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Blow-up for the sixth-order multidimensional generalized Boussinesq equation with arbitrarily high initial energy

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Abstract

In this paper, we consider the Cauchy problem for the sixth-order multidimensional generalized Boussinesq equation with double damping terms. By using the improved convexity method combined with Fourier transform, we show the finite time blow-up of solution with arbitrarily high initial energy.

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Keywords: Generalized Boussinesq equation; Damping; Cauchy problem; Blow-up; Arbitrarily high initial energy

1 Introduction

In this paper, we study the Cauchy problem of sixth-order multidimensional generalized Boussinesq equation with double damping terms

$$\begin{aligned} &u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u - \Delta^3 u + \mu \Delta^2 u_{tt} + \alpha u_t \\ &\quad - \beta \Delta u_t + u + \Delta f(u) = 0, \quad x \in R^n, t > 0, \end{aligned} \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in R^n. \tag{1.2}$$

The two damping terms are strong damping αu_t and frictional damping $\beta \Delta u_t$, where $\alpha > 0$, $\beta > 0$, $\mu \geq 0$ are constants and $u_0(x)$ and $u_1(x)$ are given initial data, $f(u)$ is a given nonlinear function with

$$\text{(H)} \quad f(z) = a|z|^{p-1}z, \quad a > 0, p > 1.$$

It is well known that Boussinesq derived some model equations in 1870s, and such models describe the propagation of small amplitude and long waves on the surface of shallow water. Boussinesq [1] was the first to give the scientific explanation of the following solitary wave equation:

$$u_{tt} = -\gamma u_{xxxx} + u_{xx} + \left(u^2\right)_{xx}, \quad x \in R, t > 0. \tag{1.3}$$

Equation (1.3) depends on the sign of γ, indeed, the case $\gamma < 0$ is called the "good" Boussinesq equation, because it is linearly stable and governs small nonlinear transverse oscillations of an elastic beam (see [2]), while equation (1.3) with $\gamma > 0$ received the name of "bad" Boussinesq equation since it possesses the linear instability. Recently, Kiselev and Tan [3] proposed a two-dimensional model which is called "hyperbolic Boussinesq system" and describes an incompressible velocity vector field, a simplified Biot–Savart law, and a simplified term modeling buoyancy. The authors showed that finite time blow-up happens for a natural class of initial data. More background introductions about Boussinesq equation can be found in references [4, 5].

An and Peirce [6] used a weakly nonlinear analysis and explored the immediate post-critical behavior of the solution of elastoplastic-microstructure models for a longitudinal movement of elastic-plastic rod, just like the following equation:

$$u_{tt} + u_{xxxx} = a\left(u_x^2\right)_x, \quad x \in R, t > 0, \tag{1.4}$$

where a is a constant. Wang and Chen [7] studied the existence and uniqueness of the global solution for the Cauchy problem of the generalized double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = g(u)_{xx}, \quad x \in R, t > 0, \tag{1.5}$$

and they proved the blow-up result of the solution by using the concavity method and under some suitable conditions. Schneider and Wayne [8] considered the following Boussinesq equation that models the water wave problem with surface tension:

$$u_{tt} - u_{xx} - u_{xxtt} - \mu u_{xxxx} + u_{xxxxtt} = \left(u^2\right)_{xx}, \quad x \in R, t > 0, \tag{1.6}$$

the model can also be derived from the 2D water wave problem. They proved that the long wave limit can be described approximately by two decoupled Kawahara equations. Wang and Xue [9] considered the Cauchy problem of equation

$$u_{tt} - u_{xx} - u_{xxtt} + \alpha u_{xxxx} + u_{xxxxtt} = \left(\beta|u|^p\right)_{xx}, \quad x \in R, t > 0, \tag{1.7}$$

where $\alpha > 0$, $\beta \neq 0$, and $p > 1$ are constants. By using the potential well method they obtained the global existence and nonexistence of the solution. Xu et al. [10] considered a sixth-order 1-D nonlinear wave equations and obtained some sufficient conditions for the global and non-global existence of solutions at three different initial energy levels, i.e., sub-critical level, critical level, and sup-critical level.

Furthermore, Song and Xue [11] considered the nonlinear viscoelastic equation

$$u_{tt} - \Delta u - \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \Delta u_t = |u|^{p-2}u, \quad x \in \Omega, t \in [0, T], \tag{1.8}$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g, they showed the finite time blow-up of some solutions whose initial data have arbitrarily high initial energy. Liu, Sun, and Wu [12] studied the initial boundary value problem for a Petrovsky-type equation with a memory term, nonlinear weak damping, and a superlinear source

$$u_{tt} - \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,d\tau + |u_t|^{m-2}u_t = |u|^{p-2}u, \quad x \in \Omega, t \in (0,T). \tag{1.9}$$

When the source is stronger than dissipations, they obtained the existence of certain weak solutions which blow up in finite time with initial data at arbitrarily high energy level. For more related works, we refer the reader to [13, 14].

In recent years, multidimensional generalized Boussinesq equation has been studied, and there are many blow-up results of the solution concerning the Boussinesq equations (see [15, 16] and the references therein). Wang and Mu [17] studied the Cauchy problem of the following generalized Boussinesq equation:

$$u_{tt} - a\Delta u_{tt} + \Delta^2 u_{tt} + \Delta^2 u - \Delta u = \Delta f(u), \quad x \in R^n, t > 0, \tag{1.10}$$

the existence and uniqueness of the global solution were obtained, and the blow-up result of the solution was proved. Wang and Wang [18] studied the Cauchy problem for the sixth-order damped Boussinesq equation

$$u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u - \Delta^3 u - r\Delta u_t = \Delta f(u), \quad x \in R^n, t > 0, \tag{1.11}$$

and they proved the global existence and asymptotic behavior of the solution provided that the initial value is suitably small. Wang [19] considered equation (1.11) and obtained the well-posedness of solution, blow-up of solution with high initial energy, and the asymptotic behavior of the solution by using the multiplier method. Piskin and Polat [20] considered the Cauchy problem of a multidimensional generalized Boussinesq-type equation with a damping term

$$u_{tt} - \Delta u - a\Delta u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - k\Delta u_t = \Delta f(u), \quad x \in R^n, t > 0, \tag{1.12}$$

and gave the existence, both locally and globally in time, the global nonexistence with high initial energy, and the asymptotic behavior of solution. Castro et al. [21] considered 2D Boussinesq equations with a velocity damping term in a strip with impermeable walls and obtained the asymptotic stability for a specific type of perturbations of a stratified solution by using a suitably weighted energy space combined with linear decay, Duhamel's formula, and "bootstrap" arguments. Wang and Su [22] studied multidimensional dissipative Boussinesq equations and obtained the sufficient conditions for global solutions and finite time blow-up solutions respectively with three different cases of initial energy. Particularly, by using some new methods and some analysis techniques, the authors gave the novel contribution for the blow-up result with initial energy at supercritical initial energy.

In fact, the study of large solutions has a long history. Many results have been obtained about the existence, uniqueness, and asymptotic behavior of large solutions. Recently, Mohammed et al. [23] discussed the existence, asymptotic boundary estimates, and uniqueness of large solutions to a fully nonlinear equation by relaxing the conditions used in most of the aforementioned papers. The monograph [24] emphasized those basic abstract methods and theories that are useful in the study of nonlinear problems. The authors gave a systematic treatment of the basic mathematical theory and constructive methods for these classes of nonlinear equations as well as their applications to various processes arising in the applied sciences. Abdelwahed et al. [25] studied the solution of a biharmonic equation with a homogeneous boundary condition and gave a method based on a dual singular method which can be used to approximate the leading singularity coefficient of the bi-Laplacian operator. Some examples are presented to show the efficiency of this method. For more related work, one can refer to [26] and the reference therein.

Note that as far as we know, in the known works (e.g., [27–31]), as seen from the above explanations, there is no obvious result on the blow-up of problem (1.1)–(1.2) in finite time with arbitrarily high initial energy. Therefore, the main purpose of the present paper is to solve the problem. By using the improved convexity method combined with Fourier transform, we obtain the finite time blow-up result for problem (1.1)–(1.2).

The structure of this paper is as follows. In Sect. 2, we give some lemmas, some necessary preliminaries, and our main result. In Sect. 3, we prove the blow-up results of solution for problem (1.1)–(1.2).

Throughout this paper, we use L^p to denote the space of $L^p(R^n)$-function with the standard norm $\|f\|_p = \|f\|_{L^p}$, $1 \leq p \leq \infty$, $\|f\| = \|f\|_2$, and denote by $(\cdot,\cdot)$ the inner product of L^2 space. H^s denotes the Sobolev space $H^s(R^n)$ with norm $\|f\|_{H^s} = \|(I-\Delta)^{\frac{s}{2}} f\|_2$, $s \in R$.

2 Preliminaries and main results

2.1 Some lemmas and some necessary preliminaries

Lemma 2.1 ([32]) *Suppose that $\Theta(t)$, $t \geq b \geq 0$ is a twice differentiable and positive function, which satisfies the inequality*

$$\ddot{\Theta}(t)\Theta(t) - \sigma\big(\dot{\Theta}(t)\big)^2 \geq 0$$

for every $t \geq b$ and some constant $\sigma > 1$. If $\Theta(b) > 0$, $\dot{\Theta}(b) > 0$, then there exists a positive constant t_* such that

$$\Theta(t) \to \infty \quad \text{for } t \to t_*^-,$$

and

$$t_* \leq \frac{\Theta(b)}{(\sigma-1)\dot{\Theta}(b)} + b.$$

Lemma 2.2 ([20] Sobolev imbedding theorem)

(1) *If $s > \frac{n}{2} + k$, where k is a nonnegative integer, then*

$$H^s \hookrightarrow C^k\big(R^n\big) \cap L^\infty;$$

(2) *If $s = \frac{n}{2}$, then for $p \in [2, +\infty)$,*

$$H^s \hookrightarrow L^p;$$

(3) *If $s < \frac{n}{2}$, then*

$$H^s \hookrightarrow L^{\frac{2n}{n-2}}.$$

We give the following well-posedness result from [20].

Theorem 2.3 *Suppose that $f(z)$ satisfies (H), $u_0 \in H^{s+1}$, and $u_1 \in H^{s+1}$ for some $s > \frac{n}{2} + 1$, $n \geq 1$, then problem (1.1)–(1.2) has a unique local solution $u(x,t)$ defined on a maximal time interval $[0, T_0)$, $T_0 > 0$, with*

$$u \in C^2\bigl([0, T_0); H^{s+1}\bigr).$$

Moreover, if

$$\sup_{t\in[0,T_0)} \bigl[\bigl\|u(\cdot,t)\bigr\|_{s+1}^2 + \bigl\|u_t(\cdot,t)\bigr\|_{s+1}^2 + \bigl\|u_{tt}(\cdot,t)\bigr\|_{s+1}^2\bigr] < \infty,$$

then $T_0 = \infty$.

Lemma 2.4 *Suppose that $f(z)$ satisfies (H), $F(u) = \int_0^u f(\tau)\,d\tau$, $u_0 \in H^2$, $(-\Delta)^{-\frac{1}{2}}u_1 \in L^2$, $u_1 \in L^2$, then for the solution $u(x,t)$ of problem (1.1)–(1.2), we have the following energy identity*:

$$\begin{aligned} E(t) = {} & \frac{1}{2}\bigl\|(-\Delta)^{-\frac{1}{2}}u_t\bigr\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\mu}{2}\|\nabla u_t\|^2 \\ & + \alpha\int_0^t \bigl\|(-\Delta)^{-\frac{1}{2}}u_\tau\bigr\|^2\,d\tau + \beta\int_0^t \|u_\tau\|^2\,d\tau + \frac{1}{2}\bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2 - \frac{a}{p+1}\|u\|_{p+1}^{p+1} \\ \equiv {} & E(0). \end{aligned} \tag{2.1}$$

Here and in the sequel, $(-\Delta)^{-\alpha}u(x) = \mathcal{F}^{-1}[|x|^{-2\alpha}\mathcal{F}u(x)]$, in which $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and the inverse transformation in R^n respectively.

Proof Multiplying equation (1.1) by $(-\Delta)^{-1}u_t$ and integrating the product with respect to x, we have

$$\bigl(u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u - \Delta^3 u + \mu\Delta^2 u_{tt} + \alpha u_t - \beta\Delta u_t + u + \Delta f(u), (-\Delta)^{-1}u_t\bigr) = 0.$$

After some computation, we obtain

$$\begin{aligned} & \bigl((-\Delta)^{-1}u_{tt} + u_{tt} + u - \Delta u + \Delta^2 u - \mu\Delta u_{tt} + \alpha(-\Delta)^{-1}u_t \\ & \quad + \beta u_t + (-\Delta)^{-1}u - f(u), u_t\bigr) = 0, \end{aligned}$$

then we can get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\biggl\{\bigl\|(-\Delta)^{-\frac{1}{2}}u_t\bigr\|^2 + \|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \mu\|\nabla u_t\|^2 \\ & \quad + 2\alpha\int_0^t \bigl\|(-\Delta)^{-\frac{1}{2}}u_\tau\bigr\|^2\,d\tau + 2\beta\int_0^t \|u_\tau\|^2\,d\tau + \bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2 - \frac{2a}{p+1}\|u\|_{p+1}^{p+1}\biggr\} \\ & = 0. \end{aligned} \tag{2.2}$$

Integrating (2.2) with respect to t over $[0,t]$, we obtain the energy identity (2.1). Thus Lemma 2.4 is proved. □

2.2 Main result

Theorem 2.5 *Suppose that $f(z)$ satisfies (H), and $u_0 \in H^1$, $u_1 \in L^2$, $(-\Delta)^{-\frac{1}{2}}u_0 \in L^2$, $(-\Delta)^{-\frac{1}{2}}u_1 \in L^2$. If*

$$\begin{aligned} &2\big((-\Delta)^{-\frac{1}{2}}u_1,(-\Delta)^{-\frac{1}{2}}u_0\big)+2(u_1,u_0)+\alpha\big\|(-\Delta)^{-\frac{1}{2}}u_0\big\|^2+\beta\|u_0\|^2+2\mu(\nabla u_1,\nabla u_0)\\ &\quad >\frac{2(p+1)}{\kappa_1}E(0), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &\kappa_1=\sup_{\xi\in(0,1)}\kappa(\xi),\\ &\kappa(\xi)=\min\Big\{\sqrt{(p+3)(p-1)}\min\{\sqrt{c\xi+1},\sqrt{\xi}\},\\ &\qquad (1-\xi)(p-1)\min\Big\{\frac{1}{\alpha},\frac{c}{\beta}\Big\},\frac{\sqrt{(p-1)(p+3)}}{\sqrt{\mu}}\Big\}, \end{aligned}$$

and c is some positive constant, then the solution $u(t)$ of problem (1.1)–(1.2) blows up in finite time.

3 Proof of the main result

In this section, our aim is to prove that finite time blow-up result of the solution of problem (1.1)–(1.2) with initial data has arbitrarily high initial energy.

Proof of Theorem 2.5 Set

$$\begin{aligned} Q(t)&=2\big((-\Delta)^{-\frac{1}{2}}u,(-\Delta)^{-\frac{1}{2}}u_t\big)+2(u,u_t)+\alpha\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2\\ &\quad+\beta\|u\|^2+2\mu(\nabla u,\nabla u_t)-mE(0), \end{aligned} \tag{3.1}$$

where m is a positive constant to be determined later. Obviously we have

$$\begin{aligned} \dot{Q}(t)&=2\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+2\big((-\Delta)^{-\frac{1}{2}}u,(-\Delta)^{-\frac{1}{2}}u_{tt}\big)+2\|u_t\|^2+2(u,u_{tt})\\ &\quad+2\alpha\big((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\big)+2\beta(u_t,u)+2\mu\|\nabla u_t\|^2+2\mu(\nabla u,\nabla u_{tt}), \end{aligned} \tag{3.2}$$

$$\big((-\Delta)^{-1}u_{tt},u\big)=\big((-\Delta)^{-\frac{1}{2}}u_{tt},(-\Delta)^{-\frac{1}{2}}u\big), \tag{3.3}$$

$$\big((-\Delta)^{-1}u_t,u\big)=\big((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\big), \tag{3.4}$$

and

$$(-\Delta)^{-1}u_{tt}+u_{tt}-\mu\Delta u_{tt}+(-\Delta)^{-1}\alpha u_t+\beta u_t=-u+\Delta u-\Delta^2u-(-\Delta)^{-1}u+f(u). \tag{3.5}$$

Multiplying (3.5) by u and integrating the product over R^n, we obtain

$$\begin{aligned} &\big((-\Delta)^{-1}u_{tt},u\big)+(u_{tt},u)-(\mu\Delta u_{tt},u)+\big((-\Delta)^{-1}\alpha u_t,u\big)+(\beta u_t,u)\\ &\quad=-\|u\|^2-\|\nabla u\|^2-\|\Delta u\|^2-\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2+a\|u\|_{p+1}^{p+1}, \end{aligned} \tag{3.6}$$

then we have

$$\big((-\Delta)^{-1}u_{tt},u\big)+(u_{tt},u)+(\mu\nabla u_{tt},\nabla u)+\big((-\Delta)^{-1}\alpha u_t,u\big)+(\beta u_t,u)$$
$$=-\|u\|^2-\|\nabla u\|^2-\|\Delta u\|^2-\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2+a\|u\|_{p+1}^{p+1}. \tag{3.7}$$

Inserting (3.7) into (3.2), then equation (3.2) can be written as follows:

$$\begin{aligned}\dot{Q}(t)&=2\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+2\|u_t\|^2+2\mu\|\nabla u_t\|^2-2\|u\|^2-2\|\nabla u\|^2\\&\quad-2\|\Delta u\|^2-2\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2+2a\|u\|_{p+1}^{p+1}.\end{aligned} \tag{3.8}$$

According (2.1), applying (3.8), then we have

$$\begin{aligned}\dot{Q}(t)&=-2(p+1)E(0)+(p+3)\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+(p+3)\|u_t\|^2+(p-1)\|u\|^2\\&\quad+(p-1)\|\nabla u\|^2+(p-1)\|\Delta u\|^2+\mu(p+3)\|\nabla u_t\|^2+(p-1)\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2\\&\quad+2\alpha(p+1)\int_0^t\big\|(-\Delta)^{-\frac{1}{2}}u_\tau\big\|^2\,d\tau+2\beta(p+1)\int_0^t\|u_\tau\|^2\,d\tau,\end{aligned} \tag{3.9}$$

thus we can obtain

$$\begin{aligned}\dot{Q}(t)&\geq-2(p+1)E(0)+(p+3)\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+(p+3)\|u_t\|^2+(p-1)\|u\|^2\\&\quad+(p-1)\|\nabla u\|^2+(p-1)\|\Delta u\|^2+\mu(p+3)\|\nabla u_t\|^2\\&\quad+(p-1)\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2.\end{aligned} \tag{3.10}$$

By using the embedding inequality $\|\Delta u\|^2\geq c\|u\|^2$, we have

$$\begin{aligned}\dot{Q}(t)&\geq-2(p+1)E(0)+(p+3)\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+(p+3)\|u_t\|^2+(p-1)\|u\|^2\\&\quad+(p-1)\|\nabla u\|^2+(p-1)c\|u\|^2+\mu(p+3)\|\nabla u_t\|^2+(p-1)\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2,\end{aligned} \tag{3.11}$$

obviously, we can get

$$\begin{aligned}\dot{Q}(t)&\geq-2(p+1)E(0)+(p+3)\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+(p+3)\|u_t\|^2\\&\quad+(p-1)\|u\|^2+(p-1)c\xi\|u\|^2\\&\quad+(p-1)c(1-\xi)\|u\|^2+\xi(p-1)\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2+(1-\xi)(p-1)\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2\\&\quad+(p-1)\|\nabla u\|^2+\mu(p+3)\|\nabla u_t\|^2,\end{aligned} \tag{3.12}$$

where $\xi\in(0,1)$ is a constant. From inequality (3.12) and using the Cauchy inequality, we see that

$$\begin{aligned}&(p+3)\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+\xi(p-1)\big\|(-\Delta)^{-\frac{1}{2}}u\big\|^2\\&\quad\geq2\sqrt{(p+3)\xi(p-1)}\big((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\big),\end{aligned} \tag{3.13}$$

$$(p+3)\|u_t\|^2+(c\xi+1)(p-1)\|u\|^2\geq2\sqrt{(p+3)(c\xi+1)(p-1)}(u_t,u), \tag{3.14}$$

and

$$(p-1)\|\nabla u\|^2 + \mu(p+3)\|\nabla u_t\|^2 \geq 2\sqrt{(p-1)\mu(p+3)}(\nabla u_t, \nabla u). \tag{3.15}$$

Substituting (3.13)–(3.15) into (3.12), we have

$$\begin{aligned}\dot{Q}(t) \geq \kappa(\xi)\biggl[&2\bigl((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\bigr) + 2(u, u_t) + \alpha\bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2 \\ &+ \beta\|u\|^2 + 2\mu(\nabla u, \nabla u_t) - \frac{2(p+1)}{\kappa(\xi)}E(0)\biggr],\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\kappa(\xi) = \min\biggl\{&\sqrt{(p+3)(p-1)}\min\{\sqrt{c\xi+1}, \sqrt{\xi}\}, \\ &(1-\xi)(p-1)\min\biggl\{\frac{1}{\alpha}, \frac{c}{\beta}\biggr\}, \frac{\sqrt{(p-1)(p+3)}}{\sqrt{\mu}}\biggr\}.\end{aligned} \tag{3.17}$$

It is obvious to see that $D(\xi) := \sqrt{(p+3)(p-1)}\min\{\sqrt{c\xi+1}, \sqrt{\xi}\}$ is strictly increasing for $0 < \xi < 1$, we have $D(0) = 0$, $D(1) = \sqrt{(p+3)(p-1)}\min\{\sqrt{c+1}, 1\}$; similarly, $G(\xi) := (1-\xi)(p-1)\min\{\frac{1}{\alpha}, \frac{c}{\beta}\}$ is strictly decreasing for $0 < \xi < 1$, $G(0) = (p-1)\min\{\frac{1}{\alpha}, \frac{c}{\beta}\}$, $G(1) = 0$. Then we have $\kappa(\xi)$ takes its maximum for $\xi = \xi_1$, where ξ_1 is the root of the equation

$$\sqrt{(p+3)(p-1)}\min\{\sqrt{c\xi+1}, \sqrt{\xi}\} = (1-\xi)(p-1)\min\biggl\{\frac{1}{\alpha}, \frac{c}{\beta}\biggr\},$$

and set

$$\kappa_1 := \sup_{\xi\in(0,1)} \kappa(\xi) = \kappa(\xi_1).$$

According to (2.3), we can get $Q(0) > 0$, and by taking $m = \frac{2(p+1)}{\kappa_1}$, we have

$$\begin{aligned}\dot{Q}(t) &\geq \kappa_1\biggl(\bigl(2(-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\bigr) + 2(u, u_t) + \alpha\bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2 \\ &\quad + \beta\|u\|^2 + 2\mu(\nabla u, \nabla u_t) - \frac{2(p+1)}{\kappa_1}E(0)\biggr) \\ &\geq \kappa_1 Q(t),\end{aligned} \tag{3.18}$$

hence we can obtain

$$Q(t) \geq Q(0)e^{\kappa t}, \quad t \geq 0,$$

therefore we have

$$\lim_{t\to\infty} Q(t) = +\infty. \tag{3.19}$$

Together with (3.18), we obtain

$$\lim_{t\to\infty}\dot{Q}(t)=+\infty.$$

Now, we let

$$\begin{aligned}\Theta(t)&=\bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2+\|u\|^2+\mu\|\nabla u\|^2+\alpha\int_0^t\bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2\,d\tau+\beta\int_0^t\|u\|^2\,d\tau\\&\quad+(T-t)\alpha\bigl\|(-\Delta)^{-\frac{1}{2}}u_0\bigr\|^2+(T-t)\beta\|u_0\|^2.\end{aligned}\tag{3.20}$$

By straightforward computations, we can get the following equality:

$$\begin{aligned}\dot{\Theta}(t)&=2\bigl((-\Delta)^{-\frac{1}{2}}u,(-\Delta)^{-\frac{1}{2}}u_t\bigr)+2(u,u_t)+2\mu(\nabla u,\nabla u_t)+\alpha\bigl\|(-\Delta)^{-\frac{1}{2}}u\bigr\|^2\\&\quad+\beta\|u\|^2-\alpha\bigl\|(-\Delta)^{-\frac{1}{2}}u_0\bigr\|^2-\beta\|u_0\|^2\\&=2\bigl((-\Delta)^{-\frac{1}{2}}u,(-\Delta)^{-\frac{1}{2}}u_t\bigr)+2(u,u_t)+2\mu(\nabla u,\nabla u_t)\\&\quad+2\alpha\int_0^t\bigl((-\Delta)^{-\frac{1}{2}}u_\tau,(-\Delta)^{-\frac{1}{2}}u\bigr)\,d\tau+2\beta\int_0^t(u_\tau,u)\,d\tau,\end{aligned}\tag{3.21}$$

then we have

$$\begin{aligned}\ddot{\Theta}&=2\bigl\|(-\Delta)^{-\frac{1}{2}}u_t\bigr\|^2+2\bigl((-\Delta)^{-\frac{1}{2}}u,(-\Delta)^{-\frac{1}{2}}u_{tt}\bigr)+2\|u_t\|^2+2(u,u_{tt})\\&\quad+2\alpha\bigl((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\bigr)+2\beta(u_t,u)+2\mu\|\nabla u_t\|^2+2\mu(\nabla u,\nabla u_{tt}).\end{aligned}\tag{3.22}$$

From equation (3.21) we can get

$$\begin{aligned}\dot{\Theta}^2&=4\biggl\{\bigl((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\bigr)^2+(u,u_t)^2+\mu^2(\nabla u_t,\nabla u)^2\\&\quad+\alpha^2\biggl(\int_0^t\bigl((-\Delta)^{-\frac{1}{2}}u_\tau,(-\Delta)^{-\frac{1}{2}}u\bigr)\,d\tau\biggr)^2\\&\quad+\beta^2\biggl(\int_0^t(u_\tau,u)\,d\tau\biggr)^2+2\bigl((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\bigr)(u_t,u)\\&\quad+2\alpha\bigl((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\bigr)\int_0^t\bigl((-\Delta)^{-\frac{1}{2}}u_\tau,(-\Delta)^{-\frac{1}{2}}u\bigr)\,d\tau\\&\quad+2\beta\bigl((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\bigr)\int_0^t(u_\tau,u)\,d\tau\\&\quad+2\mu\bigl((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}u\bigr)(\nabla u_t,\nabla u)\\&\quad+2\alpha(u_t,u)\int_0^t\bigl((-\Delta)^{-\frac{1}{2}}u_\tau,(-\Delta)^{-\frac{1}{2}}u\bigr)^2\,d\tau+2\beta(u,u_t)\int_0^t(u_\tau,u)\,d\tau\\&\quad+2\mu(u_t,u)(\nabla u_t,\nabla u)+2\alpha\beta\int_0^t\bigl((-\Delta)^{-\frac{1}{2}}u_\tau,(-\Delta)^{-\frac{1}{2}}u\bigr)\,d\tau\int_0^t(u_\tau,u)\,d\tau\\&\quad+2\alpha\mu\int_0^t\bigl((-\Delta)^{-\frac{1}{2}}u_\tau,(-\Delta)^{-\frac{1}{2}}u\bigr)\,d\tau(\nabla u_t,\nabla u)\\&\quad+2\beta\mu\int_0^t(u_\tau,u)\,d\tau(\nabla u_t,\nabla u)\biggr\}.\end{aligned}\tag{3.23}$$

Next, we estimate the terms on the right-hand side of (3.23), and we denote each term of the right-hand side of (3.23) by $I_1,\dots,I_{15}$ separately. By using Hölder's and Cauchy–Schwarz's inequalities, we have the following inequalities:

$$I_1 \le \|(-\Delta)^{-\frac{1}{2}}u_t\|^2\|(-\Delta)^{-\frac{1}{2}}u\|^2, \tag{3.24}$$

$$I_2 \le \|u_t\|^2\|u\|^2, \tag{3.25}$$

$$I_3 \le \mu^2\|\nabla u_t\|^2\|\nabla u\|^2, \tag{3.26}$$

$$I_4 \le \alpha^2\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|^2\,d\tau\int_0^t\|(-\Delta)^{-\frac{1}{2}}u\|^2\,d\tau, \tag{3.27}$$

$$I_5 \le \beta^2\int_0^t\|u_\tau\|^2\,d\tau\int_0^t\|u\|^2\,d\tau, \tag{3.28}$$

$$\begin{aligned} I_6 &\le 2\|(-\Delta)^{-\frac{1}{2}}u_t\|\|(-\Delta)^{-\frac{1}{2}}u\|\|u_t\|\|u\| \\ &\le \|(-\Delta)^{-\frac{1}{2}}u_t\|^2\|u\|^2 + \|(-\Delta)^{-\frac{1}{2}}u\|^2\|u_t\|^2, \end{aligned} \tag{3.29}$$

$$\begin{aligned} I_7 &\le 2\alpha\|(-\Delta)^{-\frac{1}{2}}u_t\|\|(-\Delta)^{-\frac{1}{2}}u\|\left(\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|(-\Delta)^{-\frac{1}{2}}u\|^2\,d\tau\right)^{\frac{1}{2}} \\ &\le \alpha\biggl(\|(-\Delta)^{-\frac{1}{2}}u_t\|^2\int_0^t\|(-\Delta)^{-\frac{1}{2}}u\|^2\,d\tau \\ &\quad + \|(-\Delta)^{-\frac{1}{2}}u\|^2\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|^2\,d\tau\biggr), \end{aligned} \tag{3.30}$$

$$\begin{aligned} I_8 &\le 2\beta\|(-\Delta)^{-\frac{1}{2}}u_t\|\|(-\Delta)^{-\frac{1}{2}}u\|\left(\int_0^t\|u_\tau\|^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|u\|^2\,d\tau\right)^{\frac{1}{2}} \\ &\le \beta\left(\|(-\Delta)^{-\frac{1}{2}}u_t\|^2\int_0^t\|u\|^2\,d\tau + \|(-\Delta)^{-\frac{1}{2}}u\|^2\int_0^t\|u_\tau\|^2\,d\tau\right), \end{aligned} \tag{3.31}$$

$$\begin{aligned} I_9 &\le 2\mu\|(-\Delta)^{-\frac{1}{2}}u_t\|\|(-\Delta)^{-\frac{1}{2}}u\|\|\nabla u_t\|\|\nabla u\| \\ &\le \mu\bigl(\|(-\Delta)^{-\frac{1}{2}}u_t\|^2\|\nabla u\|^2 + \|(-\Delta)^{-\frac{1}{2}}u\|^2\|\nabla u_t\|^2\bigr), \end{aligned} \tag{3.32}$$

$$\begin{aligned} I_{10} &\le 2\alpha\|u_t\|\|u\|\left(\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|(-\Delta)^{-\frac{1}{2}}u\|^2\,d\tau\right)^{\frac{1}{2}} \\ &\le \alpha\left(\|u_t\|^2\int_0^t\|(-\Delta)^{-\frac{1}{2}}u\|^2\,d\tau + \|u\|^2\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|^2\,d\tau\right), \end{aligned} \tag{3.33}$$

$$\begin{aligned} I_{11} &\le 2\beta\|u_t\|\|u\|\left(\int_0^t\|u_\tau\|^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|u\|^2\,d\tau\right)^{\frac{1}{2}} \\ &\le \beta\left(\|u_t\|^2\int_0^t\|u\|^2\,d\tau + \|u\|^2\int_0^t\|u_\tau\|^2\,d\tau\right), \end{aligned} \tag{3.34}$$

$$I_{12} \le 2\mu\|u_t\|\|u\|\|\nabla u\|\|\nabla u_t\| \le \mu\bigl(\|u_t\|^2\|\nabla u\|^2 + \|u\|^2\|\nabla u_t\|^2\bigr), \tag{3.35}$$

$$\begin{aligned} I_{13} &\le 2\alpha\beta\left(\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|(-\Delta)^{-\frac{1}{2}}u\|^2\,d\tau\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_0^t\|u_\tau\|^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|u\|^2\,d\tau\right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \alpha\beta\left(\int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 d\tau \int_0^t \|u\|^2 d\tau \right.\\
&\quad \left. + \int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 d\tau \int_0^t \|u_\tau\|^2 d\tau\right),
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
I_{14} &\leq 2\mu\alpha\|\nabla u_t\|\|\nabla u\|\left(\int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 d\tau\right)^{\frac{1}{2}}\\
&\leq \mu\alpha\left(\|\nabla u_t\|^2 \int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 d\tau + \|\nabla u\|^2 \int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 d\tau\right),
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
I_{15} &\leq 2\mu\beta\|\nabla u_t\|\|\nabla u\|\left(\int_0^t \|u_\tau\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u\|^2 d\tau\right)^{\frac{1}{2}}\\
&\leq \mu\beta\left(\|\nabla u_t\|^2 \int_0^t \|u\|^2 d\tau + \|\nabla u\|^2 \int_0^t \|u_\tau\|^2 d\tau\right).
\end{aligned} \tag{3.38}$$

Thus, substituting the above estimates (3.24)–(3.38) into (3.23), we find that

$$\begin{aligned}
\frac{1}{4}\dot{\Theta} &\leq \Theta(t)\left(\left\|(-\Delta)^{\frac{1}{2}}u_t\right\|^2 + \|u_t\|^2 + \alpha\int_0^t \left\|(-\Delta)^{\frac{1}{2}}u_\tau\right\|^2 d\tau\right.\\
&\quad \left. + \beta\int_0^t \|u_\tau\|^2 d\tau + \mu\|\nabla u_t\|^2\right).
\end{aligned} \tag{3.39}$$

From (3.9), (3.18), and (3.19), we can find that there exists $T_A > 0$ large enough such that, for $t > T_A$ and $t \to \infty$,

$$\begin{aligned}
\dot{Q}(t) &= -2(p+1)E(0) + (p+3)\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + (p+3)\|u_t\|^2 + (p-1)\|u\|^2\\
&\quad + (p-1)\|\nabla u\|^2 + (p-1)\|\Delta u\|^2 + \mu(p+3)\|\nabla u_t\|^2 + (p-1)\left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2\\
&\quad + 2\alpha(p+1)\int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 d\tau + 2\beta(p+1)\int_0^t \|u_\tau\|^2 d\tau \to \infty.
\end{aligned} \tag{3.40}$$

Since $\ddot{\Theta} = \dot{Q}$ and (3.19), hence we can find $\dot{\Theta}(t) > \dot{\Theta}(T_A) > 0$, where $t > T_A$, at the same time we can get from (3.9)

$$\begin{aligned}
\ddot{\Theta}(t)\Theta(t) - \frac{p+3}{4}\dot{\Theta}^2(t) \geq \Theta(t)\Bigg\{\ddot{\Theta} - (p+3)\bigg(&\left\|(-\Delta)^{\frac{1}{2}}u_t\right\|^2 + \alpha\int_0^t \left\|(-\Delta)^{\frac{1}{2}}u_\tau\right\|^2 d\tau\\
&+ \beta\int_0^t \|u_\tau\|^2 d\tau + \|u_t\|^2 + \mu\|\nabla u_t\|^2\bigg)\Bigg\}.
\end{aligned} \tag{3.41}$$

Consequently, using (3.9) again, we obtain

$$\ddot{\Theta}(t)\Theta(t) - \frac{p+3}{4}\dot{\Theta}^2(t) > 0, \quad t > T_A. \tag{3.42}$$

Set $\tilde{\Theta}(s) = \Theta(s + T_A)$, $s = t - T_A$, it is obvious that $\dot{\tilde{\Theta}} > 0$, we can verify that $\ddot{\tilde{\Theta}}(s)\tilde{\Theta}(s) - \sigma\dot{\tilde{\Theta}}^2(s) > 0$ for $s \geq 0$. According to Lemma 2.1, we have, for some $T > T_A$,

$$\lim_{t-T_A\to T^-} \tilde{\Theta}(t) = +\infty, \quad \text{where } T \leq \frac{\tilde{\Theta}(0)}{(\sigma - 1)\dot{\tilde{\Theta}}(0)},$$

similarly, we can get

$$\begin{aligned} &\lim_{t\to(T+T_A)^-} \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 + \|u\|^2 + \mu\|\nabla u\|^2 + \alpha\int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 d\tau + \beta\int_0^t \|u\|^2\, d\tau \\ &\quad + (T - t)\alpha\left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 + (T - t)\beta\|u_0\|^2 = +\infty. \end{aligned}$$

Thus Theorem 2.5 is proved. □

Remark 3.1 In the case of $\mu = 0$, the blow-up result of problem (1.1)–(1.2) holds. In the case of $\mu = 1$, the blow-up result of problem (1.1)–(1.2) holds too.

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