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Positive periodic solutions for nonlinear first-order delayed differential equations at resonance

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Abstract

This paper studies the existence of positive periodic solutions of the following delayed differential equation:

$$u' + a(t)u = f(t, u(t - \tau(t))),$$

where $a, \tau \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic functions with $\int_0^\omega a(t)\,dt = 0$, $f : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is continuous and ω-periodic with respect to t. By means of the fixed point theorem in cones, several new existence theorems are established. Our main results enrich and complement those available in the literature.

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Keywords: Positive periodic solutions; Existence; Fixed point; Resonance

1 Introduction

In the past few years, there has been considerable interest in the existence of positive periodic solutions for the first-order equation

$$u' + a(t)u = \lambda b(t) f\big(u\big(t - \tau(t)\big)\big), \tag{1.1}$$

where $a, b \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions with

$$\int_0^\omega a(t)\,dt > 0, \qquad \int_0^\omega b(t)\,dt > 0,$$

and τ is a continuous ω-periodic function. Note that when $\lambda = 0$, equation (1.1) reduces to $u' = -a(t)u$, which is well known in Malthusian population models. In real world applications, (1.1) has also been viewed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, as well as cardiac arrhythmias. See, for instance, [1–10] for some research works on this topic. Meanwhile, many researchers have also paid their attention to the differential systems corresponds to (1.1), namely,

$$u_i' + a_i(t)u_i = \lambda b_i(t) f_i(u_1, u_2, \dots, u_n), \quad i = 1, 2, \dots, n,$$

where $a_i, b_i \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions satisfying

$$\int_0^\omega a_i(t)\,dt > 0, \qquad \int_0^\omega b_i(t)\,dt > 0, \quad i = 1, 2, \ldots, n.$$

Here we refer the readers to [11–13] and the references listed therein.

Obviously, the basic assumption $\int_0^\omega a(t)\,dt > 0$ or $\int_0^\omega a_i(t)\,dt > 0$ ($i = 1, 2, \ldots, n$), usually employed to ensure the linear boundary value problem

$$u' + a(t)u = 0, \qquad u(0) = u(\omega) \tag{1.2}$$

is non-resonant, has played a key role in the arguments of the above mentioned papers. In fact, this assumption ensures that a number of tools, such as fixed point theory, bifurcation theory and so on, could be employed to study the corresponding problems and establish the desired existence results. Here the linear problem (1.2) is called non-resonant if the unique solution of it is the trivial one. It is well known if (1.2) is non-resonant then, provided h is an L^1-function, the Fredholm's alternative theorem implies that the nonhomogeneous problem

$$u' + a(t)u = h(t), \qquad u(0) = u(\omega)$$

always admits a unique solution which, moreover, can be written as

$$u(t) = \int_0^\omega G(t,s)h(s)\,ds,$$

where $G(t,s)$ is the Green's function associated to (1.2), see [7–13] for more details.

Compared with the non-resonant problems, the research of resonant problems proceeds very slowly and the related results are few. And, of course, a natural and interesting question is whether or not the corresponding nonlinear equation possesses a positive periodic solution, provided that

$$\int_0^\omega a(t)\,dt = 0,$$

which means a may change its sign on $\mathbb{R}$ and the studied problem is resonant. In the present paper, we shall give a positive answer to the above question. More concretely, several new existence and multiplicity results will be established for the resonant equation

$$u' + a(t)u = f\bigl(t, u\bigl(t - \tau(t)\bigr)\bigr). \tag{1.3}$$

To the best of our knowledge, the above problem has not been studied so far, and our results shall fill this gap.

For simplicity, we say a function $q \gg 0$ provided that $q : \mathbb{R} \to (0,\infty)$ is ω-periodic and continuous. If $q : \mathbb{R} \to [0,\infty)$ is ω-periodic and continuous with $\int_0^\omega q(t)\,dt > 0$, then it's denoted as $q \succ 0$. Thus, if we choose a function $\chi \gg 0$ such that $p := a + \chi \succ 0$, then the linear differential operator $Lu := u' + p(t)u$ is invertible since

$$\int_0^\omega p(t)\,dt = \int_0^\omega a(t)\,dt + \int_0^\omega \chi(t)\,dt = \int_0^\omega \chi(t)\,dt > 0.$$

Moreover, it is not difficult to see that u is a positive periodic solution of (1.3) if and only if it is a positive periodic solution of

$$u' + p(t)u = \chi(t)u + f\big(t, u\big(t - \tau(t)\big)\big). \tag{1.4}$$

Therefore, we shall focus on (1.4).

Throughout the paper, we make the following assumptions:

(H1) $a, \tau \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic functions with $\int_0^\omega a(t)\,dt = 0$;

(H2) There exists $\chi \gg 0$ such that $p := a + \chi \succ 0$;

(H3) $f \in C(\mathbb{R} \times [0, \infty), \mathbb{R})$ is ω-periodic with respect to t and $f(t, u) \geq -\chi(t)u$.

Remark 1.1 Obviously, since a and f are sign-changing, equation (1.3) is more general than corresponding ones studied in the existing literature. For other existence results on nonlinear differential equations at resonance, we refer the readers to [14–17] and the references listed therein.

The rest of the paper is arranged as follows. In Sect. 2, we introduce some preliminaries. Finally, in Sect. 3, we shall state and prove our main results. In addition, several remarks will be given to demonstrate the feasibility of our main results.

2 Preliminaries

Let us consider the linear boundary value problem

$$\begin{cases} u' + p(t)u = h(t), & t \in (0, \omega), \\ u(0) = u(\omega), \end{cases} \tag{2.1}$$

where p is defined as in (H2). If we denote by $K(t, s)$ the Green's function of (2.1), then a simple calculation gives

Lemma 2.1 *Suppose* (H1) *and* (H2) *hold. Let* $\delta = e^{-\int_0^\omega p(t)\,dt}$. *Then*

$$K(t, s) = \frac{e^{\int_t^s p(\theta)\,d\theta}}{\delta^{-1} - 1}, \quad t \leq s \leq t + \omega.$$

Moreover,

$$\frac{1}{\delta^{-1} - 1} \leq K(t, s) \leq \frac{\delta^{-1}}{\delta^{-1} - 1}, \quad t \leq s \leq t + \omega.$$

Let E be the Banach space of continuous ω-periodic functions equipped with the norm $\|u\| = \max_{t \in [0,\omega]} |u(t)|$. For $h \in E$, define

$$(Ah)(t) := \int_t^{t+\omega} K(t, s)h(s)\,ds. \tag{2.2}$$

Then we have

Lemma 2.2 *Suppose* (H1) *and* (H2) *hold. Then* $A : E \to E$ *is a completely continuous linear operator. Moreover, if* $h \succ 0$, *then* $(Ah)(t) > 0$ *on* $[0, \omega]$.

Proof By a standard argument, we can easily show that A is a linear completely continuous operator. In addition, Lemma 2.1 yields $K(t,s) > 0$ for any (t,s), which, together with $h \succ 0$, implies $(Ah)(t) > 0$ on $[0,\omega]$. □

Let

$$m := \frac{1}{\delta^{-1}-1}, \qquad M := \frac{\delta^{-1}}{\delta^{-1}-1}, \qquad \sigma := \frac{m}{M} < 1,$$

and

$$\mathcal{P} = \bigl\{u \in E : u(t) \geq \sigma \|u\|, t \in [0,\omega]\bigr\}.$$

Then $\mathcal{P}$ is a positive cone in E. Moreover, it is not difficult to check that (1.4) can be rewritten as an equivalent operator equation

$$u(t) = \int_t^{t+\omega} K(t,s)\bigl(\chi(s)u(s) + f\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr)\,ds =: (Tu)(t).$$

Lemma 2.3 *Suppose* (H1), (H2) *and* (H3) *hold. Then* $T(\mathcal{P}) \subseteq \mathcal{P}$ *and* $T : \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof Assume $u \in \mathcal{P}$, then $u(t) \geq \sigma\|u\|$. It follows from (H3) that $\chi(s)u(s) + f(s, u(s-\tau(s)))$ is nonnegative, and therefore

$$\begin{aligned} (Tu)(t) &= \int_t^{t+\omega} K(t,s)\bigl(\chi(s)u(s) + f\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr)\,ds \\ &\leq \int_t^{t+\omega} M\bigl(\chi(s)u(s) + f\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr)\,ds. \end{aligned} \tag{2.3}$$

Applying (H3) again, we get

$$\begin{aligned} (Tu)(t) &= \int_t^{t+\omega} K(t,s)\bigl(\chi(s)u(s) + f\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr)\,ds \\ &\geq \int_t^{t+\omega} m\bigl(\chi(s)u(s) + f\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr)\,ds \\ &= \sigma \int_t^{t+\omega} M\bigl(\chi(s)u(s) + f\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr)\,ds. \end{aligned}$$

This, together with (2.3), yields $T(\mathcal{P}) \subseteq \mathcal{P}$. Finally, by Lemma 2.2 and an argument similar to that of [12, Lemmas 2.2, 2.3] with obvious changes, we can prove that T is a completely continuous operator. □

The following lemma is crucial to prove our main results.

Lemma 2.4 (Guo–Krasnoselskii's fixed point theorem [18]) *Let E be a Banach space, and let $\mathcal{P} \subseteq E$ a cone. Assume Ω_1, Ω_2 are two open bounded subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subseteq \Omega_2$, and let $T : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that*

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.
Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we state and prove our main findings.

Theorem 3.1 *Let* (H1)–(H3) *hold. If*

$$\lim_{u\to 0+} \frac{f(t,u)}{u} = -\chi(t) \tag{3.1}$$

and

$$\lim_{u\to +\infty} \frac{f(t,u)}{u} = \infty, \quad \text{uniformly for } t \in [0,\omega], \tag{3.2}$$

then (1.3) *admits at least one positive ω-periodic solution.*

Proof For $0 < r < R < \infty$, setting

$$\Omega_1 = \{u \in E : \|u\| < r\}, \qquad \Omega_2 = \{u \in E : \|u\| < R\},$$

we have $0 \in \Omega_1$, $\bar{\Omega}_1 \subseteq \Omega_2$.

It follows from (3.1) that there exists $r > 0$ so that for any $0 < u \leq r$,

$$f(t,u) \leq cu - \chi(t)u,$$

where c is a positive constant satisfying $cM\omega < 1$. Therefore, for $u \in \mathcal{P}$ with $\|u\| = r$,

$$f(t,u) + \chi(t)u \leq cu, \quad t \in [0,\omega].$$

Moreover, $0 < \sigma\|u\| \leq u(t) \leq \|u\| = r$. Thus,

$$(Tu)(t) \leq cM\|u\| \int_t^{t+\omega} ds = cM\omega\|u\| < \|u\|,$$

which implies $\|Tu\| \leq \|u\|$, $\forall u \in \mathcal{P} \cap \partial\Omega_1$.

On the other hand, (3.2) yields the existence of $\tilde{R} > 0$ such that for any $u \geq \tilde{R}$,

$$f(t,u) \geq \eta u,$$

where $\eta > 0$ is a constant large enough such that $\sigma m\omega(\eta + \underline{\chi}) > 1$ and $\underline{\chi} = \min_{t\in[0,\omega]} \chi(t)$. Fixing $R > \max\{r, \frac{\tilde{R}}{\sigma}\}$ and letting $u \in \mathcal{P}$ with $\|u\| = R$, we get $u(t) \geq \sigma\|u\| = \sigma R > \tilde{R}$, and therefore

$$f(t,u) + \chi(t)u \geq \eta u + \chi(t)u \geq \sigma(\eta + \underline{\chi})\|u\|, \quad t \in [0,\omega].$$

Consequently, for $u \in \mathcal{P}$ with $\|u\| = R$, we can conclude

$$\begin{aligned}(Tu)(t) &= \int_t^{t+\omega} K(t,s)\big(\chi(s)u(s) + f\big(s, u\big(s-\tau(s)\big)\big)\big)\,ds \\ &\geq \sigma m(\eta + \underline{\chi})\|u\| \int_t^{t+\omega} ds = \sigma m\omega(\eta + \underline{\chi})\|u\| > \|u\|.\end{aligned}$$

Hence $\|Tu\| \geq \|u\|$, $\forall u \in \mathcal{P} \cap \partial\Omega_2$.

Consequently, by Lemma 2.4(i), T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, which is just a positive ω-periodic solution of (1.3). □

Theorem 3.2 *Let* (H1)–(H3) *hold. If*

$$\lim_{u\to 0+} \frac{f(t,u)}{u} = \infty, \quad \textit{uniformly for } t \in [0,\omega], \tag{3.3}$$

and

$$\lim_{u\to +\infty} \frac{f(t,u)}{u} = -\chi(t), \tag{3.4}$$

then (1.3) *admits at least one positive ω-periodic solution.*

Proof We follow the same strategy and notations as in the proof of Theorem 3.1. Firstly, we show that for $r > 0$ sufficiently small,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.5}$$

From (3.3) it follows that there exists $\tilde{r} > 0$ such that $f(t,u) \geq \beta u$ for any $0 < u \leq \tilde{r}$, where $\beta > 0$ is a constant large enough such that $\sigma m\omega(\beta + \underline{\chi}) > 1$. Therefore, for $0 < r \leq \tilde{r}$, if $u \in \mathcal{P}$ and $\|u\| = r$, then

$$f(t,u) + \chi(t)u \geq \beta u + \chi(t)u \geq \sigma(\beta + \underline{\chi})\|u\|, \quad t \in [0,\omega].$$

Furthermore, we obtain

$$\begin{aligned}(Tu)(t) &= \int_t^{t+\omega} K(t,s)\big(\chi(s)u(s) + f\big(s, u\big(s-\tau(s)\big)\big)\big)\,ds \\ &\geq \sigma m(\beta + \underline{\chi})\|u\| \int_t^{t+\omega} ds = \sigma m\omega(\beta + \underline{\chi})\|u\| > \|u\|.\end{aligned}$$

Thus, (3.5) is true.

Next we show for $R > 0$ sufficiently large,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.6}$$

It follows from (3.4) that there exists $\tilde{R} > 0$ so that for any $u \geq \tilde{R}$,

$$f(t,u) \leq \mu u - \chi(t)u,$$

where $\mu > 0$ satisfies $\mu M\omega < 1$. Let $R > \max\{\tilde{r}, \frac{\tilde{R}}{\sigma}\}$, then if $u \in \mathcal{P}$ and $\|u\| = R$, we can obtain

$$u(t) \geq \sigma \|u\| = \sigma R > \tilde{R},$$

and therefore,

$$f(t,u) + \chi(t)u \leq \mu u \leq \mu \|u\|, \quad t \in [0,\omega].$$

Thus for $u \in \mathcal{P}$ with $\|u\| = R$, we have

$$(Tu)(t) \leq \mu M \|u\| \int_t^{t+\omega} ds = \mu M\omega \|u\| < \|u\|,$$

which means that (3.6) is also true.

Finally, it follows from Lemma 2.4(ii) that T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, which is just a positive ω-periodic solution of (1.3). □

In the rest of this section, we shall investigate the multiplicity of positive ω-periodic solutions of (1.3). To the end, we assume:

(H4) $\lim_{u\to 0+} \frac{f(t,u)}{u} = \infty$ and $\lim_{u\to +\infty} \frac{f(t,u)}{u} = \infty$ uniformly for $t \in [0,\omega]$. In addition, there is a constant $\alpha > 0$ such that

$$\max\{f(t,u) : \sigma\alpha \leq u \leq \alpha, t \in [0,\omega]\} \leq (\epsilon - \chi(t))\alpha, \tag{3.7}$$

where $\epsilon > 0$ satisfies $\epsilon M\omega < 1$.

Theorem 3.3 *Assume that* (H1)–(H4) *hold. Then* (1.3) *admits at least two positive ω-periodic solutions.*

Proof Define

$$\Omega_3 = \{u \in E : \|u\| < \alpha\}.$$

Let Ω_1 and Ω_2 be as in the proof of Theorems 3.1 and 3.2. Then for $0 < r < \alpha < R$, we have $\bar{\Omega}_1 \subseteq \Omega_3$, $\bar{\Omega}_3 \subseteq \Omega_2$.

Since $\lim_{u\to 0+} \frac{f(t,u)}{u} = \infty$ uniformly for $t \in [0,\omega]$, by an argument similar to the proof of Theorem 3.2, we can obtain

$$\|Tu\| \geq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_1.$$

Similarly, we can show by (H4) that

$$\|Tu\| \geq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_2.$$

Clearly, the proof is completed if we prove

$$\|Tu\| \leq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.8}$$

Suppose $u \in \mathcal{P}$ and $\|u\| = \alpha$, then $\sigma\alpha \leq \sigma\|u\| \leq u(t) \leq \|u\| = \alpha$, and from (3.7) it follows

$$f(t,u) + \chi(t)u \leq f(t,u) + \chi(t)\alpha \leq \epsilon\alpha, \quad t \in [0,\omega].$$

Thus, we get

$$\begin{aligned}(Tu)(t) &= \int_t^{t+\omega} K(t,s)\big(\chi(s)u(s) + f\big(s,u\big(s-\tau(s)\big)\big)\big)\,ds \\ &\leq \epsilon M\omega\alpha = \epsilon M\omega\|u\| < \|u\|,\end{aligned}$$

and so (3.8) is satisfied.

Consequently, Lemma 2.4 implies that T has at least two fixed points u_1 and u_2, located in $\mathcal{P} \cap (\bar{\Omega}_3 \setminus \Omega_1)$ and $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_3)$, respectively. And accordingly, (1.3) admits at least two positive ω-periodic solutions. □

If we replace (H4) with

(H4)′ $\lim_{u\to 0+} \frac{f(t,u)}{u} = -\chi(t)$, $\lim_{u\to +\infty} \frac{f(t,u)}{u} = -\chi(t)$, and there exists a constant $\alpha > 0$ such that

$$\min\big\{f(t,u) : \sigma\alpha \leq u \leq \alpha, t \in [0,\omega]\big\} \geq \big(\mu - \sigma\chi(t)\big)\alpha, \tag{3.9}$$

where $\mu > 0$ satisfies $m\omega\mu > 1$.

Then we can obtain the following:

Theorem 3.4 *Let* (H1)–(H3) *and* (H4)′ *hold. Then* (1.3) *admits at least two positive ω-periodic solutions.*

Proof For $0 < r < \alpha < R$, let Ω_i $(i = 1,2,3)$ be as in the proof of Theorems 3.1 and 3.3. Then $\bar{\Omega}_1 \subseteq \Omega_3$, $\bar{\Omega}_3 \subseteq \Omega_2$. We shall follow the same strategy as in the proof of Theorem 3.3.

By (H4)′ and an argument similar to the proof of Theorems 3.1 and 3.2, we can conclude

$$\|Tu\| \leq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_1,$$
$$\|Tu\| \leq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_2.$$

Now, to apply Lemma 2.4, we only need to show

$$\|Tu\| \geq \|u\|, \quad \forall u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.10}$$

Let $u \in \mathcal{P}$ with $\|u\| = \alpha$, then $\sigma\alpha \leq \sigma\|u\| \leq u(t) \leq \|u\| = \alpha$, by (3.9) we get

$$f(t,u) + \chi(t)u \geq f(t,u) + \sigma\alpha\chi(t) \geq \mu\alpha, \quad t \in [0,\omega],$$

and then

$$\begin{aligned}(Tu)(t) &= \int_t^{t+\omega} K(t,s)\big(\chi(s)u(s) + f\big(s,u\big(s-\tau(s)\big)\big)\big)\,ds \\ &\geq m\omega\mu\alpha = m\omega\mu\|u\| > \|u\|,\end{aligned}$$

and therefore (3.10) is true. Using Lemma 2.4 again, we know T has two fixed points u_1 and u_2, located in $\mathcal{P} \cap (\bar{\Omega}_3 \setminus \Omega_1)$ and $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_3)$, respectively. Consequently, (1.3) admits at least two positive ω-periodic solutions. □

Remark 3.1 We would like to point out the results of Theorems 3.1–3.4 remain true for the special resonant equation $u' = f(t, u(t - \tau(t)))$, where $a(\cdot) \equiv 0$.

Remark 3.2 It is worth remarking that Theorems 3.1–3.4 apply to equations which could not be treated by the existing results of [7–10], and therefore our main results are novel.

4 Conclusion

By applying the fixed point theorem in cones, some new existence theorems are established for a class of first-order delayed differential equations. Our main results enrich and complement those available in the literature.

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Authors' contributions
RC carried out the analysis and proof the main results, and was a major contributor in writing the manuscript. XL participated in checking the processes of proofs, English grammar as well as typing errors in the text. All authors read and approved the final manuscript.

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