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Regularity of solutions to elliptic equations on Herz spaces with variable exponents

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Abstract

The aim of this paper is to show some regularity properties of solutions to $\mathcal{L}u = f$, where $\mathcal{L}$ is a second order nondivergence form operator and f belongs to a homogeneous Herz space $\dot{K}^{\alpha,p(\cdot)}_{q(\cdot)}$ with two variable exponents.

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1 Introduction

In this note we apply some recent estimates of singular integral operators and commutators to the study of regularity of solutions to second order elliptic equations with nondivergence principal part having discontinuous coefficients.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, and let us consider the partial differential equation

$$\mathcal{L}u \equiv a_{ij}(x)D_{x_i x_j}u = f(x), \tag{1.1}$$

where $a_{ij}, f : \Omega \to \mathbb{R}$. We require the ellipticity of the differential operator $\mathcal{L}$ imposing the positive definiteness of the coefficients matrix $\mathcal{A}(x) = \{a_{ij}(x)\}_{i,j=1,\dots,n}$. In particular, $\mathcal{L}$ is strictly elliptic if

$$\lambda(x) > \text{const} > 0 \quad \text{in } \Omega,$$

where $\lambda(x)$ is the minimum eigenvalue of $\mathcal{A}(x)$. Also, the uniform ellipticity means boundedness of the spectrum of $\mathcal{A}(x)$,

$$\frac{\Lambda(x)}{\lambda(x)} \leq \text{const} \quad \text{in } \Omega,$$

where with $\Lambda(x)$ we denote the maximum eigenvalue of $\mathcal{A}(x)$.

As usual, we say that a function $u \in C^2(\Omega)$ is a classical solution to (1.1) if it satisfies the equation at each $x \in \Omega$. It is clear that in order to deduce the solvability to (1.1) in that

rough sense, we should require some regularity assumptions on the coefficients a_{ij} beside the strict and uniform ellipticity mentioned above.

In the framework of the Schauder theory of linear elliptic operators, we have results dealing with classical solvability and regularity for equation (1.1) with Hölder continuous coefficients $a_{ij} \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, and $f \in C^{0,\alpha}(\overline{\Omega})$. However, it is well known that even the Poisson equation $\Delta u = f$ does not have a C^2-solution for continuous f.

This example justifies another definition of solution. Namely, it is common to define the concept of *strong* solution. We say that a strong solution to (1.1) is a twice weakly differentiable function, $u \in W^{2,p}(\Omega)$ for suitable $p \in (1, \infty)$, satisfying the equation $\mathcal{L}u = f$ almost everywhere (a.e.) in Ω. That new concept of solution was meant also for $f \in L^p(\Omega)$ and $a_{ij} \in L^\infty(\Omega)$.

On the other hand, the only strict ellipticity and the fact that the coefficients a_{ij} could be essentially bounded are not enough to deduce strong solvability of (1.1) as shown by Pucci. Precisely, from the studies of Pucci, it turns out that a satisfactory strong solvability theory of (1.1) can be built only with additional hypotheses on the coefficients a_{ij} of $\mathcal{L}$. These assumptions should be either regularity hypotheses on the coefficients a_{ij} or hypotheses on the spectrum of the matrix $\mathcal{A}(x)$ stronger than the strict and uniform ellipticity. For instance, if we assume that the coefficients a_{ij} are continuous in $\overline{\Omega}$,

$$a_{ij}(x) \in C^0(\overline{\Omega}),$$

the classical L^p-theory provides solvability and regularity for (1.1) in Sobolev spaces $W^{2,p}(\Omega)$ for any $p > 1$.

For further details, see, for instance, the classical monographs of Ladyzhenskaya and Ural'tseva [1], Ladyzhenskaya, Solonnikov, and Ural'tseva [2], Miranda [3], and Gilbarg and Trudinger [4].

Over the years there was developed an analogue of the L^p-theory of elliptic (and parabolic) equations, weakening the continuity assumption on the coefficients of the principal part of the operators under consideration. Precisely, it is interesting to investigate the case of *discontinuous* coefficients. We recall the so-called Cordes condition, introduced by Cordes in the study of Hölder continuity of solutions to (1.1). This condition substitutes the continuity assumption with the requirement that all the eigenvalues $\lambda_i(x)$ of the matrix $\mathcal{A}(x)$ do not spread too much. The Cordes condition allowed Talenti to obtain strong solvability in $W^{2,2}(\Omega)$ of the Dirichlet problem for the operator $\mathcal{L}$, and starting from his paper, other authors studied other boundary value problems of elliptic and parabolic type.

Another class of discontinuous coefficients was introduced by Miranda, and it consists of all functions that belong to the Sobolev space $W^{1,n}(\Omega)$. In spite of the difference between these two types of discontinuity, the approaches in the study of boundary value problems are unified by the Miranda–Talenti inequality which allows us to obtain the constants that appear in the L^2-a priori estimates.

It is important to underline that all the assumptions above permit to derive regularity results in the frameworks of $W^{2,2}$ or $W^{2,p}$ with p greater than, but sufficiently near to, 2.

Finally, it is important to investigate the L^p-theory of elliptic and parabolic operators with principal coefficients a_{ij} belonging to the Sarason class VMO of functions having vanishing mean oscillation. In two pioneering scientific notes, Chiarenza, Frasca, and Longo modified the classical methods in order to obtain L^p estimates for solutions to (1.1); the

approach followed by Chiarenza, Frasca, and Longo is strictly related to the theory of Calderón and Zygmund and uses a representation formula for the second derivatives D^2u of any solutions to (1.1) in terms of a singular integral that acts on f and a commutator that acts on the same derivatives D^2u and with density $a_{ij}(x) - a_{ij}(y)$. Thus, if the coefficients a_{ij} have a vanishing mean oscillation, guaranteed by the assumption $a_{ij} \in VMO$, then the L^p-norm of D^2u is bounded in terms of L^p-norm of f, and this result holds for any $p \in (1, \infty)$.

Taking into account that VMO contains as proper subsets both $C^0(\overline{\Omega})$ and $W^{1,n}(\Omega)$, the L^p-theory of operators with VMO principal coefficients generalized the theories known before 1990.

In this work the crucial assumption is $a_{ij} \in VMO$.

In the last years a lot of authors obtained regularity results in several spaces: for instance, we recall the regularity results obtained in the framework of Morrey spaces by Di Fazio and Ragusa in [5], generalized Morrey spaces by Scapellato in [6], and Herz spaces by Ragusa in [7] and Scapellato in [8]. In the first two papers, the authors use the representation formula for the second derivatives of a solution to the equations under consideration and apply the estimates contained in [9] and [10], respectively.

The regularity of solutions to differential problems was also extensively studied in the recent papers [11–13].

In this note, we obtain a regularity result in the framework of Herz spaces with variable exponents. The analysis of the boundedness of integral operators, the calculus of variations, the study of the regularity to solutions to partial differential equations and systems in such nonstandard functional spaces has shown a growing interest in recent years.

Our study starts from the boundedness results contained in [14] and uses the technique in [15] for interior estimates, based on the explicit representation formula for the second derivatives of a solution to (1.1).

In the next section we present some useful definitions and tools.

2 Homogeneous Herz spaces with variable exponent

Let Ω be a measurable set in $\mathbb{R}^n$. We firstly recall the definition of the Lebesgue spaces with variable exponent. For a deeper discussion of Lebesgue spaces with variable exponent, we refer the reader to [16].

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\cdot) : \Omega \to [1, \infty]$. The elements of $\mathcal{P}(\Omega)$ are called *exponent functions*. We denote by $\mathcal{B}(\Omega)$ the set of all functions $p(\cdot) \in \mathcal{P}(\Omega)$ such that the related Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.

Throughout this paper we will set

$$p_- = \operatorname{essinf}\{p(x) : x \in \Omega\}, \qquad p_+ = \operatorname{esssup}\{p(x) : x \in \Omega\}.$$

Let $p(\cdot) \in \mathcal{P}(\Omega)$ and let f be a Lebesgue measurable function. Let us define the modular functional associated with $p(\cdot)$ by

$$\rho_{p(\cdot),\Omega}(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} \, dx + \|f\|_{L^\infty(\Omega_\infty)},$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

If f is unbounded on Ω_∞ or if $f(\cdot)^{p(\cdot)} \notin L^1(\Omega \setminus \Omega_\infty)$, we define $\rho_{p(\cdot),\Omega}(f) = +\infty$. When $|\Omega_\infty| = 0$, in particular when $p_+ < \infty$, we let $\|f\|_{L^\infty(\Omega_\infty)} = 0$. When $|\Omega \setminus \Omega_\infty| = 0$, then $\rho_{p(\cdot),\Omega}(f) = \|f\|_{L^\infty(\Omega_\infty)}$. In situations where there is no ambiguity, we will simply write $\rho(f)$.

Definition 1 Let Ω be a measurable set in $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\Omega)$. Let us define $L^{p(\cdot)}(\Omega)$ to be the set of Lebesgue measurable functions f such that $\rho(\frac{f}{\lambda}) < \infty$ for some $\lambda > 0$.

Also, for every compact set $K \subseteq \Omega$, the space $L_{\mathrm{loc}}^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \{f \text{ is measurable: } f \in L^{p(\cdot)}(K)\}.$$

Definition 2 Let $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. The mixed Lebesgue sequence space with variable exponent $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the set of all sequences $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\left\| \{f_j\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{ \eta > 0 : Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{ \frac{f_j}{\zeta} \right\}_{j\in\mathbb{N}} \right) \leq 1 \right\} < \infty,$$

where

$$Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\{f_j\}_{j\in\mathbb{N}}\big) = \sum_{j=0}^{\infty} \inf\left\{ \zeta_j > 0 : \int_{\mathbb{R}^n} \left(\frac{|f_j(x)|}{\zeta_j^{\frac{1}{q(x)}}} \right)^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

We observe that, for $q_+ < \infty$, we have that

$$Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\{f_j\}_{j\in\mathbb{N}}\big) = \sum_{j=0}^{\infty} \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\chi_k := \chi_{C_k}$, $k \in \mathbb{Z}$.

Definition 3 Let $\alpha \in \mathbb{R}^n$, $q(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}$ is defined as follows:

$$\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)} = \{f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}} < \infty\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}} = \left\| \{2^{k\alpha}|f\chi_k|\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

$$= \inf\left\{ \eta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

3 Calderón–Zygmund operators, *BMO* and *VMO* spaces

In this section we review some of the standard facts on Calderón–Zygmund operators and their commutators. With $\mathscr{S}(\mathbb{R}^n)$ we denote the Schwartz class and with $\mathscr{S}'(\mathbb{R}^n)$ we indicate the space of tempered distributions. Let us start with the following definition (see [17, 18]).

Definition 4 Let T be a bounded linear operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. We say that T is a standard operator if it satisfies the following conditions:

- T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$;
- There exists a function $K(x,y)$ defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ such that

$$\left| K(x,y) \right| \leq \frac{C}{|x-y|^n},$$

where $C > 0$;
- $\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) f(y) g(x) \,\mathrm{d}x \,\mathrm{d}y$ for $f, g \in \mathscr{S}(\mathbb{R}^n)$ such that $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing$.

A standard operator T is called a γ-Calderón–Zygmund operator if K is a standard kernel satisfying

$$\left| K(x,y) - K(z,y) \right| \leq C \frac{|x-z|^\gamma}{|x-y|^{n+\gamma}},$$

$$\left| K(y,x) - K(y,z) \right| \leq C \frac{|x-z|^\gamma}{|x-y|^{n+\gamma}},$$

if $|x-z| < \frac{1}{2}|x-y|$ for some $\gamma \in\,]0,1]$.

The commutator of the Calderón–Zygmund operator is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In 1983, Journé proved that a γ-Calderón–Zygmund operator is bounded on $L^p(\mathbb{R}^n)$ (see [19]). Coifman, Rochberg, and Weiss in [20] proved that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in\,]0,1[$.

Kovácik and Rákosník in [21] introduced Lebesgue spaces and Sobolev spaces with variable exponent. For a recent treatment of Lebesgue spaces with variable exponent, we refer the reader to [16].

In the last decades, there was an increasing interest in the study of functional spaces having variable exponent thanks to the wide variety of applications, for instance, in fluid dynamics and differential equations.

In particular, Herz spaces play an important role in harmonic analysis. In this paper, we will apply the theory of regularity of solutions to partial differential equations to the main results contained in [14]. They deal with the boundedness of Calderón–Zygmund operator and their commutator on Herz spaces with two variable exponents $p(\cdot)$, $q(\cdot)$.

In order to develop a satisfactory theory of regularity of solutions to linear elliptic differential equations, following the pioneering scientific note [15], we will assume that the coefficients of the differential operators under consideration belong to the Sarason class of functions having vanishing mean oscillation. This requirement on the coefficients allows us to consider also the case that these coefficients are discontinuous.

First of all we recall the definition of *BMO* space due to John and Nirenberg (see [22]).

Definition 5 We define the space $BMO(\mathbb{R}^n)$ of functions having bounded mean oscillation as

$$BMO(\mathbb{R}^n) = \left\{ b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|b\|_* < \infty \right\},$$

where

$$\|b\|_* = \sup_{B \subseteq \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| \, dx,$$

where B is a generic ball in $\mathbb{R}^n$ and b_B stands for the integral average of the function b over the sphere B.

In the next definition we define a proper subset of the space *BMO* studied by Sarason (see [23]).

Definition 6 We define the space $VMO(\mathbb{R}^n)$ of functions having vanishing mean oscillation as

$$VMO(\mathbb{R}^n) = \left\{ b \in BMO(\mathbb{R}^n) : \lim_{r \to 0^+} \gamma_b(r) = 0 \right\},$$

where

$$\gamma_b(r) = \sup_{\rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} |b(x) - b_{B_\rho}| \, dx$$

and B_ρ varies in the class of balls in $\mathbb{R}^n$ having radius ρ. We call γ_b the *VMO*-modulus of the function b.

In a similar way, we can define the spaces $BMO(\Omega)$ and $VMO(\Omega)$ of functions defined on a domain $\Omega \subset \mathbb{R}^n$, replacing B and B_ρ by the intersections of the respective balls with Ω.

It is worth pointing out that applying the classical Poincaré inequality, it follows that $W^{1,n}(\mathbb{R}^n) \subset VMO$ and, further on, $W^{\theta,n/\theta}(\mathbb{R}^n) \subset VMO$ for $0 < \theta < 1$ as shows the function $f_\alpha(x) = |\log|x||^\alpha$ for $0 < \alpha < 1$. Straightforward calculations yield that $f_\alpha \in VMO$ for every $\alpha \in (0,1), f_\alpha \in W^{1,n}$ for $\alpha \in (0, 1 - \frac{1}{n})$, while $f_\alpha \notin W^{1,n}$ for $\alpha \in [1 - \frac{1}{n}, 1)$.

4 Main result

In the sequel, let $n \geq 3$ and

$$L := \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i x_j},$$

where

(1) $a_{ij} \in VMO \cap L^\infty(\Omega)$ for all $i,j = 1, \ldots, n$ and a.a. $x \in \Omega$;

(2) $a_{ij}(x) = a_{ji}(x)$ for all $i,j = 1, \ldots, n$ and a.a. $x \in \Omega$;

(3) $\exists \nu > 0 : \nu^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2$, a.e. in B, for all $\xi \in \mathbb{R}^n$.

In the following we set

$$\Gamma(x,t) = \frac{1}{(n-2)\omega_n(\det a_{ij}(x))^{12}} \left(\sum_{i,j=1}^n A_{ij}(x) t_i t_j \right)^{\frac{2-n}{2}}$$

for a.a. $x \in B$, for $t \in \mathbb{R}^n \setminus \{0\}$, where we denote by $A_{ij}(x)$ the entries of the inverse of the matrix $(a_{ij}(x))_{i,j=1,\ldots,n}$. Let us observe that if $\tilde{B}$ is a subset of B in which conditions (1) and

(2) hold everywhere, for fixed $x_0 \in \tilde{B}$, $\Gamma(x_0, t)$ is a fundamental solution for the operator

$$L_0 u(x) = \sum_{i,j=1}^{n} a_{ij}(x_0) u_{x_i x_j}(x).$$

Also, we set

$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i} \Gamma(x,t), \qquad \Gamma_{ij}(x,t) = \frac{\partial^2}{\partial t_i \, \partial t_j} \Gamma(x,t).$$

Furthermore, we set

$$M := \max_{i,j=1,\dots,n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x,t)}{\partial t^\alpha} \right\|_{L^\infty(\Omega \times \Sigma)}.$$

It is well known that $\Gamma_{ij}(x,t)$ are Calderón–Zygmund kernels in the t variable.

Let us start with some useful tools. For the proofs, we refer the reader to [14].

Lemma 7 *Let $p_h \in \mathcal{B}(\mathbb{R}^n)$ for $h = 1, 2$, then there exist constants $0 < t_{h1}, t_{h2} < 1$, and $C > 0$ such that, for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $R \subset B$,*

$$\frac{\|\chi_R\|_{L^{p_h(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_h(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|R|}{|B|} \right)^{t_{h1}}, \qquad \frac{\|\chi_R\|_{L^{p'_h(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_h(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|R|}{|B|} \right)^{t_{h2}},$$

where with $p'_h(\cdot)$ is the conjugate exponent function defined as

$$\frac{1}{p_h(x)} + \frac{1}{p'_h(x)} = 1, \quad x \in \mathbb{R}^n.$$

Theorem 8 *Suppose that $p_1 \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $-nt_{12} < \alpha < nt_{11}$, with t_{11}, t_{12} as in Lemma 7, then the operator T of Definition 4 is bounded from $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.*

Theorem 9 *Let $b \in BMO(\mathbb{R}^n)$. Suppose that $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $-nt_{12} < \alpha < nt_{11}$, with t_{11}, t_{12} as in Lemma 7, then the commutator $[b, T]$ is bounded from $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.*

Now we are ready to show our main result. The proof is based on the representation formula for the second derivatives of a solution in the class $W_0^{2,p}$.

Theorem 10 *Under assumptions (1), (2), (3), $p \in {]1, \infty[}$, $p_1(\cdot)$, $q(\cdot) = q_1(\cdot) = q_2(\cdot)$, and α as in Theorems 8 and 9, there exist positive numbers $C = C(n, p, p_1, q, M, \alpha)$ and $\rho_0 = \rho_0(C, n)$ such that, for any ball $B_r \Subset \Omega$ with $r < \rho_0$ and any $u \in W_0^{2,p}(B_R)$ such that $u_{x_i x_j} \in \dot{K}^{\alpha,q(\cdot)}_{p_1(\cdot)}(B_r)$ and $\mathcal{L}u = f(x)$, with $f \in \dot{K}^{\alpha,q(\cdot)}_{p_1(\cdot)}(B_r)$, we have*

$$\|u_{x_i x_j}\|_{\dot{K}^{\alpha,q(\cdot)}_{p_1(\cdot)}(B_r)} \le C \|f\|_{\dot{K}^{\alpha,q(\cdot)}_{p_1(\cdot)}(B_r)}, \quad \forall i, j = 1, \dots, n.$$

Proof Let $n \geq 3$, B be an open ball in $\mathbb{R}^n$, $(a_{ij})_{i,j=1,\ldots,n}$ be as above and $u \in W_0^{2,p}(B)$. Then, for a.a. $x \in B$, the following representation formula holds (see [15]):

$$u_{x_i x_j}(x) = \text{P.V.} \int_B \Gamma_{ij}(x, x-y) \cdot \left[\sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y)) u_{x_h x_k}(y) + Lu(y) \right] \mathrm{d}y$$

$$+ Lu(x) \int_{|t|=1} \Gamma_i(x,t) t_j \, \mathrm{d}\sigma_t. \tag{4.1}$$

From Theorem 8, it follows that, for any $h \in \dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$, the following inequality holds:

$$\left\| T(h) \right\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} \leq C \| h \|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}. \tag{4.2}$$

Furthermore, from Theorem 9, it follows that, for any $h \in \dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$, we have

$$\left\| [b,T](h) \right\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} \leq C \| b \|_* \| h \|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}. \tag{4.3}$$

Using the representation formula (4.1) and estimates (4.2), (4.3), we obtain the desired estimate. $\qquad \square$

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