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The continuum branch of positive solutions for discrete simply supported beam equation with local linear growth condition

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Abstract

In this paper, we obtain the global structure of positive solutions for nonlinear discrete simply supported beam equation

$$\begin{aligned} &\Delta^4 u(t-2) = \lambda f(t,u(t)), \quad t \in \mathbb{T}, \\ &u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{aligned}$$

with $f \in C(\mathbb{T} \times [0,\infty),[0,\infty))$ satisfying local linear growth condition and $f(t,0)=0$ uniformly for $t \in \mathbb{T}$, where $\mathbb{T} = \{2,\dots,T\}$, $\lambda > 0$ is a parameter. The main results are based on the global bifurcation theorem.

MSC: 34B15; 34B18; 34B27

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1 Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time, which often occur in numerous settings and forms, both in mathematics and in its applications to economics, statistics, biology, numerical computing, electrical circuit analysis, and other fields; see [1].

It is well known that the fourth-order two-point boundary value problem

$$\begin{aligned} &u^{(4)}(t) = f\bigl(t,u(t)\bigr), \quad t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{1.1}$$

appears in the theory of hinged beams [2, 3], so the existence and multiplicity of positive solutions for (1.1) and its discrete analog have been studied by many authors; see, for example, [3–18]. Especially, Refs. [15–18] introduce the new analytical and numerical methods for differential equations with boundary value problems.

Let a, b be integer and $[a,b]_{\mathbb{Z}} = \{a, a+1, \ldots, b\}$. In 2002, Zhang et al. [10] and He et al. [11] studied the existence of positive solutions for the following discrete analog:

$$\begin{aligned} &\Delta^4 u(t-2) = \lambda h(t) f\bigl(u(t)\bigr), \quad t \in [2,T]_{\mathbb{Z}}, \\ &u(0) = u(T+2) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{aligned} \tag{1.2}$$

here Δ is forward difference operator with $\Delta u(t) = u(t+1) - u(t)$, $h : [2,T]_{\mathbb{Z}} \to [0,\infty)$ and $f \in C([0,\infty),[0,\infty))$. It has been pointed out in [10, 11] that (1.2) is equivalent to the summing equation of the form

$$u(t) = \lambda \sum_{s=1}^{T+1} G(t,s) \sum_{j=2}^{T} G_1(s,j) a(j) f\bigl(u(j)\bigr) =: A_0 u(t), \quad t \in [0, T+2]_{\mathbb{Z}}, \tag{1.3}$$

where

$$G(t,s) = \frac{1}{T+2} \begin{cases} s(T+2-t), & 1 \le s \le t \le T+2, \\ t(T+2-s), & 0 \le t \le s \le T+1, \end{cases}$$

and

$$G_1(t,i) = \frac{1}{T} \begin{cases} (T+1-t)(i-1), & 2 \le i \le t \le T+1, \\ (T+1-i)(t-1), & 1 \le t \le i \le T. \end{cases}$$

Notice that two distinct Green's functions are used in the summing equation (1.3), which makes the construction of cones and the verification of strong positivity of A_0 become more complex and difficult.

Therefore, Ma and Xu [12] considered the discrete analog of (1.1) as follows:

$$\begin{aligned} &\Delta^4 u(t-2) = \lambda f\bigl(t, u(t)\bigr), \quad t \in [2,T]_{\mathbb{Z}}, \\ &u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{aligned} \tag{1.4}$$

and introduced the definition of *generalized positive solutions*.

Definition 1.1 A function $u : [0, T+2]_{\mathbb{Z}} \to [0,\infty)$ is called a *generalized positive solution* of (1.4), if u satisfies (1.4), and $u(t) \ge 0$ on $[1, T+1]_{\mathbb{Z}}$ and $u(t) > 0$ on $[2,T]_{\mathbb{Z}}$.

They also applied the fixed point theorem in cones to obtain some results on the existence of generalized positive solutions of (1.4); see [12]. Ma and Lu [13] applied the Dancer's global bifurcation theorem to obtain some new results on the existence and multiplicity of generalized positive solutions of discrete simply supported beam equation (1.4) with $\lambda = 1$.

However, in these papers, they assumed that the nonlinearity $f \in C([2,T]_{\mathbb{Z}} \times [0,\infty), [0,\infty))$ satisfies

$$f(t,u) \ge 0 \quad \text{on } [2,T]_{\mathbb{Z}} \times [0,\infty) \quad \text{and} \quad f(t,u) > 0 \quad \text{on } [2,T]_{\mathbb{Z}} \times (0,\infty).$$

Of course, a natural question is what would happen if f only is positive on a subinterval $[a,b]_{\mathbb{Z}} \subset [2,T]_{\mathbb{Z}}$? That is, f satisfies

$$f(t,u) \geq 0 \quad \text{on } [2,T]_{\mathbb{Z}} \times [0,\infty) \quad \text{and} \quad f(t,u) > 0 \quad \text{on } [a,b]_{\mathbb{Z}} \times [0,\infty).$$

Based on the above reasons, we shall show the global structure of positive solutions of (1.4) under the followings assumptions:

(H1) $f \in C([2,T]_{\mathbb{Z}} \times [0,\infty), [0,\infty))$ with $f(t,0) = 0$ for $t \in \mathbb{T}$;

(H2) there exist $\alpha_1 : [2,T]_{\mathbb{Z}} \to (0,\infty)$, $\beta_1 : [a,b]_{\mathbb{Z}} \to (0,\infty)$ such that

$$f(t,u) = \alpha_1(t)u(t) + \zeta(t,u), \quad (t,u) \in [2,T]_{\mathbb{Z}} \times [0,\infty),$$

and

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = \beta_1(t) \quad \text{for a subinterval } t \in [a,b]_{\mathbb{Z}} \subset [2,T]_{\mathbb{Z}},$$

where $\lim_{u\to 0} \frac{\zeta(t,u)}{u} = 0$ uniformly for $t \in [2,T]_{\mathbb{Z}}$;

(H3) there exist $\alpha_2 : [2,T]_{\mathbb{Z}} \to (0,\infty)$, $\beta_2 : [a,b]_{\mathbb{Z}} \to (0,\infty)$ such that

$$f(t,u) = \alpha_2(t)u(t) + \xi(t,u), \quad (t,u) \in [2,T]_{\mathbb{Z}} \times [0,\infty),$$

and

$$\lim_{u\to 0} \frac{f(t,u)}{u} = \beta_2(t) \quad \text{for a subinterval } t \in [a,b]_{\mathbb{Z}} \subset [2,T]_{\mathbb{Z}},$$

where $\lim_{u\to \infty} \frac{\xi(t,u)}{u} = 0$ uniformly for $t \in [2,T]_{\mathbb{Z}}$.

Let $\lambda_1(\alpha_i)$, $i = 1,2$, be the principal eigenvalue of the linear eigenvalue problem

$$\begin{aligned} &\Delta^4 u(t-2) = \lambda\alpha_i(t)u(t), \quad t \in [2,T]_{\mathbb{Z}}, \\ &u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{aligned} \tag{1.5$_i$}$$

and $\mu_1(\beta_i)$, $i = 1,2$ the principal eigenvalue of the linear eigenvalue problem

$$\begin{aligned} &\Delta^4 u(t-2) = \mu b_i(t)u(t), \quad t \in [a,b]_{\mathbb{Z}}, \\ &u(a-1) = u(b+1) = \Delta^2 u(a-2) = \Delta^2 u(b) = 0. \end{aligned} \tag{1.6$_i$}$$

Let $E = \{u : [0,T+2]_{\mathbb{Z}} \to \mathbb{R} \mid u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0\}$ be a Banach space with the norm $\|u\| = \max_{t\in[0,T+2]_{\mathbb{Z}}} |u(t)|$. Denote by Σ the closure of the set

$$\bigl\{(\lambda,u) \in [0,\infty) \times E \mid u \text{ satisfies (1.4) and } u \not\equiv 0\bigr\}$$

in $\mathbb{R} \times E$. Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. We add the points $\{(\lambda,\infty) \mid \lambda \in \mathbb{R}\}$ to our space $\mathbb{E}$. Let S^+ denote the set of generalized positive functions in $\mathbb{E}$ and $S^- = -S^+$, and $S = S^- \cup S^+$. They are disjoint and open in $\mathbb{E}$. Finally, let $\nu \in \{+,-\}$ and $\Phi^\nu = \mathbb{R} \times S^\nu$ and $\Phi = \mathbb{R} \times S$.

Our main results are the following.

Theorem 1.1 *Let* (H1), (H2) *hold. If*

$$\mu_1(\beta_1) < \lambda_1(\alpha_1), \tag{1.7}$$

then there exists a connected component $\mathcal{C}^+ \in \Sigma$ *such that*

(i) $(\mathcal{C}^+ \backslash \{(\lambda_1(\alpha_1), 0)\}) \subset \Phi^+$;

(ii) $\mathcal{C}^+$ *meets* $(\lambda_1(\alpha_1), 0)$ *and* $(\mu_1(\beta_1), \infty)$ *in* λ-*direction*;

(iii) $\operatorname{Proj}_{\mathbb{R}} \mathcal{C}^+ \supset (\mu_1(\beta_1), \lambda_1(\alpha_1))$;

(iv) *for any* $(\lambda, u^+) \in \mathcal{C}^+$, *it follows that* $u^+(t) > 0$, $t \in [2, T]_{\mathbb{Z}}$.

Corollary 1.1 *Let* (H1), (H2) *hold. If* $\mu_1(\beta_1) < \lambda < \lambda_1(\alpha_1)$, *then the problem* (1.4) *has at least one generalized positive solution.*

Theorem 1.2 *Let* (H1), (H3) *hold. If*

$$\mu_1(\beta_2) < \lambda_1(\alpha_2), \tag{1.8}$$

then there exists a connected component $\mathcal{D}^+ \in \Sigma$ *such that*

(i) $(\mathcal{D}^+ \backslash \{(\lambda_1(\alpha_2), \infty)\}) \subset \Phi^\nu$;

(ii) $\mathcal{D}^+$ *meets* $(\lambda_1(\alpha_2), \infty)$ *and* $(\mu_1(\beta_2), 0)$ *in* λ-*direction*;

(iii) $\operatorname{Proj}_{\mathbb{R}} \mathcal{D}^+ \supset (\mu_1(\beta_2), \lambda_1(\alpha_2))$;

(iv) *for any* $(\lambda, u^+) \in \mathcal{D}^+$, *it follows that* $u^+(t) > 0$, $t \in [2, T]_{\mathbb{Z}}$.

Corollary 1.2 *Let* (H1), (H3) *hold. If* $\mu_1(\beta_2) < \lambda < \lambda_1(\alpha_2)$, *then the problem* (1.4) *has at least one generalized positive solution.*

Remark 1.1 Let $\alpha_i(t) \equiv \hat{\alpha}_i$ and $\beta_i(t) \equiv \hat{\beta}_i$ ($i = 1, 2$) be constants. It is easy to compute that $\lambda_1(\hat{\alpha}_i) = \frac{(2-2\cos\frac{\pi}{T})^2}{\hat{\alpha}_i}$ and $\mu_1(\hat{\beta}_i) = \frac{(2-2\cos\frac{\pi}{b+2-a})^2}{\hat{\beta}_i}$, see [14]. So the conditions of Corollaries 1.1–1.2 are equivalent to

$$\frac{(2-2\cos\frac{\pi}{b+2-a})^2}{\hat{\beta}_1} < \lambda < \frac{(2-2\cos\frac{\pi}{T})^2}{\hat{\alpha}_1}$$

and

$$\frac{(2-2\cos\frac{\pi}{b+2-a})^2}{\hat{\beta}_2} < \lambda < \frac{(2-2\cos\frac{\pi}{T})^2}{\hat{\alpha}_2},$$

respectively.

Moreover, from the above inequalities one concludes that $\hat{\beta}_1 = \infty$, $\hat{\alpha}_1 = 0$, i.e. f is sublinear growth at zero and superlinear growth at infinity about u; $\hat{\beta}_2 = \infty$, $\hat{\alpha}_2 = 0$, i.e. f is superlinear growth at zero and sublinear growth at infinity about u.

Especially, if $[a, b]_{\mathbb{Z}} = [2, T]_{\mathbb{Z}}$, $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$, that is to say, f is linear growth at zero and infinity about u, then the main results give immediately the classical result; see the result of the case $n = 2$ in Theorem 4.1 of [14].

Remark 1.2 If $\lambda = 1$, then the problem (1.4) is the discrete analog of (1.1). Corollaries 1.1–1.2 give the sharp condition of existence results of positive solutions for the discrete analog of (1.1); see [13].

Remark 1.3 It is worth remarking that the global structure of the positive solution curves is very useful for computing the numerical solution of (1.1), for example, it can be used to estimate the value of u in advance in applying the finite difference method.

2 Preliminaries

Let $Y := \{u : [0, T+2]_{\mathbb{Z}} \to \mathbb{R}\}$ be the Banach space with the norm $\|u\| = \max_{t\in[0,T+2]_{\mathbb{Z}}} |u(t)|$. Let E be the Banach space

$$E = \bigl\{u \in Y \mid u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0\bigr\}$$

with the norm $\|u\|$.

Define a linear operator $L : E \to Y$ by

$$Lu := \Delta^4 u(t-2), \quad u \in E. \tag{2.1}$$

From [13], we can see that (1.4) is equivalent to the summing equation

$$u(t) = \lambda \sum_{s=2}^{T} G(t,s) f\bigl(s, u(s)\bigr) =: Tu(t), \quad t \in [0, T+2]_{\mathbb{Z}}, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{(s-1)(T+1-t)[2T(t-1)-(t-1)^2-s(s-2)]}{6T}, & 2 \le s \le t \le T+1, \\ \frac{(t-1)(T+1-s)[2T(s-1)-(s-1)^2-t(t-2)]}{6T}, & 1 \le t \le s \le T. \end{cases} \tag{2.3}$$

It is not difficult to verify that the Green's function $G(t,s)$ satisfies the following properties:

$$c(t)\Phi(s) \le G(t,s) \le \Phi(s) \quad \text{for } s \in [1, T+1]_{\mathbb{Z}}, t \in [1, T+1]_{\mathbb{Z}}, \tag{2.4}$$

where

$$\Phi(s) = \begin{cases} \frac{\sqrt{3}}{27T}(s-1)(T^2-(s-2)s)^{3/2}, & 1 \le s \le \frac{T}{2}+1, \\ \frac{\sqrt{3}}{27T}(T+1-s)(2T(s-1)-(s-2)s)^{3/2}, & \frac{T}{2}+1 < s \le T+1, \end{cases}$$

$$c(t) = \begin{cases} \frac{3\sqrt{3}[T^2-t(t-2)](t-1)}{2(T^2+1)^{3/2}}, & 1 \le t \le \frac{T}{2}+1, \\ \frac{3\sqrt{3}(T+1-t)[2T(t-1)-t(t-2)]}{2(T^2+1)^{3/2}}, & \frac{T}{2}+1 < s \le T+1. \end{cases}$$

Moreover, let $[a,b]_{\mathbb{Z}}$ be a subinterval of $[2,T]_{\mathbb{Z}}$, then, for any $(t,s) \in [a,b]_{\mathbb{Z}} \times [1, T+1]_{\mathbb{Z}}$, we have

$$G(t,s) \ge \sigma \Phi(s),$$

here

$$\sigma = \min\left\{\frac{3\sqrt{3}(a-1)[T^2-a(a-2)]}{2(T^2+1)^{3/2}}, \frac{3\sqrt{3}(b-1)[T^2-b(b-2)]}{2(T^2+1)^{3/2}},\right.$$
$$\frac{3\sqrt{3}(T+1-a)[2T(a-1)-a(a-2)]}{2(T^2+1)^{3/2}},$$
$$\left.\frac{3\sqrt{3}(T+1-b)[2T(b-1)-b(b-2)]}{2(T^2+1)^{3/2}}\right\}. \tag{2.5}$$

Define the cone P in E by

$$P = \left\{u \in E : u(t) \geq 0 \text{ on } [1, T+1]_{\mathbb{Z}} \text{ and } \min_{t\in[a,b]_{\mathbb{Z}}} u(t) \geq \sigma \|u\|\right\}.$$

By a standard argument, it is easy to verify that $T : P \to P$ is completely continuous.

Lemma 2.1 ([13, Lemma 3.2]) *Let $\alpha_i : [2, T]_{\mathbb{Z}} \to (0, \infty)$, $i = 1, 2$. Then the principal eigenvalue $\lambda_1(\alpha_i)$ of the eigenvalue problems $(1.5)_i$ is positive and the corresponding eigenfunctions $\varphi_i(t)$ is positive on $[2, T]_{\mathbb{Z}}$.*

By the same method with obvious changes, we can see that the problem

$$\Delta^4 u(t-2) = \lambda f\bigl(t, u(t)\bigr) \quad t \in [a, b]_{\mathbb{Z}},$$
$$u(a-1) = u(b+1) = \Delta^2 u(a-2) = \Delta^2(b) = 0,$$

is equivalent to the summing equation

$$u(t) = \lambda \sum_{s=a}^{b} K(t,s) f\bigl(s, u(s)\bigr), \quad t \in [a-2, b+2]_{\mathbb{Z}},$$

where

$$K(t,s) = \begin{cases} \frac{(s-a+1)(b+1-t)[2(b+2-a)(t-a+1)-(t-a+1)^2-(s-a)(s-a+2)]}{6(b+2-a)}, & a \leq s \leq t \leq b+1, \\ \frac{(t-a+1)(b+1-s)[2(b+2-a)(s-a+1)-(s-a+1)^2-(t-a)(t-a+2)]}{6(b+2-a)}, & a-1 \leq t \leq s \leq b. \end{cases} \tag{2.6}$$

Moreover, by a similar argument to [13, Lemma 3.2], we can obtain the following.

Lemma 2.2 *Let $\beta_i : [a, b]_{\mathbb{Z}} \to (0, \infty)$, $i = 1, 2$. Then the principal eigenvalue $\lambda_1(\beta_i)$, $i = 1, 2$, of the eigenvalue problems $(1.6)_i$ is positive and the corresponding eigenfunctions $\psi_i(t)$ is positive on $[a, b]_{\mathbb{Z}}$.*

3 The proof of the main results

To apply the unilateral global bifurcation results [19–22], we extend f by an odd function $g : [2, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ such that

$$g(t,s) = \begin{cases} f(t,s), & (t,s) \in [2,T]_{\mathbb{Z}} \times [0, \infty), \\ -f(t,-s), & (t,s) \in [2,T]_{\mathbb{Z}} \times (-\infty, 0). \end{cases} \tag{3.1}$$

Now let us consider an auxiliary family of equations

$$\begin{aligned}&\Delta^4 u(t-2) = \lambda g\bigl(t, u(t)\bigr), \quad t \in [2, T]_{\mathbb{Z}},\\&u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0.\end{aligned} \tag{3.2}$$

It follows from (H2) that

$$g(t,u) = \alpha_1(t)u + \zeta(t,u). \tag{3.3}$$

Note that

$$\lim_{|u|\to 0} \frac{\zeta(t,u)}{u} = 0 \quad \text{uniformly for } t \in [2,T]_{\mathbb{Z}}.$$

Proof of Theorem 1.1 (i) Let us consider

$$\begin{aligned}&\Delta^4 u(t-2) = \lambda\alpha_1(t)u(t) + \lambda\zeta(t,u), \quad t \in [2, T]_{\mathbb{Z}},\\&u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0,\end{aligned} \tag{3.4}$$

as a bifurcation problem for the trivial solution $u \equiv 0$. Problem (3.4) can be converted to the summing equation

$$u(t) = \sum_{s=2}^{T} G(t,s)\bigl[\lambda\alpha_1(s)u(s) + \lambda\zeta\bigl(s,u(s)\bigr)\bigr],$$

where $G(t,s)$ is defined by (2.3).

Furthermore, we note that $\|L^{-1}(\zeta(\cdot,u(\cdot)))\| = \|\sum_{s=2}^{T} G(\cdot,s)\zeta(s,u(s))\| = o(\|u\|)$ for u near 0 in E, since

$$\bigl\|L^{-1}\bigl(\zeta\bigl(\cdot,u(\cdot)\bigr)\bigr)\bigr\| = \max_{t\in[0,T+2]_{\mathbb{Z}}} \left|\sum_{s=2}^{T} G(t,s)\zeta\bigl(s,u(s)\bigr)\right| \le C_0\bigl\|\zeta\bigl(\cdot,u(\cdot)\bigr)\bigr\|.$$

From Lemma 2.1, the algebraic multiplicity of $\lambda_1(\alpha_1)$ equals 1, the pair $(\lambda_1(\alpha),0)$ is a bifurcation point of problem (3.4). Therefore, according to a revised version of [19, Theorem 6.2.1], there exists a component, denoted by $\mathcal{C} \subset \Sigma$, emanating from $(\lambda_1(\alpha_1),0)$. Moreover, (3.4) enjoys all the structural requirements for applying the unilateral global bifurcation theory of López-Gómez [19, Sects. 6.4–6.5], and thanks to the global alternative of Rabinowitz (see, e.g., [20, Corollary 6.3.2]), either $\mathcal{C}$ is unbounded in $\mathbb{E}$, or $(\lambda_j(\alpha_1),0) \in \mathcal{C}$ for some $\lambda_j(\alpha_1) \neq \lambda_1(\alpha_1)$, or contains a point $(\lambda,u) \in \mathbb{R} \times (E_0\backslash\{0\})$, here $\lambda_j(\alpha_1)$ is another eigenvalue of $(1.5)_1$, and $E = \operatorname{span}\{\varphi_1\} \oplus E_0$.

Although the unilateral bifurcation results of [20, Theorems 1.27 and 1.40] cannot be applied here, among other things because they are false as originally stated (cf. the counterexample of Dancer [21]), the reflection argument of [20] and a similar argument to Theorem 6.4.3 of [19] can be applied to conclude that $\mathcal{C} = \mathcal{C}^+ + \mathcal{C}^-$, where $\mathcal{C}^+$ is the component of positive solutions emanating from $(\lambda_1(\alpha_1),0)$, because of

$$-u = \lambda L^{-1}(-u) + \lambda L^{-1}\zeta\bigl(\cdot,-u(\cdot)\bigr) = -\bigl[\lambda L^{-1}u + \lambda L^{-1}\zeta\bigl(\cdot,u(\cdot)\bigr)\bigr], \quad u \in E.$$

Moreover, $\mathcal{C}^\nu$ must be unbounded and, $\mathcal{C}^\nu\backslash\{(\lambda_1(\alpha_1),0)\} \subset \Phi^+$.

(ii) It is clear that any solution of (3.4) of the form (λ, u) yields a solution u of (1.4). We will show $\mathcal{C}^+ \subset \mathbb{R} \times E$ meets $(\lambda_1(\alpha_1), 0)$ and $(\mu_1(\beta_1), \infty)$ in λ-direction. To do this, it is enough to show that $\mathrm{Proj}_{\mathbb{R}}\,\mathcal{C}^+ \supset (\mu_1(\beta_1), \lambda_1(\alpha_1))$. Let $(\lambda_n, u_n) \in \mathcal{C}^+$ satisfy

$$|\lambda_n| + \|u_n\| \to \infty. \tag{3.5}$$

We note that $\lambda_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (3.4) for $\lambda = 0$ and $\mathcal{C}^+ \cap (\{0\} \times E) = \emptyset$.

Now we show that

$$\bigl(\mu_1(\beta_1), \lambda_1(\alpha_1)\bigr) \subseteq \bigl\{\lambda \in \mathbb{R} \mid \exists (\lambda, u) \in \mathcal{C}^+\bigr\}.$$

We divide the proof into two steps.

Step 1 We show that there exists a constant number $M > 0$ such that

$$\lambda_n \subset (0, M], \quad \forall n \in \mathbb{N}^*.$$

Since (λ_n, u_n) is the solution of (3.4), it follows that

$$\begin{aligned} &\Delta^4 u_n(t-2) = \lambda_n \frac{f(t, u_n(t))}{u_n(t)} u_n(t), \quad t \in [2, T]_{\mathbb{Z}}, \\ &u_n(1) = u_n(T+1) = \Delta^2 u_n(0) = \Delta^2 u_n(T) = 0. \end{aligned} \tag{3.6}$$

From (H2), there exists a constant $\varrho_1 > 0$ such that $\frac{f(t,u)}{u} \geq \varrho_1 > 0$ uniformly for $t \in [a,b]_{\mathbb{Z}}$. From (2.2)–(2.5), we have

$$\min_{t \in [a,b]_{\mathbb{Z}}} u_n(t) \geq \sigma \|u\|. \tag{3.7}$$

Then, for any $t \in [a,b]_{\mathbb{Z}}$, we get

$$\begin{aligned} u_n(t) &= \lambda_n \sum_{s=2}^{T} G(t,s) \frac{f(s, u_n(s))}{u_n(s)} u_n(s) \\ &\geq \lambda_n \sum_{s=a}^{b} G(t,s) \frac{f(s, u_n(s))}{u_n(s)} u_n(s) \\ &\geq \lambda_n \sigma^2 \varrho_1 \sum_{s=a}^{b} \Phi(s) \|u_n\|. \end{aligned}$$

Therefore

$$0 < \lambda_n \leq \sigma^{-2} \left[\varrho_1 \sum_{s=a}^{b} \Phi(s) \right]^{-1} =: M.$$

Step 2 We show that $\mathrm{Proj}_{\mathbb{R}}\,\mathcal{C}^+ \supset (\mu_1(\beta_1), \lambda_1(\alpha_1))$.

From Step 1 and (3.5), it follows that

$$\|u_n\| \to \infty. \tag{3.8}$$

This combining (3.7) and (3.8) implies

$$\min_{t\in[a,b]_{\mathbb{Z}}} u_n(t) \to \infty.$$

Let us consider the problem (3.6) and divide (3.6) by $\|u_n\|$ and set $v_n = \frac{u_n}{\|u_n\|}$. Since v_n is bounded in E, after taking a subsequence if necessary, we have $v_n \to v$ for some $v \in E$ with $\|v\| = 1$. Moreover, let $y_n = v_n(t)|_{[a,b]_{\mathbb{Z}}}$ and $y_n \to y = v(t)|_{[a,b]_{\mathbb{Z}}}$, then there exist constants $r_1, r_2 \geq 0$, $r_3, r_4 \leq 0$ such that

$$\begin{aligned} &\Delta^4 y_n(t-2) = \lambda_n \frac{f(t,u_n(t))}{u_n(t)} y_n(t), \quad t \in [a,b]_{\mathbb{Z}}, \\ &y_n(a-1) = r_1, \qquad y_n(b+1) = r_2, \qquad \Delta^2 y_n(a-2) = r_3, \qquad \Delta^2 y_n(b) = r_4. \end{aligned} \tag{3.9}$$

From (H2), it follows that

$$y(t) = \sum_{s=a}^{b} K(t,s)\tilde{\lambda}\beta_1(s)y(s) + R(t), \quad t \in [a-2,b+2]_{\mathbb{Z}},$$

where $\tilde{\lambda} = \lim_{n\to\infty} \lambda_n$, $K(t,s)$ is defined by (2.6), and

$$\begin{aligned} R(t) = {} & \frac{(2r_1 - r_3)(a-1) + (r_4 - 2r_2)(b+1)}{6(b+2-a)(b+a)} t^3 \\ & + \frac{(6r_2 - 3r_4)(a-1)^2 + (3r_3 - 6r_1)(b+1)^2}{6(b+2-a)(b+a)} t^2 \\ & + \frac{(r_3 - 2r_1)(a-1)^3 + (2r_2 - r_4)(b+1)^3 - 6(a-1)r_1 + 6(b+1)r_2}{6(b+2-a)(b+a)} t \\ & + \frac{(3r_4 - 3r_3 + 6r_1 - 6r_2)(a-1)^2(b+1)^2 - 6r_2(a-1)^2 + 6r_1(b+1)^2}{6(b+2-a)(b+a)}, \end{aligned}$$

again choosing a subsequence and relabeling if necessary. Thus

$$\begin{aligned} &\Delta^4 y(t-2) = \tilde{\lambda}\beta_1(t)y(t), \quad t \in [a,b]_{\mathbb{Z}}, \\ &y(a-1) = r_1, \qquad y(b+1) = r_2, \qquad \Delta^2 y(a-2) = r_3, \qquad \Delta^2 y(b) = r_4. \end{aligned} \tag{3.10}$$

We claim that $y \in \mathcal{C}^+$. Suppose on the contrary that $y \notin \mathcal{C}^+$. Since $y \not\equiv 0$ is a solution of (3.10) and there exists $c \in [a,b]_{\mathbb{Z}}$ such that $y(c)y(c+1) \leq 0$, which together with the fact $y_n \in E$ implies that y changes its sign in $[a,b]_{\mathbb{Z}}$. This contradicts the facts that $y_n \to y$ in E and $y_n \in \mathcal{C}^+$. Therefore $y \in \mathcal{C}^+$. Moreover, let us consider the problem (3.10) and the problem

$$\begin{aligned} &\Delta^4 \psi_1(t-2) = \mu_1(\beta_1)\beta_1(t)\psi_1(t), \quad t \in [a,b]_{\mathbb{Z}}, \\ &\psi_1(a-1) = \psi_1(b+1) = \Delta^2\psi_1(a-2) = \Delta^2\psi(b) = 0. \end{aligned} \tag{3.11}$$

Multiplying $\psi_1(t)$ in (3.10) and $y(t)$ in (3.11), then summing from $t = a$ to b and subtracting, it follows that $\Delta^2 y(b)\psi_1(b) + \Delta^2\psi_1(b-1)y(b+1) + \psi_1(a)\Delta^2 y(a-2) + y(a-1)\Delta^2\psi_1(a-$

$1) = (\tilde{\lambda} - \mu_1(\beta_1))\sum_{t=a}^{b}\beta_1(t)y(t)\psi_1(t)$. It follows from $y(t) > 0$, $\Delta^2 y(t-1) < 0$ on $[2,T]_{\mathbb{Z}}$ and $\psi_1(t) \geq 0$, $\Delta^2\psi_1(t-1) \leq 0$ on $[a-1,b+1]_{\mathbb{Z}}$ that

$$\big(\tilde{\lambda} - \mu_1(\beta_1)\big)\sum_{t=a}^{b}\beta_1(t)y(t)\psi_1(t) < 0.$$

That is, $\tilde{\lambda} < \mu_1(\beta_1)$. Thus, $\mathrm{Proj}_{\mathbb{R}}\,\mathcal{C}^+ \supset (\mu_1(\beta_1), \lambda_1(\alpha_1))$.

Hence, the conclusions of (ii)–(iv) are true. □

Let $\lambda_1(\alpha_2)$ is the principal eigenvalue of $(1.5)_2$, then from Lemma 2.1, $\lambda_1(\alpha_2)$ is isolated, having geometric multiplicity 1. Let E_0 be a closed subspace of E such that $E = \mathrm{span}\{\varphi_2\} \oplus E_0$, where φ_2 is defined as Lemma 2.1 and $\|\varphi_2\| = 1$. Let $B_r(0) = \{u \in E \mid \|u\| < r\}$.

Proof of Theorem 1.2 (i) Let $\xi \in C([2,T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$ be such that

$$g(t,u) = \alpha_2(t)u + \xi(t,u).$$

Note that

$$\lim_{|u|\to\infty}\frac{\xi(t,u)}{u} = 0 \quad \text{uniformly for } t \in [2,T]_{\mathbb{Z}}. \tag{3.12}$$

Let us consider

$$\begin{aligned} &\Delta^4 u(t-2) = \lambda\alpha_2(t)u(t) + \lambda\zeta(t,u), \quad t \in [2,T]_{\mathbb{Z}}, \\ &u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{aligned} \tag{3.13}$$

as a bifurcation problem from the infinity. Problem (3.13) can be converted to the equivalent equation

$$u(t) = \sum_{s=2}^{T} G(t,s)\big[\lambda\alpha_2(s)u(s) + \lambda\xi\big(s,u(s)\big)\big],$$

where $G(t,s)$ is defined by (2.3). Furthermore, we note that

$$\big\|L^{-1}\big(\xi\big(\cdot,u(\cdot)\big)\big)\big\| = \left\|\sum_{s=2}^{T} G(\cdot,s)\xi\big(s,u(s)\big)\right\| = o\big(\|u\|\big)$$

for u near ∞ in E, since

$$\big\|L^{-1}\big(\xi\big(\cdot,u(\cdot)\big)\big)\big\| = \max_{t\in[0,T+2]_{\mathbb{Z}}}\left|\sum_{s=2}^{T} G(t,s)\xi\big(s,u(s)\big)\right| \leq C_1\big\|\xi\big(\cdot,u(\cdot)\big)\big\|.$$

By a similar argument to [22] and the structural requirements for applying the unilateral global bifurcation theory of López-Gómez [19, Sects. 6.4–6.5], we can conclude to the following.

Let $\lambda_1(\alpha_2)$ be the principal eigenvalue of $(1.5)_2$, such that

$$\deg\bigl(I - L\bigl(\lambda_1(\alpha_2) - \varepsilon\bigr), B_r(0), 0\bigr) \neq \deg\bigl(I - L\bigl(\lambda_1(\alpha_2) + \varepsilon\bigr), B_r(0), 0\bigr)$$

for any $\varepsilon > 0$ small enough, then Σ possesses two unbounded components $\mathcal{D}^+$ and $\mathcal{D}^-$, which meet $(\lambda_1(\alpha_2), \infty)$. Moreover, if $\Lambda_* \subset \mathbb{R}$ is an interval such that $\Lambda_* \cap \Lambda_2 = \{\lambda_1(\alpha_2)\}$ and $\mathcal{M}$ is a neighborhood of $(\lambda_1(\alpha_2), \infty)$ whose projection on $\mathbb{R}$ lies in Λ_* and whose projection on E is bounded away from 0, here Λ_2 denotes the set of real eigenvalues of $(1.5)_2$, then at least one of the following three properties is satisfied by $\mathcal{D}^\nu$ for $\nu \in \{+, -\}$.

(1) $\mathcal{D}^\nu - \mathcal{M}$ is bounded in $\mathbb{R} \times E$ in which $\mathcal{D}^\nu - \mathcal{M}$ meets $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ or

(2) $\mathcal{D}^\nu - \mathcal{M}$ is unbounded,

(3) it contains a point $(\lambda, v) \in \mathbb{R} \times (E_0 \backslash \{0\})$.

If (2) occurs and $\mathcal{D}^\nu - \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $\mathcal{D}^\nu - \mathcal{M}$ meets (μ, ∞), where $\lambda_1(\alpha_2) \neq \mu \in \Lambda_2$.

By applying a similar argument to [19, Sects. 6.4–6.5] and [22], it is easy to verify that $\mathcal{D}^\nu$ must be unbounded and, $\mathcal{D}^\nu \subset \{(\lambda_1(\alpha_2), \infty)\} \cup \Phi^+$.

(ii) It is clear that any solution of (3.13) of the form (λ, u) yields a solution u of (1.4). We will show $\mathcal{D}^+$ meets $(\lambda_1(\alpha_2), \infty)$ and $(\mu_1(\beta_2), 0)$ in the λ-direction. To do this, we only need to show that $\mathrm{Proj}_{\mathbb{R}}\, \mathcal{D}^+ \supset (\mu_1(\beta_2), \lambda_1(\alpha_2))$. Let $(\lambda_n, u_n) \in \mathcal{D}^+$ satisfy $\|u_n\| \to 0$. Then $\lambda_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (3.13) for $\lambda = 0$ and $\mathcal{D}^+ \cap (\{0\} \times E) = \emptyset$.

Now we show that

$$\bigl(\mu_1(\beta_2), \lambda_1(\alpha_2)\bigr) \subseteq \bigl\{\lambda \in \mathbb{R} \mid \exists(\lambda, u) \in \mathcal{D}^+\bigr\}.$$

By a similar method to proving *Step 1* of Theorem 1.1, there exists a constant M_1, such that $0 < \lambda_n \leq M_1$. We only need to prove that

$$\lambda < \mu_1(\beta_2).$$

From $\|u_n\| \to 0$, let us consider (3.9), from (H3), it follows that

$$y(t) = \sum_{s=a}^{b} K(t,s)\hat{\lambda}\beta_2(s)y(s) + R(t), \quad t \in [a,b]_{\mathbb{Z}},$$

where $\hat{\lambda} = \lim_{n\to\infty} \lambda_n$, $R(t)$ and $K(t,s)$ are defined as Theorem 1.1, again choosing a subsequence and relabeling if necessary. Thus

$$\begin{aligned} &\Delta^4 y(t-2) = \hat{\lambda}\beta_2(t)y(t), \quad t \in [a,b]_{\mathbb{Z}}, \\ &y(a-1) = r_1, \qquad y(b+1) = r_2, \qquad \Delta^2 y(a-2) = r_3, \qquad \Delta^2 y(b) = r_4. \end{aligned} \tag{3.14}$$

We claim that $y \in \mathcal{D}^+$. Suppose on the contrary that $y \notin \mathcal{D}^+$. $y \neq 0$ is a solution of (3.14); there exists $c \in [a,b]_{\mathbb{Z}}$ such that $y(c)y(c+1) \leq 0$, and this together with the fact $y_n \in E$ implies that y changes its sign in $[a,b]_{\mathbb{Z}}$. This contradicts the facts that $y_n \to y$ in E and $y_n \in \mathcal{D}^+$. Therefore $y \in \mathcal{D}^+$. Moreover, let us consider the problem (3.14) and the problem

$$\begin{aligned} &\Delta^4 \psi_2(t-2) = \mu_1(\beta_2)\beta_2(t)\psi_2(t), \quad t \in [a,b]_{\mathbb{Z}}, \\ &\psi_2(a-1) = \psi_2(b+1) = \Delta^2\psi_2(a-2) = \Delta^2\psi_2(b) = 0. \end{aligned} \tag{3.15}$$

Multiplying $\psi_2(t)$ in (3.14) and $y(t)$ in (3.15), then summing from $t = a$ to b and subtracting, it follows that $\Delta^2 y(b)\psi_2(b) + \Delta^2\psi_2(b-1)y(b+1) + \psi_2(a)\Delta^2 y(a-2) + y(a-1)\Delta^2\psi_2(a-1) = (\hat{\lambda} - \mu_1(\beta_2))\sum_{t=a}^{b}\beta_2(t)y(t)\psi_2(t)$. It follows from $y(t) > 0$, $\Delta^2 y(t-1) < 0$ on $[2,T]_{\mathbb{Z}}$ and $\psi_2(t) \geq 0$, $\Delta^2\psi_2(t-1) \leq 0$ on $[a-1,b+1]_{\mathbb{Z}}$ that

$$\big(\hat{\lambda} - \mu_1(\beta_2)\big)\sum_{t=a}^{b}\beta_2(t)y(t)\psi_2(t) < 0.$$

That is, $\hat{\lambda} < \mu_1(\beta_2)$.

Therefore, the conclusions of (ii)–(iv) are true. □

From the proofs of Theorems 1.1–1.2, we can directly give the conclusions of Corollaries 1.1–1.2.

Example Let us consider the following problem:

$$\begin{aligned} &\Delta^4 u(t-2) = f\big(t,u(t)\big), \quad t \in [2,9]_{\mathbb{Z}}, \\ &u(1) = u(10) = \Delta^2 u(0) = \Delta^2 u(9) = 0, \end{aligned} \tag{3.16}$$

where

$$f(t,u) = \begin{cases} g(t)u, & (t,u) \in [1,9]_{\mathbb{Z}} \times [0, \frac{6}{6+\sqrt{3}-8\cos\frac{\pi}{12}}), \\ g(t)\frac{6}{6+\sqrt{3}-8\cos\frac{\pi}{12}}(u - \frac{8\cos\frac{\pi}{12}-\sqrt{3}}{6+\sqrt{3}-8\cos\frac{\pi}{12}}), & (t,s) \in [1,9]_{\mathbb{Z}} \times [\frac{6}{6+\sqrt{3}-8\cos\frac{\pi}{12}}, \infty), \end{cases}$$

and

$$g(t) = \begin{cases} 0, & t = 1, \\ 6 + \sqrt{3} - 8\cos\frac{\pi}{12}, & t \in [2,4]_{\mathbb{Z}}, \\ 0, & t \in [5,9]_{\mathbb{Z}}. \end{cases}$$

Clearly, $f(t,0) = 0$ uniformly in $t \in [1,T]_{\mathbb{Z}}$. Let $a = 2$, $b = 4$, then by computation $\hat{\beta}_1 = 6$ and $\mu_1(\hat{\beta}_1) = \frac{6-4\sqrt{2}}{6} < 1$, $\alpha_1(t) = g(t)$ and $\lambda_1(\alpha_1) > 1$. From Corollary 1.1 and Remark 1.2, the problem (3.16) has at least one generalized positive solution.

Let us consider the problem (3.16) with the nonlinearity

$$f(t,u) = \begin{cases} \frac{t}{50}u(12-\sqrt{2}-u) + \frac{1-t}{50}(12-\sqrt{2})u, & (t,u) \in [2,9]_{\mathbb{Z}} \times [0,1), \\ \frac{12-\sqrt{2}-t}{50}, & (t,s) \in [2,9]_{\mathbb{Z}} \times [1, 12-\sqrt{2}), \\ \frac{u-t}{50}, & (t,s) \in [2,9]_{\mathbb{Z}} \times [12-\sqrt{2}, \infty). \end{cases}$$

Clearly, $f(t,0) = 0$ for uniformly $t \in [2,9]_{\mathbb{Z}}$. Let $a = 2$, $b = 4$, then by simple computation $\beta_2(t) = 12 - \sqrt{2} - t$, $\mu_1(\beta_2) < 1$ and $\hat{\alpha}_2 = \frac{1}{50}$ and $\lambda_1(\hat{\alpha}_2) = 50(2 - 2\cos\frac{\pi}{9})^2 > 1$. From Corollary 1.2 and Remark 1.2, the problem (3.16) has at least one generalized positive solution.

4 Conclusions

By using the positive property of Green's function and the unilateral global bifurcation theorem, we obtain the global structure of positive solutions for a class of nonlinear discrete simply supported beam equation with the nonlinearity satisfying local linear growth conditions. The main results extend the existent results of positive solutions and generalize many related problems in the literature.

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