Zuo et al. *Boundary Value Problems* (2018) 2018:180
https://doi.org/10.1186/s13661-018-1100-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Superlinear Kirchhoff-type problems of the fractional p-Laplacian without the (AR) condition

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Abstract

In this paper, we study the following superlinear p-Kirchhoff-type equation:

$$\begin{cases} \mathcal{M}(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy)(-\triangle)_p^s u(x) - \lambda|u|^{p-2}u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

Under suitable assumptions on $g(x,u)$ without the (AR) condition, the existence of infinitely many solutions for the Kirchhoff equation of a fractional p-Laplacian is obtained by using the fountain theorem. Our conclusions generalize and extend some existing results.

MSC: 49J35; 35A15; 35S15; 35R11; 47G20

Keywords: Fountain theorem; Kirchhoff-type equation; Fractional p-Laplacian; Ambrosetti–Rabinowitz condition

1 Introduction and the main result

The Kirchhoff-type equation has wide applications in population dynamics, optimization, anomalous diffusion, continuum mechanics, etc. In recent years, research on this subject has been very active; see, for example, [1–10] and the references therein. This paper discuss the existence of infinitely many weak solutions for the following problem:

$$\begin{cases} \mathcal{M}(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy)(-\triangle)_p^s u(x) - \lambda|u|^{p-2}u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 < s < 1 < p < \infty$, $ps < N$. λ is a real parameter and $(-\triangle)_p^s$ is the fractional p-Laplacian operator defined by

$$(-\triangle)_p^s u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dx\,dy,$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x-y| < \varepsilon\}$. The function $\mathcal{M}$ satisfies:

(M_1) $\mathcal{M} \in C(\mathbb{R}_0^+, \mathbb{R}^+)$, $\inf_{t\in\mathbb{R}_0^+} \mathcal{M}(t) \ge m_0 > 0$, m_0 is a positive constant.

(M_2) There exists $\theta \in [1, \frac{N}{N-ps})$, such that

$$\theta \widetilde{\mathcal{M}}(t) = \theta \int_0^t \mathcal{M}(s)\,ds \geq \mathcal{M}(t)t, \quad \forall t \in \mathbb{R}_0^+.$$

In the study of the problems (1.1), the following Ambrosetti–Rabinowitz condition [11] is used widely:

$$0 < \mu G(x,t) = \mu \int_0^t g(x,\tau)\,d\tau \leq g(x,t)t, \quad x \in \Omega, t \geq r, \tag{AR}$$

where $\mu > p\theta$ and $r > 0$.

It is well known that (AR) condition is very important for variational method, but cannot be satisfied in many cases. There have been some contributions attempting to replace this condition by new ones, we can consult the references [12–16]. Motivated by this work, in this paper we investigate the existence of infinitely many solutions of problem (1.1) without the (AR) condition. Our result extends Theorem 1 of [17] and Theorem 1.1 of [18].

We assume that $g : \overline{\Omega} \times \mathbb{R}$ is a continuous function satisfying:

(g_1) There exist constants $1 < \eta_1 < \eta_2 < \cdots < \eta_m < p < \alpha < p_s^*$, and functions $\nu_0(x) \in L^\beta(\Omega)$, where $p_s^* = \frac{pN}{N-ps}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\nu_i(x) \in L^{\frac{\alpha}{\alpha-\eta_i}}(\Omega), i = 1, \ldots, m$, and $\nu_{m+1} > 0$ is a constant such that

$$\bigl|g(x,u)\bigr| \leq \bigl|\nu_0(x)\bigr| + \sum_{i=1}^m \bigl|\nu_i(x)\bigr||u|^{\eta_i - 1} + \nu_{m+1}|u|^{\alpha-1}, \quad (x,u) \in \Omega \times \mathbb{R}.$$

(g_2) There are two constants $\mu > p\theta$ and $\varpi_0 > 0$ such that

$$G(x,t) = \int_0^t g(x,s)\,ds \leq \frac{1}{\mu} g(x,t)t + \varpi_0 |t|^p \quad \text{for any } x \in \Omega, t \in \mathbb{R}.$$

(g_3) $\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^{p\theta}} \to +\infty$ uniformly for $x \in \Omega$.

(g_4) $g(x,t)$ is odd for t, i.e. $g(x,-t) = -g(x,t)$ for any $x \in \Omega$ and $t \in \mathbb{R}$.

Remark (i) Note that condition (g_2) is different from the (AR) condition, and is weaker than the condition of [17, 19] and [18].

(ii) The function $g(x,t) = |t|^{p\theta-2} t \ln(1 + |t|)$ satisfies the conditions (g_2) and (g_3), but it does not satisfy the (AR) condition.

Now we state our main result.

Theorem 1.1 *Set $s \in (0,1)$, $N > ps$. If (M_1)–(M_2) and (g_1)–(g_4) hold. Then, for any $\lambda \in \mathbb{R}$, the problem* (1.1) *has infinitely many weak solutions $\{u_k\}$ in X_0 with unbounded energy.*

The definition of weak solution will be given in the next section. The framework of this paper is as follows. In Sect. 2 we give the variational framework. Section 3 verifies the Cerami condition. In Sect. 4, we establish the existence of infinitely many weak solutions for problem (1.1) by the fountain theorem.

2 Variational framework

In this section, we first review some basic variational frameworks and main Lemmas that will be used in the next section for problem (1.1). Denote $W = \mathbb{R}^{2N}\backslash\mathcal{O}$, where $\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N}$, and $\mathcal{C}(\Omega) = \mathbb{R}^N\backslash\Omega$. Define a normed linear space X by

$$X = \left\{u \in L^p(\Omega) \Big| \int_W \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy < \infty\right\}$$

with norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_W \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}.$$

Then X is a normed linear space and $C_0^\infty(\Omega) \subset X$ (see [10], Lemma 2.1). Define a subspace $X_0 \subset X$ by

$$X_0 = \left\{u \in X : u(x) = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\right\}.$$

Under the equivalent norm

$$\|u\|_{X_0} = \left(\int_W \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}, \quad u \in X_0.$$

X_0 is a uniformly convex reflexive Banach space ([10], Lemma 2.4).

Now, we give the definition of weak solutions for problem (1.1).

Definition 2.1 We say that $u \in X_0$ is a weak solution of problem (1.1), if

$$\begin{aligned}&\mathcal{M}\left(\|u\|_{X_0}^p\right)\int_W \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy\\&\quad - \lambda\int_\Omega \left|u(x)\right|^{p-2}u(x)\phi(x)\,dx - \int_\Omega g\left(x,u(x)\right)\phi(x)\,dx = 0, \quad \forall\phi \in X_0.\end{aligned}$$

Subsequently we review some of the properties of the eigenvalue problem and the spectrum of the operator. Consider the problem:

$$\begin{cases}(-\triangle)_p^s u = \lambda_k |u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{2.1}$$

there is a divergent positive eigenvalue sequence.

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \lambda_{k+1} \le \cdots,$$

whose eigenvalues are the critical values of the functional

$$T_p(u) = \|u\|_{X_0}^p, \quad u \in \Sigma = \left\{u \in X_0 : \int_\Omega |u|^p\,dx = 1\right\}.$$

We notice that the first eigenvalue $\lambda_1 := \inf_{u\in\Sigma} T_p(u) > 0$. The corresponding eigenfunctions will be denoted by e_j. More details can be found in [20].

Let $X_j = \operatorname{span}\{e_j\}$, define

$$X_0 = \overline{\bigoplus_{i=1}^{\infty} X_i}, \qquad Y_k = \bigoplus_{i=1}^{k} X_i, \qquad Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}, \quad k = 1, 2, \ldots.$$

And let $W_k := \{u \in Y_k : \|u\|_{X_0} \leq \rho_k\}$, $N_k := \{u \in Z_k : \|u\|_{X_0} = \gamma_k\}$, where $\rho_k > \gamma_k > 0$.

Lemma 2.2 (Fountain theorem, [21]) *Consider an even functional $T \in C^1(X_0, \mathbb{R})$. Assume for each $k \in \mathbb{N}$, there exist $\rho_k > \gamma_k > 0$ such that*

(Φ_1) $a_k := \max_{u\in Y_k, \|u\|_{X_0}=\rho_k} T(u) \leq 0$,

(Φ_2) $b_k := \inf_{u\in Z_k, \|u\|_{X_0}=\gamma_k} T(u) \to +\infty, k \to +\infty$,

(Φ_3) *T satisfies the $(PS)_c$ condition for every $c > 0$.*

Then T has an unbounded sequence of critical values.

Define the energy functional $T : X_0 \to \mathbb{R}$ corresponding to the problem (1.1) by

$$T(u) = I(u) - J(u) - H(u),$$

where

$$I(u) = \frac{1}{p}\widetilde{\mathcal{M}}\left(\|u\|_{X_0}^p\right), \qquad J(u) = \frac{\lambda}{p}\int_\Omega |u|^p\,dx, \qquad H(u) = \int_\Omega G(x,u)\,dx.$$

Lemma 2.3 ([10]) *If (M_1) holds, then $I : X_0 \to \mathbb{R}$ is of class $C^1(X_0, \mathbb{R})$, and*

$$\langle I'(u), \phi\rangle = \mathcal{M}\left(\|u\|_{X_0}^p\right)\int_W \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy$$

for all $u, \phi \in X_0$.

Lemma 2.4 ([22], Lemma 2) *Assume that g is a continuous function. Let (g_1) holds, then the functional H is of class $C^1(X_0, \mathbb{R})$, and*

$$\langle H'(u), \phi\rangle = \int_\Omega g(x,u)\phi\,dx$$

for all $u, \phi \in X_0$.

Combining Lemma 2.3 with Lemma 2.4, we get $T \in C^1(X_0, \mathbb{R})$ and

$$\begin{aligned}\langle T'(u), \phi\rangle = {}& \mathcal{M}\left(\|u\|_{X_0}^p\right)\int_W \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ & - \lambda\int_\Omega \left|u(x)\right|^{p-2}u(x)\phi(x)\,dx - \int_\Omega g\big(x,u(x)\big)\phi(x)\,dx = 0\end{aligned}$$

for any $u, \phi \in X_0$. Clearly, weak solutions of problem (1.1) are the critical points of energy functional T.

3 Verification of compactness conditions

We firstly state two definitions.

Definition 3.1 ([23, 24]) Let $T \in C^1(X_0, \mathbb{R})$, we say that T satisfies the $(PS)_c$ condition at the level $c \in \mathbb{R}$, if any sequence $\{u_n\}_n \subset X_0$ such that

$$T(u_n) \to c, \quad \sup_{\|\phi\|_{X_0=1}} \{|\langle T'(u_n), \phi\rangle|\} \to 0 \quad \text{as } n \to \infty,$$

possesses a convergent subsequence in X_0; T satisfies the (PS) condition if T satisfies the $(PS)_c$ for all $c \in \mathbb{R}$.

Definition 3.2 ([25, 26]) Let $T \in C^1(X, \mathbb{R})$, we say that T satisfies the $(Ce)_c$ condition at the level $c \in \mathbb{R}$, if any sequence $\{u_n\}_n \subset X_0$ such that

$$T(u_n) \to c, \quad (1 + \|u_n\|) \sup_{\|\phi\|_{X_0=1}} \{|\langle T'(u_n), \phi\rangle|\} \to 0 \quad \text{as } n \to \infty,$$

possesses a convergent subsequence in X_0; T satisfies the (Ce) condition if T satisfies the $(Ce)_c$ for all $c \in \mathbb{R}$.

When T fulfills the (AR) condition, we know the corresponding energy functional T fulfills the Palais-Smale compactness assumptions, however we dropped the (AR) condition, we show that T fulfills the (Ce) condition.

Lemma 3.3 *Let $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying conditions (M_1)–(M_2) and (g_1)–(g_3). Then T fulfills the* (Ce) *condition at level $c \in \mathbb{R}$.*

Proof Set $c \in \mathbb{R}$. Suppose $\{u_n\}$ satisfies

$$T(u_n) \to c, \quad (1 + \|u_n\|) \sup_{\|\phi\|_{X_0=1}} \{|\langle T'(u_n), \phi\rangle|\} \to 0 \tag{3.1}$$

as $n \to \infty$.

Step 1. We prove the sequence $\{u_n\}$ is bounded in X_0.

Arguing by contradiction, if $\{u_n\}_{n\in\mathbb{N}}$ is unbounded in X_0. Up to subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, suppose

$$\|u_n\|_{X_0} \to +\infty. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\sup_{\|\phi\|_{X_0=1}} \{|\langle T'(u_n), \phi\rangle|\} \to 0. \tag{3.3}$$

Thus

$$\|u_n\|_{X_0} \sup_{\|\phi\|_{X_0=1}} \{|\langle T'(u_n), \phi\rangle|\} \to 0 \tag{3.4}$$

as $n \to \infty$. For any $n \in \mathbb{N}$, we consider $\nu_n := \frac{u_n}{\|u_n\|_{X_0}}$, then $\|\nu_n\|_{X_0} = 1$, so $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded in X_0. Similarly to Lemma 1 in [22]. Going if necessary to a subsequence, there exists ν_∞ such that

$$\nu_n \to \nu_\infty \quad \text{in } L^p(\mathbb{R}^N), \tag{3.5}$$

$$\nu_n \to \nu_\infty \quad \text{in } L^\alpha(\mathbb{R}^N), \tag{3.6}$$

$$\nu_n \to \nu_\infty \quad \text{in } \mathbb{R}^N, \tag{3.7}$$

as $n \to \infty$. We discuss two cases.

Case 1. $\nu_\infty \equiv 0$.

By (g_2), (M_2) and (3.1)–(3.2), we obtain

$$\begin{aligned}
&\frac{1}{\|u_n\|_{X_0}^p}\left(T(u_n) - \frac{1}{\mu}T'(u_n)u_n\right) \\
&\quad\geq \frac{1}{\|u_n\|_{X_0}^p}\left(\frac{1}{p}\widetilde{\mathcal{M}}\big(\|u_n\|_{X_0}^p\big) - \frac{1}{\mu}\mathcal{M}\big(\|u_n\|_{X_0}^p\big)\|u_n\|_{X_0}^p \right.\\
&\qquad\left. + \lambda\left(\frac{1}{\mu} - \frac{1}{p}\right)\|u_n\|_p^p - \int_\Omega \left(G\big(x, u_n(x)\big) - \frac{1}{\mu}g\big(x, u_n(x)\big)u_n(x)\right)dx\right) \\
&\quad\geq \frac{1}{\|u_n\|_{X_0}^p}\left(\left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\mathcal{M}\big(\|u_n\|_{X_0}^p\big)\|u_n\|_{X_0}^p\right) - \lambda\left(\frac{1}{\mu} - \frac{1}{p}\right)\int_\Omega \nu_n^p\,dx - \varpi_0\int_\Omega |\nu_n|^p\,dx \\
&\quad\geq m_0\left(\frac{1}{p\theta} - \frac{1}{\mu}\right),
\end{aligned}$$

which implies $0 \geq m_0(\frac{1}{p\theta} - \frac{1}{\mu})$. This is a contradiction.

Case 2. $\nu_\infty \not\equiv 0$.

Setting $\Omega_1 = \{x \in \Omega : \nu_\infty \neq 0\}$, it is easy to see that $|\Omega_1| > 0$ and

$$\big|u_n(x)\big| = |\nu_n|\|u_n\|_{X_0} \to +\infty \quad \text{on } \Omega_1 \tag{3.8}$$

as $n \to \infty$, thanks to (3.2), (3.7). From (3.1) and (3.2), we get $\frac{T(u_n)}{\|u_n\|_{X_0}^{p\theta}} \to 0$, that is,

$$\begin{aligned}
0 = \lim_{n\to\infty} \frac{1}{\|u_n(x)\|_{X_0}^{p\theta}}&\left(\frac{1}{p}\widetilde{\mathcal{M}}\big(\|u_n\|_{X_0}^p\big) - \frac{\lambda}{p}\|u_n\|_p^p - \int_{\Omega_1} G(x, u_n)\,dx\right. \\
&\left. - \int_{\Omega\setminus\Omega_1} G(x, u_n)\,dx\right).
\end{aligned} \tag{3.9}$$

Note that

$$0 < \lambda_1 = \min_{u\in X_0\setminus\{0\}} \frac{\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\int_\Omega |u(x)|^p\,dx},$$

which implies

$$\|u\|_{L^p(\Omega)}^p \leq \frac{1}{\lambda_1}\|u\|_{X_0}^p. \tag{3.10}$$

Because of (M_2), we get

$$\widetilde{\mathcal{M}}(t) \le \widetilde{\mathcal{M}}(1)t^{\theta}, \quad \forall t \in [1,+\infty). \tag{3.11}$$

By (3.9)–(3.11), we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\frac{1}{p}\widetilde{\mathcal{M}}\bigl(\|u_n\|_{X_0}^p\bigr) - \frac{\lambda}{p}\|u_n\|_p^p - \int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad - \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr) \\ &\le \frac{1}{p}\widetilde{\mathcal{M}}(1) - \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\frac{\lambda}{p}\|u_n\|_p^p + \int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad + \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr). \end{aligned} \tag{A}$$

When $\lambda \ge 0$, from (A), we have

$$\begin{aligned} 0 &\le \frac{1}{p}\widetilde{\mathcal{M}}(1) - \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\frac{\lambda}{p}\|u_n\|_p^p + \int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad + \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr) \\ &\le \frac{1}{p}\widetilde{\mathcal{M}}(1) - \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad + \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr). \end{aligned} \tag{3.12}$$

When $\lambda < 0$, from (A) and (3.10), we get

$$\begin{aligned} 0 &\le \frac{1}{p}\widetilde{\mathcal{M}}(1) - \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\frac{\lambda}{p}\|u_n\|_p^p + \int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad + \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr) \\ &\le \frac{1}{p}\widetilde{\mathcal{M}}(1) - \lim_{n\to\infty} \frac{\lambda}{p\lambda_1\|u_n\|_{X_0}^{p\theta-p}} - \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad + \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr) \\ &= \frac{1}{p}\widetilde{\mathcal{M}}(1) - \lim_{n\to\infty} \frac{1}{\|u_n\|_{X_0}^{p\theta}} \biggl(\int_{\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \\ &\quad + \int_{\Omega\setminus\Omega_1} G\bigl(x,u_n(x)\bigr)\,dx \biggr). \end{aligned} \tag{3.13}$$

It follows from (g_3) and (3.8) that

$$\frac{G(x,u_n(x))}{\|u_n\|_{X_0}^{p\theta}} = \frac{G(x,u_n(x))}{|u_n(x)|^{p\theta}}\frac{|u_n(x)|^{p\theta}}{\|u_n\|_{X_0}^{p\theta}} = \frac{G(x,u_n(x))}{|u_n(x)|^{p\theta}}\bigl|v_n(x)\bigr|^{p\theta} \to +\infty \quad \text{a.e. } x \in \Omega_1$$

as $n \to \infty$. Making use of the Fatou lemma, we obtain

$$\lim_{n\to\infty} \int_{\Omega_1} \frac{G(x,u_n(x))}{|u_n(x)|^{p\theta}}\,dx \to +\infty. \tag{3.14}$$

By (g_3), we know

$$\lim_{t\to+\infty} G(x,t) = +\infty \tag{3.15}$$

uniformly for every $x \in \overline{\Omega}$. Therefore, (3.15) means that there are two positive constants t_1 and D such that

$$G(x,t) \geq D \tag{3.16}$$

for any $x \in \overline{\Omega}$ and $|t| > t_1$. In addition, since the continuity of G on $\overline{\Omega} \times \mathbb{R}$, we get

$$G(x,t) \geq \min_{(x,t)\in\overline{\Omega}\times[-t_1,t_1]} G(x,t), \quad \forall |t| \leq t_1. \tag{3.17}$$

Hence, in view of (3.16) and (3.17), we have

$$G(x,t) \geq \min\Big\{D, \min_{(x,t)\in\overline{\Omega}\times[-t_1,t_1]} G(x,t)\Big\} := \varrho, \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.18}$$

By (3.2) and (3.18), we obtain

$$\lim_{n\to\infty} \int_{\Omega\setminus\Omega_1} \frac{G(x,u_n(x))}{|u_n(x)|^{p\theta}}\,dx \geq 0. \tag{3.19}$$

Combining (3.19) and (3.12)–(3.14), we have a contradiction.

Step 2. We prove $\{u_n\} \to u$ in X_0 for some u.

Let $K(x-y) = |x-y|^{-N-ps}$. For every fixed $\varphi \in X_0$, define

$$Q_\varphi(v) = \int_{\mathbb{R}^{2N}} \big|\varphi(x) - \varphi(y)\big|^{p-2}\big(\varphi(x) - \varphi(y)\big)\big(v(x) - v(y)K(x-y)\,dx\,dy$$

for any $v \in X_0$. By the Hölder inequality and the continuity of Q_φ, we get

$$\big|Q_\varphi(v)\big| \leq \|\varphi\|_{X_0}^{p-1}\|v\|_{X_0}, \quad \forall v \in X_0.$$

Since $u_n \rightharpoonup u$ in X_0, we have

$$\lim_{n\to\infty} Q_u(u_n - u) = 0. \tag{3.20}$$

Obviously, $\langle T'(u_n), u_n - u\rangle \to 0$. Then we have

$$\begin{aligned}\big\langle T'(u_n), u_n - u\big\rangle &= \mathcal{M}\big(\big\|u_n(x)\big\|_{X_0}^p\big)Q_{u_n}(u_n - u) - \lambda \int_\Omega |u_n|^{p-2}u_n(u_n - u)\,dx \\ &\quad - \int_\Omega g(x,u_n)(u_n - u)\,dx \to 0\end{aligned} \tag{3.21}$$

as $n \to \infty$. Due to the reflexivity of X_0, similarly to Lemma 8 in [28], there is a subsequence, still denoted by $\{u_n\}_n$, such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X_0, \\ &u_n \to u \quad \text{in } L^r(\Omega), 1 \le r < p_s^*, \\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^N \end{aligned}$$

as $n \to \infty$. So, $g(x,u_n)(u_n - u) \to 0$ a.e. in Ω as $n \to \infty$. The sequence $\{g(x,u_n)(u_n - u)\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. By the Vitali Convergence Theorem (see [27]),

$$\lim_{n\to\infty} \int_\Omega g(x,u_n)(u_n - u)\,dx = 0.$$

By (3.21), we have

$$\mathcal{M}\big(\|u_n(x)\|_{X_0}^p\big) Q_{u_n}(u_n - u) \to 0$$

as $n \to 0$. It follows from (M_1),

$$Q_{u_n}(u_n - u) \to 0 \tag{3.22}$$

as $n \to \infty$. Combining (3.20) with (3.22), we get

$$\big(Q_{u_n}(u_n - u) - Q_u(u_n - u)\big) \to 0 \tag{3.23}$$

as $n \to 0$. Using the Simon inequalities

$$\begin{aligned} &\big(|\varrho|^{p-2}\varrho - |\sigma|^{p-2}\sigma\big) \cdot (\varrho - \sigma) \ge D_P|\varrho - \sigma|^p, \quad p \ge 2, \\ &\big(|\varrho|^{p-2}\varrho - |\sigma|^{p-2}\sigma\big) \cdot (\varrho - \sigma) \ge \widehat{D_P}\frac{|\varrho - \sigma|^2}{(|\varrho| + |\sigma|)^{2-p}}, \quad 1 < p < 2, \end{aligned}$$

for all $\varrho, \sigma \in \mathbb{R}^N$, where $D_P, \widehat{D_P} > 0$ only rely on p.

If $p \ge 2$, By (3.23), for n large enough,

$$\begin{aligned} \|u_n - u\|_{X_0}^p \le{}& D_P \int_{\mathbb{R}^{2N}} \big(\big|u_n(x) - u_n(y)\big|^{p-2}\big(u_n(x) - u_n(y)\big) \\ &- \big|u(x) - u(y)\big|^{p-2}\big(u(x) - u(y)\big)\big) \\ &\times \big(u_n(x) - u(x) - u_n(y) + u(y)\big)K(x-y)\,dx\,dy \\ ={}& D_P\big(Q_{u_n}(u_n - u) - Q_u(u_n - u)\big) = o(1). \end{aligned}$$

Then $\|u_n - u\|_{X_0}^p \to 0$.

If $1 < p < 2$, though the Hölder inequality, the Simon inequality, and (3.23), we have

$$\|u_n - u\|_{X_0}^p \le \widehat{D_P}\big(Q_{u_n}(u_n - u) - Q_u(u_n - u)\big)^{\frac{p}{2}} \big(\|u_n\|_{X_0}^p + \|u\|_{X_0}^p\big)^{\frac{2-p}{2}}$$

$$\leq \widehat{D_P}\big(Q_{u_n}(u_n-u)-Q_u(u_n-u)\big)^{\frac{p}{2}}\big(\|u_n\|_{X_0}^{p(2-p)/2}+\|u\|_{X_0}^{p(2-p)/2}\big)$$
$$= C\big(Q_{u_n}(u_n-u)-Q_u(u_n-u)\big)^{\frac{p}{2}} = o(1),$$

as $n \to \infty$, where $C > 0$. Combining the above two cases, thus, $u_n \to u$ in X_0. The proof of Lemma 3.3 is completed. □

4 Proof of Theorem 1.1

Similarly to [5], by a direct calculation, we have the following lemma.

Lemma 4.1 *Set $1 \leq q < p_s^*$ and, for every $k \in \mathbb{N}$, let*

$$\mu_k := \sup\big\{\|u\|_q : u \in Z_k, \|u\|_{X_0} = 1\big\}.$$

Then $\mu_k \to 0$ as $k \to \infty$.

Proof of Theorem 1.1 We only need to verify the conditions (Φ_1) and (Φ_2) of Lemma 2.2.

Verification of (Φ_1): Since Y_k is finite dimensional, there exist positive constants $B_{k,q}$ and $\widetilde{B_{k,q}}$ depending on k, q, such that for each $u \in Y_k$

$$B_{k,q}\|u\|_{X_0} \leq \|u\|_q \leq \widetilde{B_{k,q}}\|u\|_{X_0}. \tag{4.1}$$

In view of (g_3), for every $c > \frac{\widetilde{\mathcal{M}}(1)}{pB_{k,p\theta}^{p\theta}}$, there exists $\delta_c > 0$ such that

$$G(x,t) \geq c|t|^{p\theta} \tag{4.2}$$

for all $x \in \overline{\Omega}, |t| > \delta_c$. According to the Weierstrass theorem, we get

$$G(x,t) \geq m_c := \min_{x\in\overline{\Omega},|t|\leq\delta_c} G(x,t) \tag{4.3}$$

for any $|t| \leq \delta_c$. We claim that $m_c \leq 0$, since $G(x,0) = 0$ for all $x \in \overline{\Omega}$. Therefore, by (4.2) and (4.3), we obtain

$$G(x,t) \geq c|t|^{p\theta} - H_c \tag{4.4}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, for suitable positive constant $H_c \geq c\delta_c - m_c$.

By (4.1), (4.4) and (3.11), we get

$$\begin{aligned} T(u) &\leq \frac{1}{p}\widetilde{\mathcal{M}}\big(\|u\|_{X_0}^p\big) - \frac{\lambda}{p}\|u\|_p^p - c\|u\|_{p\theta}^{p\theta} + H_c|\Omega| \\ &\leq \frac{\widetilde{\mathcal{M}}(1)}{p}\|u\|_{X_0}^{p\theta} - \frac{\lambda}{p}\|u\|_p^p - c\|u\|_{p\theta}^{p\theta} + H_c|\Omega| \\ &\leq \frac{\widetilde{\mathcal{M}}(1)}{p}\|u\|_{X_0}^{p\theta} - cB_{k,p\theta}^{p\theta}\|u\|_{X_0}^{p\theta} - \frac{\lambda}{p}B_{k,p}^p\|u\|_{X_0}^p + H_c|\Omega|. \end{aligned}$$

So, we have, for any $u \in Y_k$ with $\|u\|_{X_0} = \rho_k \geq 1$ for big enough, such that $T \leq 0$. The condition (Φ_1) holds.

Verification of (Φ_2): There exists $\gamma_k > 0$ such that

$$b_k := \inf\{T(u) : u \in Z_k, \|u\|_{X_0} = \gamma_k\} \to +\infty$$

as $k \to +\infty$. It follows from (g_1), (M_1) and the Hölder inequality that

$$\begin{aligned}
T(u) &\geq \frac{1}{p}\widetilde{\mathcal{M}}\left(\|u\|_{X_0}^p\right) - \frac{\lambda}{p}\|u\|_p^p - \|v_0\|_\beta\|u\|_\alpha - \sum_{i=1}^m \|v_i(x)\|_{\frac{\alpha}{\alpha-\eta_i}}\|u\|_\alpha^{\eta_i} - v_{m+1}\|u\|_\alpha^\alpha \\
&\geq C_{K,\lambda}\|u\|_{X_0}^p - \|v_0\|_\beta \left\|\frac{u}{\|u\|_{X_0}}\right\|_\alpha \|u\|_{X_0} \\
&\quad - \sum_{i=1}^m \|v_i(x)\|_{\frac{\alpha}{\alpha-\eta_i}} \left\|\frac{u}{\|u\|_{X_0}}\right\|_\alpha^{\eta_i} \|u\|_{X_0}^{\eta_i} - v_{m+1}\left\|\frac{u}{\|u\|_{X_0}}\right\|_\alpha^\alpha \|u\|_{X_0}^\alpha \\
&\geq C_{K,\lambda}\|u\|_{X_0}^p - \mu_k\|v_0\|_\beta\|u\|_{X_0} - \sum_{i=1}^m \|v_i(x)\|_{\frac{\alpha}{\alpha-\eta_i}}\mu_k^{\eta_i}\|u\|_{X_0}^{\eta_i} - v_{m+1}\mu_k^\alpha\|u\|_{X_0}^\alpha,
\end{aligned}$$

where μ_k is defined in Lemma 4.1 and

$$C_{K,\lambda} = \begin{cases} \frac{1}{2}m_0 & \text{if } \lambda \leq 0, \\ \frac{1}{2}(m_0 - \frac{\lambda}{\lambda_1}) & \text{if } 0 < \lambda < \lambda_1, \\ \frac{1}{2}(m_0 - \frac{\lambda}{\lambda_k}) & \text{if } 0 < \lambda_k < \lambda_{k+1}. \end{cases}$$

We define γ_k as

$$\gamma_k = \left(\frac{pC_{K,\lambda}}{\alpha v_{m+1}\mu_k^\alpha}\right)^{\frac{1}{\alpha-p}}.$$

Thus $\gamma_k \to +\infty$ as $k \to \infty$. Note that $\alpha > p$, so for every $\|u\|_{X_0} = \gamma_k$, we get

$$\begin{aligned}
T(u) &\geq \|u\|_{X_0}^p\left(C_{K,\lambda} - v_{m+1}\mu_k^\alpha\|u\|_{X_0}^{\alpha-p}\right) - \mu_k\|v_0\|_\beta\gamma_k - \sum_{i=1}^m \|v_i(x)\|_{\frac{\alpha}{\alpha-\eta_i}}\mu_k^{\eta_i}\gamma_k^{\eta_i} \\
&= \left(1 - \frac{p}{\alpha}\right)C_{K,\lambda}\gamma_k^p - \mu_k\|v_0\|_\beta\gamma_k - \sum_{i=1}^m \|v_i(x)\|_{\frac{\alpha}{\alpha-\eta_i}}\mu_k^{\eta_i}\gamma_k^{\eta_i} \to +\infty
\end{aligned}$$

as $k \to \infty$. Therefore, the condition (Φ_2) is satisfied. The proof is completed. □

5 Conclusion

In this article, the existence of infinitely many solutions to Eq. (1.1) is established by using the variational methods (i.e. the fountain theorem). We consider fractional p-Kirchhoff problems with more general nonlinearity g in Ω, which improves the previous results. In order to overcome new difficulties, we need to adopt special techniques and methods in our paper.

Acknowledgements
The authors thank the anonymous referees for invaluable comments and insightful suggestions which improved the presentation of this manuscript.

Funding
The work is supported by the Fundamental Research Funds for Central Universities (2017B19714 and 2017B07414), Natural Science Foundation of Jiangsu Province (BK20180500) and Natural Science Foundation of Jilin Engineering Normal University (XYB201814).

Abbreviation
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 21 August 2018 Accepted: 26 November 2018 Published online: 04 December 2018

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