an* R_δ*-set.*

Proof In Theorem 4.3, it has been shown that solution sets for problem (4.1) on compact intervals are R_δ-sets. Next we consider an inverse system as in the proof of Theorem 4.2 and obtain that the solution set of problem (1.1) is an R_δ-set. □

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