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On the estimations of the small eigenvalues of Sturm–Liouville operators with periodic and antiperiodic boundary conditions

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Abstract
We give a new approach for the estimations of the eigenvalues of non-self-adjoint Sturm–Liouville operators with periodic and antiperiodic boundary conditions. Moreover, we give error estimations, and finally we present some numerical examples.

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1 Introduction and preliminary facts

Let $L_k(q)$ for $k = 0, 1$ be the operators generated in $L_2[0, 1]$ by the differential expression $-y'' + q(x)y$ and the boundary conditions

$$y(1) = e^{i\pi k} y(0), \qquad y'(1) = e^{i\pi k} y'(0), \tag{1}$$

that is, periodic and antiperiodic boundary conditions, where q is a complex-valued summable function on $[0, 1]$. It is well known that the spectra of these operators are discrete, and for large enough n, there are two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues (counted with multiplicity) in the neighborhood of $(\pi n)^2$. See basics and detailed classical results in [4, 13, 14, 16], and the references therein.

Note also that these boundary conditions are the most commonly studied ones among the regular but not strongly regular boundary conditions. Therefore there exist a lot of studies about the investigation of Riesz basis property. Let us mention only the pioneer papers about it. The first results about the Riesz basis property of such operators were obtained by Kerimov and Mamedov [12]. They established that, if $q \in C^4[0, 1], q(1) \neq q(0)$, then the root functions of the operator $L_0(q)$ form a Riesz basis in $L_2[0, 1]$. The first result in terms of the Fourier coefficients of the potential q was obtained by Dernek and Veliev [6], and this result was essentially improved by Shkalikov and Veliev [19].

There are also many studies about the numerical estimations of the small eigenvalues of the Sturm–Liouville operators with periodic and antiperiodic boundary conditions.

Some popular methods that have been used are the finite difference method, finite element method, Prüfer transformations, and the shooting method. For example, Andrew considered the computations of the eigenvalues by using the finite element method [1] and the finite difference method [2]. Then these results were extended by Condon [5] and by Vanden Berghe et al. [21]. Ji and Wong used Prüfer transformation and the shooting method in their works [10, 11, 23]. Malathi et al. [15] used the shooting method and direct integration method for computing eigenvalues of the periodic Sturm–Liouville problems. One of the interesting approaches was given by Dinibütün and Veliev [7]. They considered the matrix form of the operator $T(p)$ generated in $L_2[0,1]$ by the differential expression $-y'' + q(x)y$ and the boundary conditions

$$y(2\pi) = y(0), \qquad y'(2\pi) = y'(0),$$

where the potential q is in the form $q(x) = p(x) + \sum_{n:|n|>s} q_n e^{inx}$ and $p(x) = \sum_{n:|n|\leq s} q_n e^{inx}$, and they gave an approximation with very small errors for the eigenvalues of the periodic Sturm–Liouville problems.

The eigenvalues of $L_0(0)$ and $L_1(0)$ are $(2n\pi)^2$ and $((2n+1)\pi)^2$ for $n \in \mathbb{Z}$, respectively, and all eigenvalues of $L_0(0)$ and $L_1(0)$ are double except 0. The eigenvalues of the operators $L_0(q)$ and $L_1(q)$ are called the periodic and antiperiodic eigenvalues, and if q is a real periodic function, then they are denoted by $\lambda_{2n}(q)$ and $\lambda_{2n+1}(q)$ for $n \in \mathbb{Z}$, respectively, where

$$\lambda_0(q) < \lambda_{-1}(q) \leq \lambda_1(q) < \lambda_{-2}(q) \leq \lambda_2(q) < \lambda_{-3}(q) \leq \lambda_3(q) < \lambda_{-4}(q) \leq \lambda_4(q) < \cdots$$

[9, see p. 27]. The investigation of the periodic and antiperiodic eigenvalues is crucial because the spectrum of the Hill operator $L(q)$ generated in $L_2[-\infty,\infty]$ by the differential expression $-y'' + q(x)y$ consists of the intervals $[\lambda_{n-1}(q), \lambda_{-n}(q)]$ for $n = 1, 2, \ldots$, for real periodic potentials. Moreover, these intervals are the closure of the stable intervals of the equation

$$-y''(x) + q(x)y(x) = \lambda y(x). \tag{2}$$

The intervals $(\lambda_{-n}, \lambda_n)$ for $n = 1, 2, \ldots$ are the gaps in the spectrum. These intervals with $(-\infty, \lambda_0)$ are the instable intervals of (2) [9, see pages 32 and 82]. The length of the nth gap in the spectrum of $L(q)$ (the length of the $(n+1)$th instable interval of (2)) is

$$\delta_n(q) := \lambda_n(q) - \lambda_{-n}(q).$$

Therefore the investigation of the periodic and antiperiodic eigenvalues is, at the same time, the investigation of the spectrum of the operator $L(q)$ and the stable intervals of (2) for real periodic potentials.

We are interested in the numerical estimations of the small eigenvalues of the operators $L_0(q)$ and $L_1(q)$. In this paper we give a new approach to get subtle estimations for the small periodic and antiperiodic eigenvalues when the complex-valued summable potential is in the form $q(x) = 2\sum_{k=1}^{\infty} q_k \cos 2\pi kx$, where $q_k := (q, e^{i2\pi kx})$ is the Fourier coefficient of q and $(.,.)$ denotes the inner product in $L_2[0,1]$. Without loss of generality, we assume that $q_0 = 0$.

We essentially use the following equation obtained from [6] (see (15) and (17) of [6]):

$$\left[\lambda - (2\pi n)^2 - A(\lambda)\right]\left[\lambda - (2\pi n)^2 - A'(\lambda)\right] - \left(q_{2n} + B(\lambda)\right)\left(q_{-2n} + B'(\lambda)\right) = 0, \tag{3}$$

where the terms in this equation are defined in (14) and (15). Nevertheless, even if we have obtained this equation from [6], the method of investigation is absolutely different. In [6, 19, 22], and [24], they use asymptotic formulas for the large eigenvalues which cannot be used for the small eigenvalues. Note that the asymptotic behaviors of large eigenvalues were investigated in detail (see [3, 8, 16, 17, 20], and the references therein). In this paper we consider the small eigenvalues by numerical methods.

We will focus only on the operator $L_0(q)$. The investigation of $L_1(q)$ is the same. For simplicity of reading, first let us give the brief scheme of the proofs of the main results. To consider the small eigenvalues, first we prove (see Theorem 1) that the small eigenvalues satisfy equation (3), and using this equation we show that the eigenvalue $\lambda_{n,j}$ is the root of one of the equations:

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[\left(A(\lambda) + A'(\lambda)\right) - \sqrt{\Delta(\lambda)}\right], \tag{4}$$

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[\left(A(\lambda) + A'(\lambda)\right) + \sqrt{\Delta(\lambda)}\right], \tag{5}$$

where $\Delta(\lambda) = (A(\lambda) - A'(\lambda))^2 + 4(q_{2n} + B(\lambda))(q_{-2n} + B'(\lambda))$. To use numerical methods, we take finite summations instead of the infinite series in expressions (4) and (5) and show that the eigenvalues are close to the roots of the equations obtained by taking these finite summations. To find the roots of these equations, many numerical methods can be used such as the fixed point iteration and the Newton method. Since it is not necessary to compute the derivatives of the functions $f_j(x)$, $j = 1, 2$, defined in (16), we choose the fixed point iteration method. Then, using the Banach fixed point theorem, we prove that each of these equations containing the finite summations has a unique solution on the convenient set (see Theorem 2). Moreover, we give error estimations. Finally we present some examples.

For simplicity of calculations, we assume that

$$\sum_{k=-\infty}^{\infty} |q_k| := c < \infty, \qquad \sum_{k=-s}^{s} |q_k| := b, \qquad \sup_x \left|q(x)\right| := M, \qquad \sup_n |q_n| := Q. \tag{6}$$

It is well known that [18]

$$\left|\lambda_n(q) - \lambda_n(0)\right| \le M, \quad \lambda_n(0) = (2\pi n)^2, \forall n \in \mathbb{Z}.$$

Therefore, we have

$$(2\pi n)^2 - M \le \lambda_n(q) \le (2\pi n)^2 + M,$$

and for $n \neq k$ we have that

$$\left|\lambda_n - (2\pi n_k)^2\right| \ge \left|(2\pi n)^2 - (2\pi n_k)^2\right| - M \ge \left|4\pi^2(n - n_k)(n + n_k)\right| - M \ge 2\rho(n), \tag{7}$$

where $2\rho(n) = 4\pi^2(2n-1) - M$.

2 Main results

Let us introduce some notations and relations we use from [6]. One of the main equations is

$$\big(\lambda_{n,j} - (2\pi n)^2 - A_m(\lambda_{n,j})\big)\big(\Psi_{n,j}, e^{i2\pi nx}\big) - \big(q_{2n} + B_m(\lambda_{n,j})\big)\big(\Psi_{n,j}, e^{-i2\pi nx}\big) = R_m \tag{8}$$

(see (15) in [6]), where $A_m(\lambda_{n,j}) = \sum_{k=1}^{m} a_k(\lambda_{n,j})$, $B_m(\lambda_{n,j}) = \sum_{k=1}^{m} b_k(\lambda_{n,j})$,

$$\begin{aligned}
a_k(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_k))^2]},\\
b_k(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{2n-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_k))^2]},\\
R_m(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_{m+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}(q(x)\Psi_{n,j}(x), e^{i2\pi(n-n_1-\cdots-n_{m+1})x})}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_{m+1}))^2]}.
\end{aligned} \tag{9}$$

Here the sums are taken under the conditions

$$n_s \neq 0, \quad \sum_{j=1}^{s} n_j \neq 0, 2n$$

for $s = 1, 2, \ldots, m+1$. Another main equation is

$$\big(\lambda_{n,j} - (2\pi n)^2 - A'_m(\lambda_{n,j})\big)\big(\Psi_{n,j}, e^{-i2\pi nx}\big) - \big(q_{-2n} + B'_m(\lambda_{n,j})\big)\big(\Psi_{n,j}, e^{i2\pi nx}\big) = R'_m \tag{10}$$

(see (17) in [6]), where $A'_m(\lambda_{n,j}) = \sum_{k=1}^{m} a'_k(\lambda_{n,j})$, $B'_m(\lambda_{n,j}) = \sum_{k=1}^{m} b'_k(\lambda_{n,j})$,

$$\begin{aligned}
a'_k(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_k))^2]},\\
b'_k(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-2n-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_k))^2]},\\
R'_m(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_{m+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}(q(x)\Psi_{n,j}(x), e^{i2\pi(n+n_1+\cdots+n_{m+1})x})}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_{m+1}))^2]}.
\end{aligned} \tag{11}$$

Here the sums are taken under the conditions

$$n_s \neq 0, \quad \sum_{j=1}^{s} n_j \neq 0, -2n$$

for $s = 1, 2, \ldots, m+1$.

Now letting m tend to infinity in (8) and (10), we obtain

$$\big(\lambda_{n,j} - (2\pi n)^2 - A(\lambda_{n,j})\big)u_{n,j} = \big(q_{2n} + B(\lambda_{n,j})\big)v_{n,j}, \tag{12}$$

$$\big(\lambda_{n,j} - (2\pi n)^2 - A'(\lambda_{n,j})\big)v_{n,j} = \big(q_{-2n} + B'(\lambda_{n,j})\big)u_{n,j}, \tag{13}$$

where

$$A(\lambda_{n,j}) = \sum_{k=1}^{\infty} a_k(\lambda_{n,j}), \qquad B(\lambda_{n,j}) = \sum_{k=1}^{\infty} b_k(\lambda_{n,j}), \tag{14}$$

$$A'(\lambda_{n,j}) = \sum_{k=1}^{\infty} a'_k(\lambda_{n,j}), \qquad B'(\lambda_{n,j}) = \sum_{k=1}^{\infty} b'_k(\lambda_{n,j}) \tag{15}$$

and

$$u_{n,j} = \left(\Psi_{n,j}, e^{i2\pi nx}\right), \qquad v_{n,j} = \left(\Psi_{n,j}, e^{-i2\pi nx}\right). \tag{16}$$

To prove one of the main results, Theorem 1, we use the following lemma.

Lemma 1 *If*

$$\rho(n) > 2c, \tag{17}$$

then the following hold:

(a)

$$\lim_{m\to\infty} R_m(\lambda_{n,j}) = 0, \qquad \lim_{m\to\infty} R'_m(\lambda_{n,j}) = 0$$

for $j = 1, 2$, *where* $R_m(\lambda_{n,j})$ *and* $R'_m(\lambda_{n,j})$ *are defined by* (9) *and* (11), *respectively.*

(b)

$$|u_{n,j}|^2 + |v_{n,j}|^2 > 0$$

for $j = 1, 2$, *where* $u_{n,j}$ *and* $v_{n,j}$ *are defined by* (16).

Proof (a) By the definition of R_m we have

$$\left|R_m(\lambda_{n,j})\right| \le \sum_{n_1,n_2,\ldots,n_{m+1}} \frac{|q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}(q(x)\Psi_{n,j}(x), e^{i2\pi(n-n_1-\cdots-n_{m+1})x})|}{|\lambda_{n,j}-(2\pi(n-n_1))^2|\cdots|\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_{m+1}))^2|}.$$

Taking into account that $\|\Psi_{n,j}\| = 1$ and that

$$\left|\left(q\Psi_{n,j}, e^{i2\pi(n-n_1-\cdots-n_{m+1})x}\right)\right| \le \|q\Psi_{n,j}\| \left\|e^{i2\pi(n-n_1-\cdots-n_{m+1})x}\right\| \le M,$$

we obtain by (6) and (7) that

$$\left|R_m(\lambda_{n,j})\right| \le M\left(\frac{c}{2\rho(n)}\right)^{m+1}.$$

Thus this with (17) implies $R_m(\lambda_{n,j}) \to 0$ as $m \to \infty$ for $j = 1, 2$. Similarly, we prove the same result for $R'_m(\lambda_{n,j})$.

(b) Suppose to the contrary $u_{n,j} = 0$, $v_{n,j} = 0$. Since the root functions of $L_0(0)$ form an orthonormal basis, we have the decomposition

$$\Psi_{n,j} = u_{n,j}e^{i2\pi nx} + v_{n,j}e^{-i2\pi nx} + h_{n,j}(x)$$

for the normalized eigenfunction $\Psi_{n,j}$ corresponding to the eigenvalue $\lambda_{n,j}$ of $L_0(q)$, where

$$h_{n,j}(x) = \sum_{k\in\mathbb{Z},k\neq\pm n}^{\infty} \left(\Psi_{n,j}, e^{i2\pi kx}\right)e^{i2\pi kx}.$$

To get a contradiction, it is enough to show that $\|\Psi_{n,j}\| < 1$ for $j = 1, 2$. By Parseval's equality, we have

$$\|\Psi_{n,j}\|^2 = \|h_{n,j}\|^2 = \sum_{k\in\mathbb{Z},k\neq\pm n}^{\infty} \left|\left(\Psi_{n,j}, e^{i2\pi kx}\right)\right|^2.$$

Now using (9) in [6], (7), and Bessel inequality and taking into account that $c \geq M$, we obtain by (17) that

$$\begin{aligned}\sum_{k\in\mathbb{Z},k\neq\pm n}^{\infty} \left|\left(\Psi_{n,j}, e^{i2\pi kx}\right)\right|^2 &= \sum_{k\in\mathbb{Z},k\neq\pm n}^{\infty} \frac{|(q\Psi_{n,j}, e^{i2\pi kx})|^2}{|\lambda_{n,j} - (2\pi k)^2|^2} \\ &\leq \frac{1}{(2\rho(n))^2} \sum_{k\in\mathbb{Z},k\neq\pm n}^{\infty} \left|\left(q\Psi_{n,j}, e^{i2\pi kx}\right)\right|^2 \\ &\leq \frac{M^2}{(2\rho(n))^2} \leq \frac{c^2}{(2\rho(n))^2} < \frac{1}{16},\end{aligned}$$

which contradicts $\|\Psi_{n,j}\| = 1$ and completes the proof of the lemma. □

Now we state one of the main results:

Theorem 1 *If (17) holds, then $\lambda_{n,j}$ is an eigenvalue of L_0 if and only if it is a root of equation (3).*

Moreover, $\lambda \in U(n) := [(2\pi n)^2 - M, (2\pi n)^2 + M]$ is a double eigenvalue of L_0 if and only if it is a double root of (3).

Proof By (12) and (13), we have the following cases:

Case 1. If $u_{n,j} = 0$, then by Lemma 1(b) we have $v_{n,j} \neq 0$. Therefore by (12) and (13) we obtain $(q_{2n} + B(\lambda_{n,j})) = 0$ and $(\lambda_{n,j} - (2\pi n)^2 - A'(\lambda_{n,j})) = 0$, which means that (3) holds.

Case 2. If $v_{n,j} = 0$, then again by Lemma 1(b) we have $u_{n,j} \neq 0$. It follows from (12) and (13) that $(\lambda_{n,j} - (2\pi n)^2 - A(\lambda_{n,j})) = 0$ and $(q_{-2n} + B'(\lambda_{n,j})) = 0$, which means that (3) again holds.

Case 3. If $u_{n,j} \neq 0$ and $v_{n,j} \neq 0$, then multiplying equations (12) and (13) side by side and then canceling $u_{n,j}v_{n,j}$, we obtain (3). Thus in any case $\lambda_{n,j}$ is a root of (3).

Now we prove that the roots of (3) lying in $U(n)$ are the eigenvalues of $L_0(q)$. Let $F(\lambda)$ be the left-hand side of (3) which can be written as

$$F(\lambda) := \left(\lambda - (2\pi n)^2\right)^2 - \left(A(\lambda) + A'(\lambda)\right)\left(\lambda - (2\pi n)^2\right)$$

$$+ A(\lambda)A'(\lambda) - \bigl(q_{2n} + B(\lambda)\bigr)\bigl(q_{-2n} + B'(\lambda)\bigr)$$

and

$$G(\lambda) := \bigl(\lambda - (2\pi n)^2\bigr)^2.$$

It is easy to verify that the inequality

$$\bigl|F(\lambda) - G(\lambda)\bigr| < \bigl|G(\lambda)\bigr|$$

holds for all λ from the boundary of $U(n)$. Since the function $G(\lambda)$ has two roots in the set $U(n)$, by Rouche's theorem, $F(\lambda)$ has also two roots in $U(n)$. Therefore L_0 has two eigenvalues (counting with multiplicities) lying in $U(n)$ that are the roots of (3). On the other hand, (3) has preciously two roots (counting with multiplicities) in $U(n)$. Thus $\lambda \in U(n)$ is an eigenvalue of L_0 if and only if (3) holds.

If $\lambda \in U(n)$ is a double eigenvalue of L_0, then L_0 has no other eigenvalues in $U(n)$ and hence (3) has no other roots. This implies that λ is a double root of (3). By the same way one can prove that if λ is a double root of (3), then it is a double eigenvalue of L_0. □

Now let us substitute $t := \lambda - (2\pi n)^2$ in $F(\lambda) = 0$. Then

$$t^2 - \bigl(A(\lambda) + A'(\lambda)\bigr)t + A(\lambda)A'(\lambda) - \bigl(q_{2n} + B(\lambda)\bigr)\bigl(q_{-2n} + B'(\lambda)\bigr) = 0.$$

The solutions of this equation are

$$t_{1,2} = \frac{(A(\lambda) + A'(\lambda)) \pm \sqrt{\Delta(\lambda)}}{2},$$

where

$$\Delta(\lambda) = \bigl(A(\lambda) + A'(\lambda)\bigr)^2 - 4\bigl[A(\lambda)A'(\lambda) - \bigl(q_{2n} + B(\lambda)\bigr)\bigl(q_{-2n} + B'(\lambda)\bigr)\bigr],$$

which can be written in the form

$$\Delta(\lambda) = \bigl(A(\lambda) - A'(\lambda)\bigr)^2 + 4\bigl(q_{2n} + B(\lambda)\bigr)\bigl(q_{-2n} + B'(\lambda)\bigr).$$

By Theorem 1, the eigenvalue $\lambda_{n,j}$ is a root either of equation (4) or of equation (5). To use numerical methods, we take finite summations instead of the infinite series in expressions (4) and (5), and get

$$\lambda = (2\pi n)^2 + f_j(\lambda) \tag{18}$$

for $j = 1$ and $j = 2$, where

$$f_j(\lambda) = \frac{1}{2}\bigl[\bigl(A_{m,s}(\lambda) + A'_{m,s}(\lambda)\bigr) + (-1)^j\sqrt{\Delta_{m,s}(\lambda)}\bigr],$$

$$A_{m,s}(\lambda) := \sum_{k=1}^{m} a_{k,s}(\lambda_{n,j}), \qquad A'_{m,s}(\lambda) := \sum_{k=1}^{m} a'_{k,s}(\lambda_{n,j}),$$

$$\begin{aligned}
a_{k,s}(\lambda_{n,j}) &:= \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_k))^2]}, \\
a'_{k,s}(\lambda_{n,j}) &:= \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_k))^2]}, \\
\Delta_{m,s}(\lambda) &:= \left(A_{m,s}(\lambda)-A'_{m,s}(\lambda)\right)^2 + 4\left(q_{2n}+B_{m,s}(\lambda)\right)\left(q_{-2n}+B'_{m,s}(\lambda)\right), \\
B_{m,s}(\lambda) &:= \sum_{k=1}^{m} b_{k,s}(\lambda_{n,j}), \qquad B'_{m,s}(\lambda) := \sum_{k=1}^{m} b'_{k,s}(\lambda_{n,j}), \\
b_{k,s}(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{2n-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_k))^2]}, \\
b'_{k,s}(\lambda_{n,j}) &= \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-2n-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_k))^2]}.
\end{aligned} \tag{19}$$

Now we prove that the eigenvalues of L_0 are close to the roots of (18) for the complex-valued summable potential q in the form $q(x) = 2\sum_{k=1}^{\infty} q_k \cos 2\pi kx$. We have the following relations for such potential:

$$\begin{aligned}
&a_{k,s}(\lambda_{n,j}) = a'_{k,s}(\lambda_{n,j}), \qquad b_{k,s}(\lambda_{n,j}) = b'_{k,s}(\lambda_{n,j}), \\
&A_{m,s}(\lambda) = A'_{m,s}(\lambda), \qquad B_{m,s}(\lambda) = B'_{m,s}(\lambda), \\
&a_k(\lambda_{n,j}) = a'_k(\lambda_{n,j}), \qquad b_k(\lambda_{n,j}) = b'_k(\lambda_{n,j}), \\
&A(\lambda) = A'(\lambda), \qquad B(\lambda) = B'(\lambda), \\
&\Delta_{m,s}(\lambda) = 4\left(q_{2n}+B_{m,s}(\lambda)\right)^2, \qquad \Delta(\lambda) = 4\left(q_{2n}+B(\lambda)\right)^2.
\end{aligned} \tag{20}$$

Theorem 2 *If* (17) *holds, then for all x and y from* $[(2\pi n)^2 - M, (2\pi n)^2 + M]$ *the relations*

$$\left|f_j(x) - f_j(y)\right| < C_n|x-y|, \qquad C_n = \frac{Qb}{2\rho(n)(\rho(n)-b)} < \frac{1}{4}, \tag{21}$$

hold for $j = 1,2$, *and for each* j, *equation* (18) *has a unique solution* $r_{n,j}$ *on* $[(2\pi n)^2 - M, (2\pi n)^2 + M]$.

Moreover,

$$|\lambda_{n,j} - r_{n,j}| \le \frac{2Qc^{m+1}}{2^m(\rho(n))^m(2\rho(n)-c)(1-C_n)} \tag{22}$$

for $j = 1,2$ *and* $s \ge m$.

Proof First let us prove (21) by using the mean-value theorem. For this we estimate $|f'_j(\lambda)|$. By (19) and (20) we have

$$\begin{aligned}
\left|f'_j(\lambda)\right| &= \left|\frac{1}{2}\left(\frac{d}{d\lambda}A_{m,s}(\lambda) + \frac{d}{d\lambda}A'_{m,s}(\lambda)\right) + (-1)^j \frac{1}{4}\frac{\frac{d}{d\lambda}\Delta_{m,s}(\lambda)}{\sqrt{\Delta_{m,s}(\lambda)}}\right| \\
&\le \left|\frac{d}{d\lambda}A_{m,s}(\lambda)\right| + \frac{1}{4}\left|\frac{\frac{d}{d\lambda}\Delta_{m,s}(\lambda)}{\sqrt{\Delta_{m,s}(\lambda)}}\right|.
\end{aligned} \tag{23}$$

For the first term of (23), we obtain

$$\begin{aligned}
&\left|\frac{d}{d\lambda}A_{m,s}(\lambda)\right| \\
&\quad = \left|\sum_{k=1}^{m}\frac{d}{d\lambda}a_{k,s}(\lambda)\right| \\
&\quad = \left|\sum_{k=1}^{m}\sum_{n_1,n_2,\ldots,n_k=-s}^{s}\frac{d}{d\lambda}\frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda-(2\pi(n-n_1))^2]\cdots[\lambda-(2\pi(n-n_1-\cdots-n_k))^2]}\right| \\
&\quad \le \sum_{n_1=-s}^{s}\frac{|q_{n_1}q_{-n_1}|}{(2\rho(n))^2}+\sum_{n_1,n_2=-s}^{s}\frac{2|q_{n_1}q_{n_2}q_{-n_1-n_2}|}{(2\rho(n))^3}+\cdots \\
&\qquad + \sum_{n_1,n_2,\ldots,n_m=-s}^{s}\frac{m|q_{n_1}q_{n_2}\cdots q_{n_m}q_{-n_1-n_2-\cdots-n_m}|}{(2\rho(n))^{m+1}} \\
&\quad = \sum_{k=1}^{m}\sum_{n_1,n_2,\ldots,n_k=-s}^{s}\frac{k|q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}|}{(2\rho(n))^{k+1}} \\
&\quad \le \frac{Qb}{(2\rho(n))^2}+\frac{2Qb^2}{(2\rho(n))^3}+\cdots+\frac{mQb^m}{(2\rho(n))^{m+1}} \\
&\quad = \frac{Qb}{(2\rho(n))^2}\sum_{i=0}^{m-1}(i+1)\left(\frac{b}{2\rho(n)}\right)^i \\
&\quad \le \frac{Qb}{(2\rho(n))^2}\frac{1}{1-\frac{b}{\rho(n)}}=\frac{Qb}{4\rho(n)(\rho(n)-b)}.
\end{aligned}$$

Similarly, for the second term of (23), we get

$$\begin{aligned}
\left|\frac{\frac{d}{d\lambda}\Delta_{m,s}(\lambda)}{\sqrt{\Delta_{m,s}(\lambda)}}\right| &= \left|\frac{\frac{d}{d\lambda}[4(q_{2n}+B_{m,s}(\lambda))^2]}{\sqrt{4(q_{2n}+B_{m,s}(\lambda))^2}}\right| = \left|\frac{8(q_{2n}+B_{m,s}(\lambda))\frac{d}{d\lambda}B_{m,s}(\lambda)}{2(q_{2n}+B_{m,s}(\lambda))}\right| \\
&= 4\left|\frac{d}{d\lambda}B_{m,s}(\lambda)\right| \le \frac{Qb}{\rho(n)(\rho(n)-b)}.
\end{aligned}$$

Therefore, by the geometric series formula we obtain

$$\left|f_j'(\lambda)\right| \le \frac{Qb}{2\rho(n)(\rho(n)-b)} = C_n < \frac{1}{4}.$$

Since the inequality

$$\left|f_j'(\lambda)\right| \le C_n < 1 \tag{24}$$

is satisfied for all $x, y \in [(2\pi n)^2 - M, (2\pi n)^2 + M]$, (21) holds by the mean value theorem and equation (18) has a unique solution $r_{n,j}$ on $[(2\pi n)^2 - M, (2\pi n)^2 + M]$ for each j ($j = 1, 2$) by the contraction mapping theorem.

Now let us prove (22). Let us define the function

$$F_j(x) := x - (2\pi n)^2 - f_j(x). \tag{25}$$

By the definition of $\{r_{n,j}\}$, we write

$$F_j(r_{n,j}) = 0$$

for $j = 1, 2$. Therefore, by (4), (5), and (20), we obtain

$$\begin{aligned}
&\bigl|F_j(\lambda_{n,j}) - F_j(r_{n,j})\bigr| \\
&\quad = \bigl|F_j(\lambda_{n,j})\bigr| \\
&\quad = \left|\lambda_{n,j} - (2\pi n)^2 - \frac{1}{2}\Bigl[\bigl(A_{m,s}(\lambda_{n,j}) + A'_{m,s}(\lambda_{n,j})\bigr) + (-1)^j\sqrt{\Delta_{m,s}(\lambda_{n,j})}\Bigr]\right| \\
&\quad = \frac{1}{2}\Bigl|\Bigl[\bigl(A(\lambda_{n,j}) + A'(\lambda_{n,j})\bigr) + (-1)^j\sqrt{\Delta(\lambda_{n,j})}\Bigr] - \Bigl[\bigl(A_{m,s}(\lambda_{n,j}) + A'_{m,s}(\lambda_{n,j})\bigr) \\
&\qquad + (-1)^j\sqrt{\Delta_{m,s}(\lambda_{n,j})}\Bigr]\Bigr| \\
&\quad \le \frac{1}{2}\bigl|A(\lambda_{n,j}) - A_{m,s}(\lambda_{n,j})\bigr| + \frac{1}{2}\bigl|A'(\lambda_{n,j}) - A'_{m,s}(\lambda_{n,j})\bigr| + \frac{1}{2}\Bigl|\sqrt{\Delta(\lambda_{n,j})} - \sqrt{\Delta_{m,s}(\lambda_{n,j})}\Bigr| \\
&\quad = \bigl|A(\lambda_{n,j}) - A_{m,s}(\lambda_{n,j})\bigr| + \bigl|B(\lambda_{n,j}) - B_{m,s}(\lambda_{n,j})\bigr|.
\end{aligned} \tag{26}$$

First let us estimate the first term of the right-hand side of (26). For $s \ge m$, we obtain

$$\begin{aligned}
&\bigl|A(\lambda_{n,j}) - A_{m,s}(\lambda_{n,j})\bigr| \\
&\quad \le \bigl|A(\lambda_{n,j}) - A_m(\lambda_{n,j})\bigr| + \bigl|A_m(\lambda_{n,j}) - A_{m,s}(\lambda_{n,j})\bigr| \\
&\quad = \left|\sum_{k=1}^{\infty} a_k(\lambda_{n,j}) - \sum_{k=1}^{m} a_k(\lambda_{n,j})\right| + \left|\sum_{k=1}^{m} a_k(\lambda_{n,j}) - \sum_{k=1}^{m} a_{k,s}(\lambda_{n,j})\right| \\
&\quad = \left|\sum_{k=m+1}^{\infty} a_k(\lambda_{n,j})\right| + \left|\sum_{k=1}^{m}\bigl[a_k(\lambda_{n,j}) - a_{k,s}(\lambda_{n,j})\bigr]\right| \\
&\quad \le 2\Biggl\{\sum_{n_1,n_2,\ldots,n_{m+1}} \frac{|q_{n_1}q_{n_2}\cdots q_{n_{m+1}}q_{-n_1-n_2-\cdots-n_{m+1}}|}{|\lambda_{n,j} - (2\pi(n-n_1))^2|\cdots|\lambda_{n,j} - (2\pi(n-n_1-\cdots-n_{m+1}))^2|} \\
&\qquad + \sum_{n_1,n_2,\ldots,n_{m+2}} \frac{|q_{n_1}q_{n_2}\cdots q_{n_{m+2}}q_{-n_1-n_2-\cdots-n_{m+2}}|}{|\lambda_{n,j} - (2\pi(n-n_1))^2|\cdots|\lambda_{n,j} - (2\pi(n-n_1-\cdots-n_{m+2}))^2|} + \cdots\Biggr\} \\
&\quad \le \frac{2Qc^{m+1}}{(2\rho(n))^{m+1}} + \frac{2Qc^{m+2}}{(2\rho(n))^{m+2}} + \cdots \\
&\quad = \frac{2Qc^{m+1}}{(2\rho(n))^{m+1}}\frac{1}{1 - \frac{c}{2\rho(n)}} = \frac{Qc^{m+1}}{(2\rho(n))^m(2\rho(n) - c)}.
\end{aligned} \tag{27}$$

Similarly, for the second term of the right-hand side of (26), we obtain

$$\bigl|B(\lambda_{n,j}) - B_{m,s}(\lambda_{n,j})\bigr| \le \frac{Qc^{m+1}}{(2\rho(n))^m(2\rho(n) - c)}. \tag{28}$$

Thus, by (26)–(28) we get

$$\bigl|F_j(\lambda_{n,j}) - F_j(r_{n,j})\bigr| \le \frac{2Qc^{m+1}}{(2\rho(n))^m(2\rho(n) - c)}. \tag{29}$$

To apply the mean value theorem, we estimate $|F_j'(\lambda)|$:

$$\left|F_j'(\lambda)\right| = \left|1 - f_j'(\lambda)\right| \geq \left|1 - \left|f_j'(\lambda)\right|\right| \geq 1 - C_n. \tag{30}$$

By the mean value formula, (29), and (30), we obtain

$$\begin{aligned}
&\left|F_j(\lambda_{n,j}) - F_j(r_{n,j})\right| = \left|F_j'(\xi)\right||\lambda_{n,j} - r_{n,j}|, \quad \xi \in \left[(2\pi n)^2 - M, (2\pi n)^2 + M\right], \\
&|\lambda_{n,j} - r_{n,j}| = \frac{|F_j(\lambda_{n,j}) - F_j(r_{n,j})|}{|F_j'(\xi)|} \leq \frac{2Qc^{m+1}}{(2\rho(n))^m(2\rho(n) - c)(1 - C_n)}.
\end{aligned}$$

□

Now let us approximate $r_{n,j}$ by the fixed point iterations:

$$x_{n,i+1} = (2\pi n)^2 + f_1(x_{n,i}) \tag{31}$$

and

$$y_{n,i+1} = (2\pi n)^2 + f_2(y_{n,i}), \tag{32}$$

where $f_j(x)$ $(j = 1, 2)$ is defined in (19). Using (20) we get

$$\begin{aligned}
\left|f_j(\lambda_{n,j})\right| &= \frac{1}{2}\left|\left(A_{m,s}(\lambda) + A'_{m,s}(\lambda)\right) + (-1)^j\sqrt{\Delta_{m,s}(\lambda)}\right| \\
&\leq \frac{1}{2}\left(2\left|A_{m,s}(\lambda)\right| + 2\left|q_{2n} + B_{m,s}(\lambda)\right|\right) \\
&\leq |q_{2n}| + \left|A_{m,s}(\lambda)\right| + \left|B_{m,s}(\lambda)\right|.
\end{aligned} \tag{33}$$

Since the potential q has the form $q(x) = 2\sum_{k=1}^{\infty} q_k \cos 2\pi kx$, we obtain that

$$x_{n,i} = x_{-n,i}, \qquad y_{n,i} = y_{-n,i}.$$

Now, let us estimate $|A_{m,s}(\lambda)|$. The estimation of $|B_{m,s}(\lambda)|$ is similar.

$$\begin{aligned}
&\left|A_{m,s}(\lambda)\right| \\
&\quad = \left|\sum_{k=1}^{m} a_{k,s}(\lambda_{n,j})\right| \\
&\quad = \left|\sum_{k=1}^{m}\ \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j} - (2\pi(n - n_1))^2]\cdots[\lambda_{n,j} - (2\pi(n - n_1 - \cdots - n_k))^2]}\right| \\
&\quad \leq \left|\sum_{k=1}^{m}\ \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{(2\delta(n))^k}\right| \\
&\quad = \left|\sum_{n_1=-s}^{s} \frac{q_{n_1}q_{-n_1}}{(2\delta(n))} + \sum_{n_1,n_2=-s}^{s} \frac{q_{n_1}q_{n_2}q_{-n_1-n_2}}{(2\delta(n))^2} + \cdots\right. \\
&\qquad \left. + \sum_{n_1,n_2,\ldots,n_m=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_m}q_{-n_1-n_2-\cdots-n_m}}{(2\delta(n))^m}\right|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{Qb}{2\delta(n)} + \frac{Qb^2}{(2\delta(n))^2} + \cdots + \frac{Qb^m}{(2\delta(n))^m} \\ &= \frac{Qb}{2\delta(n)} \frac{1}{1 - \frac{b}{2\delta(n)}} = \frac{Qb}{(2\delta(n) - b)}. \end{aligned} \tag{34}$$

Similarly, we obtain

$$\left|B_{m,s}(\lambda)\right| \leq \frac{Qb}{(2\delta(n) - b)}$$

and

$$\left|f_j(\lambda_{n,j})\right| \leq |q_{2n}| + \frac{2Qb}{2\delta(n) - b}. \tag{35}$$

On the other hand, writing $4\pi^2(2n-1)$ instead of $2\delta(n)$ in (35), we get

$$\begin{aligned} \left|f_j\left((2\pi n)^2\right)\right| &\leq |q_{2n}| + \left|A_{m,s}\left((2\pi n)^2\right)\right| + \left|B_{m,s}\left((2\pi n)^2\right)\right| \\ &\leq |q_{2n}| + \frac{2Qb}{4\pi^2(2n-1) - b} \end{aligned} \tag{36}$$

since $|(2\pi n)^2 - (2\pi k)^2| \geq 4\pi^2(2n-1)$ for $n = 1, 2, \ldots$.

Theorem 3 *If* (17) *holds, then the following estimations hold for the sequences* $\{x_{n,i}\}$ *and* $\{y_{n,i}\}$ *defined by* (31) *and* (32):

$$|x_{n,i} - r_{n,1}| \leq (C_n)^i \left(\frac{|q_{2n}|}{1 - C_n} + \frac{2Qb}{(1 - C_n)(4\pi^2(2n-1) - b)} \right), \tag{37}$$

$$|y_{n,i} - r_{n,2}| \leq (C_n)^i \left(\frac{|q_{2n}|}{1 - C_n} + \frac{2Qb}{(1 - C_n)(4\pi^2(2n-1) - b)} \right) \tag{38}$$

for $i = 1, 2, 3, \ldots$, *where* C_n *is defined in* (21).

Proof Without loss of generality, we can take $x_{n,0} = (2\pi n)^2$. By (21), (25), and (31), we have

$$\begin{aligned} |x_{n,i} - r_{n,1}| &= \left|(2\pi n)^2 + f_1(x_{n,i-1}) - \left((2\pi n)^2 + f_1(r_{n,1})\right)\right| \\ &= \left|f_1(x_{n,i-1}) - f_1(r_{n,1})\right| < C_n|x_{n,i-1} - r_{n,1}| < (C_n)^i|x_{n,0} - r_{n,1}|. \end{aligned}$$

Therefore it is enough to estimate $|x_{n,0} - r_{n,1}|$. By definitions of $r_{n,j}$ and $x_{n,0}$ we obtain

$$r_{n,1} - x_{n,0} = f_1(r_{n,1}) + (2\pi n)^2 - x_{n,0} = f_1(r_{n,1}) - f_1(x_{n,0}) + f_1\left((2\pi n)^2\right),$$

and by the mean value theorem there exists $x \in [(2\pi n)^2 - M, (2\pi n)^2 + M]$ such that $f_1(r_{n,1}) - f_1(x_{n,0}) = f_1'(x)(r_{n,1} - x_{n,0})$. The last two equalities imply that $(r_{n,1} - x_{n,0})(1 - f_1'(x)) = f_1((2\pi n)^2)$. Hence by (24) and (36) we get

$$|r_{n,1} - x_{n,0}| \leq \frac{|f_1((2\pi n)^2)|}{1 - C_n} \leq \frac{|q_{2n}|}{1 - C_n} + \frac{2Qb}{(1 - C_n)(4\pi^2(2n-1) - b)}$$

Table 1 Estimations of eigenvalues for $q(x) = 2\cos(2\pi x)$

n	i	$x_{n,i}$	$\lvert x_{n,i+1} - x_{n,i}\rvert$	$y_{n,i}$	$\lvert y_{n,i+1} - y_{n,i}\rvert$
±1	0	1.000000000000	38.472044558692		
	1	39.472044558692	0.002070157686	1.000000000000	38.478417489727
	2	39.469974401006	0.000000147576	39.478417489727	0.000000111099
	3	39.469974548582	0.000000000011	39.478417378628	0.000000000000
	4	39.469974548572	0.000000000000	39.478417378628	
	5	39.469974548572			
±2	0	1.000000000000	156.911370091955	1.000000000000	
	1	157.911370091955	0.002765277401	157.914499110787	156.914499110787
	2	157.908604814554	0.000000070999	157.913670814745	0.000828296043
	3	157.908604885553	0.000000000002	157.913670814754	0.000000000009
	4	157.908604885551	0.000000000000	157.913670814754	0.000000000000
	5	157.908604885551			
±3	0	1.000000000000	354.299384450125	1.000000000000	354.304171727348
	1	355.299384450125	0.002755519393	355.304171727348	0.006652872309
	2	355.302139969517	0.000000036084	355.310824599657	0.000000170751
	3	355.302139933433	0.000000000001	355.310824428906	0.000000000004
	4	355.302139933434	0.000000000000	355.310824428910	0.000000000000
	5	355.302139933434		355.310824428910	
±4	0	1.000000000000		1.000000000000	630.653667378752
	1	631.651859197264	630.651859197264	631.653667378752	0.004632935559
	2	631.651867232362	0.000008035097	631.658300314310	0.000000060666
	3	631.651867232298	0.000000000064	631.658300253644	0.000000000001
	4	631.651867232298	0.000000000000	631.658300253645	0.000000000000
	5			631.658300253645	
±5	0	1.000000000000	985.958854447771	1.000000000000	985.959735986823
	1	986.958854447771	0.000717077657	986.959735986823	0.003518611957
	2	986.958137370115	0.000000003802	986.963254598781	0.000000027872
	3	986.958137373917	0.000000000000	986.963254570909	0.000000000000
	4	986.958137373917	0.000000000000	986.963254570909	0.000000000000
	5	986.958137373917		986.963254570909	
±6	0	1.000000000000	1420.222019515655	1.000000000000	1420.222516542946
	1	1421.222019515655	0.000934235956	1421.222516542946	0.002819974119
	2	1421.221085279699	0.000000003547	1421.225336517065	0.000000014953
	3	1421.221085283246	0.000000000000	1421.225336502112	0.000000000000
	4	1421.221085283246		1421.225336502112	
±7	0	1.000000000000	1933.441758498203	1.000000000000	1933.442066670380
	1	1934.441758498203	0.000984567325	1934.442066670380	0.002344430688
	2	1934.440773930878	0.000000002808	1934.444411101068	0.000000008901
	3	1934.440773933686	0.000000000000	1934.444411092167	0.000000000000
	4	1934.440773933686		1934.444411092167	

and

$$|x_{n,i} - r_{n,1}| \le (C_n)^i \left(\frac{|q_{2n}|}{1 - C_n} + \frac{2Qb}{(1 - C_n)(4\pi^2(2n-1) - b)} \right).$$

One can easily show in a similar way to (37) that

$$|y_{n,i} - r_{n,2}| \le (C_n)^i \left(\frac{|q_{2n}|}{1 - C_n} + \frac{2Qb}{(1 - C_n)(4\pi^2(2n-1) - b)} \right)$$

for iteration (32). □

Thus by (22), (37), and (38) we have the approximations $x_{n,i}$ and $y_{n,i}$ for $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively, with the errors

$$|\lambda_{n,1} - x_{n,i}| < \frac{2Qc^{m+1}}{2^m(\rho(n))^m(2\rho(n) - c)(1 - C_n)}$$

Table 2 Estimations of eigenvalues for $q(x) = 2\cos(2\pi x) + 2\cos(4\pi x)$

n	i	$x_{n,i}$	$\lvert x_{n,i+1} - x_{n,i}\rvert$	$y_{n,i}$	$\lvert y_{n,i+1} - y_{n,i}\rvert$
±1	0	1.000000000000	37.473844750118	1.000000000000	39.423036168470
	1	38.473844750118	0.001155787367	40.423036168470	0.048751667415
	2	38.472688962751	0.000000045838	40.471787835885	0.000001266531
	3	38.472689008589	0.000000000002	40.471786569354	0.000000000033
	4	38.472689008587	0.000000000000	40.471786569387	0.000000000000
	5	38.472689008587		40.471786569387	
±2	0	1.000000000000	156.708686755063	1.000000000000	156.910954517251
	1	157.708686755063	0.197853424408	157.910954517251	0.002765125738
	2	157.906540179470	0.000006158197	157.913719642989	0.000000003732
	3	157.906534021274	0.000000000192	157.913719639257	0.000000000000
	4	157.906534021465	0.000000000000	157.913719639257	
	5	157.906534021465			
±3	0	1.000000000000	354.274676914265	1.000000000000	354.304635333209
	1	355.274676914265	0.025887337209	355.304635333209	0.009362588088
	2	355.300564251475	0.000000407817	355.313997921297	0.000000335592
	3	355.300563843658	0.000000000006	355.313997585704	0.000000000012
	4	355.300563843665	0.000000000000	355.313997585716	0.000000000000
	5	355.300563843665		355.313997585716	
±4	0	1.000000000000	630.645567421755	1.000000000000	630.653044860730
	1	631.645567421755	0.005035432636	631.653044860730	0.007368399068
	2	631.650602854392	0.000000048178	631.660413259797	0.000000129629
	3	631.650602806214	0.000000000000	631.660413130168	0.000000000002
	4	631.650602806214	0.000000000000	631.660413130170	0.000000000000
	5	631.650602806214		631.660413130170	
±5	0	1.000000000000	985.956043584523	1.000000000000	985.959230332354
	1	986.956043584523	0.001039199076	986.959230332354	0.005608239586
	2	986.957082783600	0.000000006685	986.964838571940	0.000000058572
	3	986.957082776914	0.000000000000	986.964838513367	0.000000000001
	4	986.957082776914		986.964838513368	0.000000000000
				986.964838513368	
±6	0	1.000000000000	1420.220438496119	1.000000000000	1420.222125234640
	1	1421.220438496119	0.000257471532	1421.222125234640	0.004478190927
	2	1421.220181024587	0.000000001190	1421.226603425567	0.000000030964
	3	1421.220181025778	0.000000000001	1421.226603394604	0.000000000000
	4	1421.220181025777	0.000000000000	1421.226603394604	0.000000000000
		1421.220181025777		1421.226603394604	
±7	0	1.000000000000	1933.440746482760	1.000000000000	1933.441756076886
	1	1934.440746482760	0.000763911135	1934.441756076886	0.003710663673
	2	1934.439982571624	0.000000002660	1934.445466740559	0.000000018236
	3	1934.439982574285	0.000000000000	1934.445466722323	0.000000000000
	4	1934.439982574285		1934.445466722323	

$$+ (C_n)^i\left(\frac{|q_{2n}|}{1-C_n} + \frac{2Qb}{(1-C_n)(4\pi^2(2n-1)-b)}\right),$$

$$|\lambda_{n,2} - y_{n,i}| < \frac{2Qc^{m+1}}{2^m(\rho(n))^m(2\rho(n)-c)(1-C_n)}$$

$$+ (C_n)^i\left(\frac{|q_{2n}|}{1-C_n} + \frac{2Qb}{(1-C_n)(4\pi^2(2n-1)-b)}\right).$$

By these error formulas it is clear that the error will be very small if m grows.

3 Numerical examples

In this section we estimate the small eigenvalues for the potentials $q_1(x) := 2\cos(2\pi x)$ and $q_2(x) := 2\cos(2\pi x) + 2\cos(4\pi x)$ by iterations (31) and (32). Note that $q_1(x)$ is a famous Mathieu potential and $q_2(x)$ is the generalization of the Mathieu potential. Therefore we consider these potentials in our examples.

Example 1 For $q(x) = 2\cos(2\pi x)$, $m = 3$, and $s = 5$ with the initial approximations $x_{n,0} = 1$ and $y_{n,0} = 1$, we have Table 1 for the estimations of the small eigenvalues of $L_0(q)$. The fixed point iterations continue until the tolerance $1e - 18$. Usually it takes only 4 or 5 iterations to get this tolerance for any initial value $x_{n,0} \neq 0$, which means that the iterations converge very rapidly. In this table $x_{n,i}$ and $y_{n,i}$ denote the estimations for $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively, where i is the number of the iterations.

We see from Table 1 that the eigenvalues $\lambda_{n,1}$ and $\lambda_{n,2}$ are close to each other and they are close to $(2\pi n)^2$.

Example 2 For $q(x) = 2\cos(2\pi x) + 2\cos(4\pi x)$, $m = 3$, and $s = 5$ with the initial approximations $x_{n,0} = 1$ and $y_{n,0} = 1$, we have Table 2 for the estimations of the small eigenvalues of $L_0(q)$. $x_{n,i}$ is the estimation for $\lambda_{n,1}$ and $y_{n,i}$ is the estimation for $\lambda_{n,2}$, where i is the number of the iterations. Again, the fixed point iterations continue until the tolerance $1e - 18$ and converge very fast.

From Table 2 we can see that the first eigenvalues $\lambda_{1,1}$ and $\lambda_{1,2}$ are far from each other but the other eigenvalues $\lambda_{n,1}$ and $\lambda_{n,2}$ are close to each other and they are close to $(2\pi n)^2$.

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References
1. Andrew, A.L.: Correction of finite element eigenvalues for problems with natural or periodic boundary conditions. BIT Numer. Math. **28**(2), 254–269 (1988)
2. Andrew, A.L.: Correction of finite difference eigenvalues of periodic Sturm–Liouville problems. J. Aust. Math. Soc. Ser. B **30**(4), 460–469 (1989)
3. Birkhoff, G.D.: Boundary value problems and expansion problem of ordinary linear differential equations. Trans. Am. Math. Soc. **9**, 373–395 (1908)
4. Brown, B.M., Eastham, M.S.P., Schmidt, K.M.: Periodic Differential Operators. Operator Theory: Advances and Applications., vol. 230. Springer, Basel (2013)
5. Condon, D.J.: Corrected finite difference eigenvalues of periodic Sturm–Liouville problems. Appl. Numer. Math. **30**(4), 393–401 (1999)
6. Dernek, N., Veliev, O.A.: On the Riesz basisness of the root functions of the nonself-adjoint Sturm–Liouville operators. Isr. J. Math. **145**, 113–123 (2005)
7. Dinibütün, S., Veliev, O.A.: On the estimations of the small periodic eigenvalues. Abstr. Appl. Anal. **2013**, Article ID 145967 (2013)
8. Dunford, N., Schwartz, J.T.: Linear Operators, Part 3, Spectral Operators. Wiley, New York (1988)
9. Eastham, M.S.P.: The Spectral Theory of Periodic Differential Equations. Scottish Academic Press, Edinburg (1973)
10. Ji, X., Wong, Y.S.: Prüfer method for periodic and semiperiodic Sturm–Liouville eigenvalue problems. Int. J. Comput. Math. **39**, 109–123 (1991)
11. Ji, X.Z.: On a shooting algorithm for Sturm–Liouville eigenvalue problems with periodic and semi-periodic boundary conditions. J. Comput. Phys. **111**(1), 74–80 (1994)
12. Kerimov, N.B., Mamedov, K.R.: On the Riesz basis property of the root functions in certain regular boundary value problems. Math. Notes **64**(4), 483–487 (1998)

13. Levy, D.M., Keller, J.B.: Instability intervals of Hill's equation. Commun. Pure Appl. Math. **16**, 469–476 (1963)
14. Magnus, W., Winkler, S.: Hill Equation. Wiley, New York (1969)
15. Malathi, V., Suleiman, M.B., Taib, B.B.: Computing eigenvalues of periodic Sturm–Liouville problems using shooting technique and direct integration method. Int. J. Comput. Math. **68**(1–2), 119–132 (1998)
16. Marchenko, V.: Sturm–Liouville Operators and Applications. Birkhauser, Basel (1986)
17. Naimark, M.A.: Linear Differential Operators. George G. Harap & Company, (1967)
18. Pöschel, J., Trubowitz, E.: Inverse Spectral Theory. Academic Press, Boston (1987)
19. Shkalikov, A.A., Veliev, O.A.: On the Riesz basis property of the eigen- and associated functions of periodic and antiperiodic Sturm–Liouville problems. Math. Notes **85**(5), 647–660 (2009)
20. Tamarkin, Y.D.: Some general problem of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions. Math. Z. **27**, 1–54 (1927)
21. Vanden Berghe, G., Van Daele, M., De Meyer, H.: A modified difference scheme for periodic and semiperiodic Sturm–Liouville problems. Appl. Numer. Math. **18**(1–3), 69–78 (1995)
22. Veliev, O.A., Duman, M.T.: The spectral expansion for a nonselfadjoint hill operators with a locally integrable potential. J. Math. Anal. Appl. **265**(1), 76–90 (2002)
23. Wong, Y.S., Ji, X.Z.: On shooting algorithm for Sturm–Liouville eigenvalue problems with periodic and semi-periodic boundary conditions. Appl. Math. Comput. **51**(2–3), 87–104 (1992)
24. Yılmaz, B., Veliev, O.A.: Asymptotic formulas for Dirichlet boundary value problems. Studia Sci. Math. Hung. **42**(2), 153–171 (2005)