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# Maximum principle and its application to multi-index Hadamard fractional diffusion equation

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**Abstract**
This study establishes some new maximum principle which will help to investigate an IBVP for multi-index Hadamard fractional diffusion equation. With the help of the new maximum principle, this paper ensures that the focused multi-index Hadamard fractional diffusion equation possesses at most one classical solution and that the solution depends continuously on its initial boundary value conditions.

**Keywords:** Maximum principle; Hadamard fractional derivative; Uniformly elliptic operator; Uniqueness and continuous dependence

## 1 Introduction

As is known, the maximum principle is one of the most effective tools to investigate ordinary (partial, evolution, fractional) differential equations. In the absence of any clear information about the solution, some properties of the solution can be obtained using the maximum principle. Recently, the maximum principle and its effective application in investigating fractional differential equations have received great attention from scholars. In [1], the authors studied the IBVP for the single-term and the multi-term as well as the distributed order time-fractional diffusion equations with Riemann–Liouville and Caputo type time-fractional derivatives. Meanwhile, they proved the weak maximum principle and established the uniqueness of solutions to the IBVP with Dirichlet boundary conditions. The maximum principles for classical solution and weak solution of a time-space fractional diffusion equation with the fractional Laplacian operator were considered in [2]. In [3], Korbol and Luchko generalized the mathematical model of variable-order space-time fractional diffusion equation to analyze some financial data and considered the option pricing as an application of this model. In [4], the authors established the maximum principle for the multi-term time-space Riesz–Caputo fractional differential equation, uniqueness and continuous dependence of the solution as well as presented a numerical method for the specified equation. In recent years, the study of maximum principle has attracted a lot of attention, we refer the reader to papers [5–10] and the references therein.

The importance of Hadamard fractional calculus has risen. For its recent study and development, we refer to [11–20]. The maximum principle for IBVP with the Hadamard fractional derivative has just been awakened. Only in [21], Kirane and Torebek obtained the extreme principles for the Hadamard fractional derivative and applied the extreme principles to develop some Hadamard fractional maximum principles, by which the authors show the uniqueness and continuous dependence of the solution of a class of Hadamard time-fractional diffusion equations.

In this article, we study the following multi-index Hadamard fractional diffusion equation:

$$\mathbb{P}\left({}^{H}D_t\right)\nu(x,t) = -\mathbb{L}\nu(x,t) + C(x,t)\nu(x,t) + \Psi(x,t), \quad (x,t) \in \Omega \times (1,T]. \tag{1.1}$$

Here, $\mathbb{L}\nu$ is a uniformly elliptic operator

$$\mathbb{L}\nu = -\sum_{i,j=1}^{n} \phi_{i,j}(x,t)\frac{\partial^2 \nu}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \varphi_i(x,t)\frac{\partial \nu}{\partial x_i}. \tag{1.2}$$

Moreover, we suppose that the functions $\varphi_i$, $\phi_{i,j}$ $(i,j = 1,2,\dots,n)$ are continuous on $\bar{\Omega} \times [1,T]$ and equipped with $\phi_{i,j} = \phi_{j,i}$ on $\Omega \times (1,T]$. In addition, for a positive constant $\eta$,

$$\sum_{i,j=1}^{n} \phi_{i,j}(x,t)\theta_i\theta_j \geq \eta\|\theta\|_2^2 \quad \forall (x,t) \in \Omega \times (1,T] \text{ and } \theta \in \mathbb{R}^N. \tag{1.3}$$

Clearly, the matrix $A = (\phi_{i,j})_{n\times n}$ is positive definite and symmetric. $\mathbb{P}({}^{H}D_t)$ is a multi-term Hadamard fractional derivative defined by

$$\mathbb{P}\left({}^{H}D_t\right) = {}^{H}D_t^{p} + \sum_{i=1}^{m} \vartheta_i {}^{H}D_t^{p_i}, \quad 0 < p_m < \cdots < p_1 < p \leq 1, 0 \leq \vartheta_i, i = 1,\dots,m, m \in \mathbb{N}^*.$$

Besides this, we also suppose that $C$ and $\vartheta_i$ are continuous on $\Omega \times (1,T]$ equipped with $\vartheta_i(x,t) \geq 0$ and $C(x,t) \leq 0$.

The structure of the article is as follows: In Sect. 2 we give basic concepts and the definitions of Hadamard fractional calculus, and also give some lemmas, which will be needed in our subsequent proof. Further, the maximum principle of IBVP for the multi-index Hadamard fractional differential equation is derived in Sect. 3. In Sect. 4, some applications are demonstrated, i.e., the uniqueness and continuous dependence of solution to the multi-index linear (nonlinear) Hadamard fractional diffusion equations are discussed.

## 2 Preliminaries

Now, we list some basic definitions and lemmas needed in our subsequent proof.

From paper [22], Hadamard fractional integral and derivative of order $p$ are defined as

$$\left({}^{H}I_t^{p}g\right)(t) = \frac{1}{\Gamma(p)}\int_1^t \left(\log\frac{t}{y}\right)^{p-1}\frac{g(y)}{y}\,dy$$

and

$$\left({}^{H}D_t^{p}g\right)(t) = \frac{1}{\Gamma(n-p)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{y}\right)^{n-p-1}\frac{g(y)}{y}\,dy, \quad n-1 < p < n,$$

where $n = [p] + 1$ and $\log(\cdot) = \log_e(\cdot)$, respectively.

**Lemma 2.1** ([22]) *If $a, p, q > 0$, then*

$$\left({}^H I_a^p \left(\log\frac{t}{a}\right)^{q-1}\right)(y) = \frac{\Gamma(q)}{\Gamma(q+p)}\left(\log\frac{y}{a}\right)^{q+p-1},$$
$$\left({}^H D_a^p \left(\log\frac{t}{a}\right)^{q-1}\right)(y) = \frac{\Gamma(q)}{\Gamma(q-p)}\left(\log\frac{y}{a}\right)^{q-p-1}.$$

**Lemma 2.2** ([21]) *For $0 < p < 1$, if $g \in C^1([1,T])$ attains its maximum at $t_0 \in [1,T]$, then*

$$\left({}^H D_t^p g\right)(t_0) \geq \frac{(\log t_0)^{-p}}{\Gamma(1-p)} g(t_0)$$

*holds. Further, if $g(t_0) \geq 0$, then*

$$\left({}^H D_t^p g\right)(t_0) \geq 0.$$

**Lemma 2.3** ([10]) *Suppose that a function $g \in C^2(\bar{\Omega})$ attains its maximum at $x_0 \in \Omega$, then*

$$\left(\sum_{i,j=1}^{n} \phi_{i,j}(x)\frac{\partial^2 g}{\partial x_i \partial x_j}\right)\Bigg|_{x=x_0} \leq 0$$

*and*

$$\left(\sum_{i=1}^{n} \varphi_i(x)\frac{\partial g}{\partial x_i}\right)\Bigg|_{x=x_0} = 0$$

*hold.*

## 3 Maximum principle

In this subsection, we develop some maximum principle of IBVP for the multi-index Hadamard fractional diffusion equation, by means of which we shall show the uniqueness and continuous dependence of the solution of the multi-index Hadamard fractional diffusion equation.

First, consider the multi-index Hadamard fractional diffusion equation (1.1) with the initial-boundary conditions:

$$v(x,1) = a(x), \quad x \in \Omega, \tag{3.1}$$
$$v(x,t) = b(x,t), \quad (x,t) \in \partial\Omega \times [1,T], \tag{3.2}$$

where $\Omega \in \mathbb{R}^N$ is an open domain with a smooth boundary $\partial\Omega$. Denote

$$W_* = \left\{ v(x,t) \,\Big|\, \frac{\partial^2 v}{\partial x_i \partial x_j} \in C(\bar{\Omega}) \text{ and } \frac{\partial v}{\partial t} \in C\big([1,T]\big)\right\}. \tag{3.3}$$

**Theorem 3.1** *Let $\Psi(x,t)$, $C(x,t)$ be nonpositive on $\Omega \times (1,T]$ and $v(x,t) \in W_*$ be a solution of IBVP (1.1) and (3.1)–(3.2). It follows that*

$$\max v(x,t) \leq \max\left\{\max_{x\in\Omega} a(x), \max_{(x,t)\in\partial\Omega\times[1,T]} b(x,t), 0\right\}.$$

*Proof* First of all, suppose that the statement is violated, then there exists $(x_0, t_0) \in \Omega \times (1, T]$ such that $v(x,t)$ attains the maximum value $v(x_0, t_0)$ and satisfies

$$v(x_0,t_0) \geq \max\Bigl\{\max_{x\in\Omega} a(x), \max_{(x,t)\in\partial\Omega\times[1,T]} b(x,t), 0\Bigr\} = N > 0.$$

Let $\delta = v(x_0,t_0) - N > 0$. For $\forall(x,t) \in \bar{\Omega} \times [1,T]$, let us introduce the auxiliary function

$$\zeta(x,t) = v(x,t) + \frac{\delta}{2}\biggl(1 - \frac{\log t}{\log T}\biggr).$$

From the definition of $\zeta$, we get

$$\zeta(x,t) \leq v(x,t) + \frac{\delta}{2}, \quad (x,t) \in \bar{\Omega} \times [1,T],$$

and

$$\zeta(x_0,t_0) > v(x_0,t_0) = \delta + N > \delta + v(x,t) > \zeta(x,t) + \frac{\delta}{2}, \quad (x,t) \in \Omega \times \{1\} \cup \partial\Omega \times [1,T].$$

The last inequality means that $\zeta(x,t)$ cannot get the maximum on $\Omega \times \{1\} \cup \partial\Omega \times [1,T]$. Without loss of generality, put $(x^*,t^*)$ to be a maximum point of $\zeta(x,t)$ on $\bar{\Omega} \times [1,T]$, then we have

$$\zeta\bigl(x^*,t^*\bigr) > \zeta(x_0,t_0) > \delta + N > 0, \quad x^* \in \Omega, 1 < t^* \leq T.$$

It follows from Lemma 2.3 that

$$\begin{aligned}
&\mathbb{L}\zeta(x,t)\big|_{(x,t)=(x^*,t^*)} \\
&\quad= \Biggl(-\sum_{i,j=1}^{n} \phi_{i,j}\bigl(x,t^*\bigr)\frac{\partial^2(v(x,t^*) + \frac{\delta}{2}(1 - \frac{\log t^*}{\log T}))}{\partial x_i \partial x_j} \\
&\qquad+ \sum_{i=1}^{n} \varphi_i\bigl(x,t^*\bigr)\frac{\partial(v(x,t^*) + \frac{\delta}{2}(1 - \frac{\log t^*}{\log T}))}{\partial x_i}\Biggr)\Bigg|_{x=x^*} \\
&\quad= -\Biggl(\sum_{i,j=1}^{n} \phi_{i,j}\bigl(x,t^*\bigr)\frac{\partial^2 v(x,t^*)}{\partial x_i \partial x_j}\Biggr)\Bigg|_{x=x^*} + \Biggl(\sum_{i=1}^{n} \varphi_i\bigl(x,t^*\bigr)\frac{\partial v(x,t^*)}{\partial x_i}\Biggr)\Bigg|_{x=x^*} \\
&\quad\geq 0.
\end{aligned}$$

According to Lemma 2.2 and $\vartheta_i(x,t) \geq 0$, we know

$$\begin{aligned}
\mathbb{P}\bigl({}^H D_t\bigr)\zeta\bigl(x^*,t^*\bigr) &= {}^H D_t^p \zeta\bigl(x^*,t^*\bigr) + \sum_{i=1}^{m} \vartheta_i\bigl(x^*,t^*\bigr){}^H D_t^{p_i} \zeta\bigl(x^*,t^*\bigr) \\
&\geq \frac{(\log t^*)^{-p}}{\Gamma(1-p)}\zeta\bigl(x^*,t^*\bigr) + \sum_{i=1}^{m} \vartheta_i\bigl(x^*,t^*\bigr)\frac{(\log t^*)^{-p}}{\Gamma(1-p)}\zeta\bigl(x^*,t^*\bigr) \\
&> 0.
\end{aligned}$$

By the definition of $\zeta(x,t)$ and Lemma 2.1, we obtain

$$\begin{aligned}
&\big(\mathbb{P}\big({}^H D_t\big)v(x,t) + \mathbb{L}v(x,t) - C(x,t)v(x,t)\big)\big|_{(x^*,t^*)} \\
&\quad = \mathbb{P}\big({}^H D_t\big)\zeta\big(x^*,t^*\big) + \frac{\delta}{2\log T}\left(\frac{1}{\Gamma(2-p)}\big(\log t^*\big)^{1-p} + \sum_{i=1}^{m}\vartheta_i\frac{1}{\Gamma(2-p_i)}\big(\log t^*\big)^{1-p_i}\right) \\
&\qquad + \mathbb{L}\zeta\big(x^*,t^*\big) - C\big(x^*,t^*\big)\left(\zeta\big(x^*,t^*\big) - \frac{\delta}{2}\left(1-\frac{\log t^*}{\log T}\right)\right) \\
&\quad \geq \frac{\delta}{2\log T}\left(\frac{1}{\Gamma(2-p)}\big(\log t^*\big)^{1-p} + \sum_{i=1}^{m}\vartheta_i\frac{1}{\Gamma(2-p_i)}\big(\log t^*\big)^{1-p_i}\right) - C\big(x^*,t^*\big)\frac{\delta}{2}\frac{\log t^*}{\log T} \\
&\quad > 0,
\end{aligned}$$

which is not in accordance with $\Psi(x^*,t^*) \leq 0$. □

In the same way, we can prove the following.

**Theorem 3.2** *Let functions $\Psi$, $C$ be nonnegative on $\Omega \times (1,T]$ and $v(x,t) \in W_*$ be a solution of IBVP (1.1) and (3.1)–(3.2), it follows that*

$$v(x,t) \geq \min\Big\{\min_{x\in\Omega} a(x), \min_{(x,t)\in\partial\Omega\times[1,T]} b(x,t), 0\Big\}.$$

## 4 Application of the maximum principle

**Theorem 4.1** *Let $C(x,t)$ be nonpositive on $\Omega \times (1,T]$ and $v(x,t) \in W_*$ be a solution of IBVP (1.1) and (3.1)–(3.2). Then*

$$\|v\|_{C(\bar{\Omega}\times[1,T])} \leq \max\{N_0,N_1\} + 2\frac{(\log T)^p}{\Gamma(1+p)}N \tag{4.1}$$

*holds, where*

$$N_0 = \|a\|_{C^2(\bar{\Omega})}, \qquad N_1 = \|b\|_{C^1(\partial\Omega\times(1,T])}, \qquad N = \|\Psi\|_{C(\bar{\Omega}\times[1,T])}.$$

*Proof* For $\forall(x,t) \in \bar{\Omega} \times [1,T]$, set the auxiliary function

$$\psi(x,t) = v(x,t) - \frac{N}{\Gamma(1+p)}(\log t)^p,$$

then $\psi(x,t)$ is a solution of (1.1) with the function

$$\begin{aligned}
\Psi_1(x,t) &= \Psi(x,t) - N - \sum_{i=1}^{m}\vartheta_i(x,t)\frac{N}{\Gamma(p_i+1-p)}(\log t)^{p_i-p} + C(x,t)\frac{N}{\Gamma(1+p)}(\log t)^p, \\
b_1(x,t) &= b(x,t) - \frac{N}{\Gamma(1+p)}(\log t)^p
\end{aligned}$$

instead of $\Psi(x,t)$ and $b(x,t)$, respectively. Since $\Psi_1(x,t) \leq 0$, we apply the maximum principle (Theorem 3.1) to $\psi(x,t)$, we can get

$$\psi(x,t) \leq \max\left\{N_0, N_1 + \frac{N}{\Gamma(1+p)}(\log T)^p\right\}.$$

Therefore,

$$v(x,t) \leq \max\{N_0, N_1\} + 2\frac{N}{\Gamma(1+p)}(\log T)^p, \quad (x,t) \in \bar{\Omega} \times [1,T]. \tag{4.2}$$

Again, set another auxiliary function

$$\varpi(x,t) = v(x,t) + \frac{N}{\Gamma(1+p)}(\log t)^p,$$

and applying the minimum principle (Theorem 3.2), we obtain

$$v(x,t) \geq -\max\{N_0, N_1\} - 2\frac{N}{\Gamma(1+p)}(\log T)^p, \quad (x,t) \in \bar{\Omega} \times [1,T]. \tag{4.3}$$

Inequalities (4.2) and (4.3) together complete the proof of the theorem. □

**Theorem 4.2** *The solution of problem* (1.1) *and* (3.1)–(3.2) *depends continuously on the data given. That is, if*

$$\|\Psi - \bar{\Psi}\|_{C(\bar{\Omega}\times[1,T])} \leq \epsilon, \qquad \|a - \bar{a}\|_{C^2(\bar{\Omega})} \leq \epsilon_0, \qquad \|b - \bar{b}\|_{C^1(\partial\Omega\times[1,T])} \leq \epsilon_1,$$

*then the estimate*

$$\|v - \bar{v}\|_{C(\bar{\Omega}\times[1,T])} \leq \max\{\epsilon_0, \epsilon_1\} + 2\frac{(\log T)^p}{\Gamma(1+p)}\epsilon \tag{4.4}$$

*for the corresponding classical solution $v(x,t)$ and $\bar{v}(x,t)$ holds true.*

The last inequality (4.4) is a simple consequence of norm estimate (4.1). Applying Theorem 4.1 and replacing $\Psi$, $a$, and $b$ by $\Psi - \bar{\Psi}$, $a - \bar{a}$, and $b - \bar{b}$ in problem (1.1), (3.1), and (3.2), respectively, one can easily prove Theorem 4.2.

**Theorem 4.3** *Assume that $\Psi(x,t) \leq 0$, $C(x,t) \leq 0$, $\forall(x,t) \in \bar{\Omega} \times [1,T]$, and $v(x,t) \in W_*$ is a solution of IBVP* (1.1) *and* (3.1)–(3.2). *If $a(x) \leq 0$, $x \in \Omega$, and $b(x,t) \leq 0$, $(x,t) \in \partial\Omega \times [1,T]$, then*

$$v(x,t) \leq 0, \quad (x,t) \in \bar{\Omega} \times [1,T].$$

**Theorem 4.4** *If the inequality is reversed in Theorem* 4.3, *then the inequality of the conclusion is also reversed.*

From Theorems 4.3 and 4.4, the following remark holds.

*Remark* 4.1 If functions $\Psi$, $C$, $a$, $b$ are zero in Theorem 4.3 (or 4.4), then $v(x,t)$ is also zero on $\bar{\Omega} \times [1,T]$.

Now, let us consider the uniqueness of solution for the multi-index nonlinear Hadamard fractional diffusion equation

$$\mathbb{P}\left({}^H D_t\right)v(x,t) = -\mathbb{L}v(x,t) + C(x,t)v(x,t) + \Psi(x,t,v), \quad (x,t) \in \Omega \times (1,T] \tag{4.5}$$

with initial boundary value conditions (3.1)–(3.2).

**Theorem 4.5** *If the smooth function $\Psi(x,t,v)$ of diffusion equation (4.5) is nonincreasing with respect to the third variable and $C(x,t) \leq 0$, then the multi-index nonlinear Hadamard fractional diffusion problem (4.5) and (3.1)–(3.2) has at most one solution $v(x,t) \in W_*$.*

*Proof* Let $v_1, v_2 \in W_*$ be two solutions of Eq. (4.5) with initial boundary value conditions (3.1)–(3.2). Define an auxiliary function on $\bar{\Omega} \times [1,T]$

$$\mathfrak{P}(x,t) = v_1(x,t) - v_2(x,t).$$

Then $\mathfrak{P}$ satisfies the equation

$$\begin{cases} \mathbb{P}({}^H D_t)\mathfrak{P}(x,t) + \mathbb{L}\mathfrak{P}(x,t) - C(x,t)\mathfrak{P}(x,t) \\ \quad = \Psi(x,t,v_1) - \Psi(x,t,v_2), \quad (x,t) \in \Omega \times (1,T], \\ \mathfrak{P}(x,1) = 0, \quad x \in \Omega, \\ \mathfrak{P}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [1,T]. \end{cases} \tag{4.6}$$

It follows from the assumptions on $\Psi$ that

$$\Psi(\cdot,v_1) - \Psi(\cdot,v_2) = \frac{\partial \Psi}{\partial v}(\tilde{v})(v_1 - v_2) = \frac{\partial \Psi}{\partial v}(\tilde{v})\mathfrak{P}(x,t) \leq 0, \tag{4.7}$$

where $\tilde{v} = \lambda v_1 + (1-\lambda)v_2$ for some $0 \leq \lambda \leq 1$.

Since $\Psi$ is nonincreasing with respect to the third variable, i.e., $\frac{\partial \Psi}{\partial v} \leq 0$, it follows from Theorem 4.3 that, for the multi-index nonlinear Hadamard fractional diffusion problem (4.6),

$$\mathfrak{P}(x,t) \leq 0, \quad (x,t) \in \bar{\Omega} \times [1,T]. \tag{4.8}$$

In the same way, applying Theorem 4.3 to function $-\mathfrak{P}(x,t)$, for $(x,t) \in \bar{\Omega} \times [1,T]$, the inequality

$$-\mathfrak{P}(x,t) \leq 0 \tag{4.9}$$

holds. Thus, (4.8) and (4.9) imply $\mathfrak{P}(x,t) = 0$. This completes the proof. □

It is obvious to observe from the proof process of Theorem 4.5.

*Remark* 4.2 If $\frac{\partial \Psi}{\partial v}(\tilde{v}) + C \leq 0$, then the conclusion of Theorem 4.5 holds.

**Corollary 4.1** *If the function $C$ is nonpositive on $\bar{\Omega} \times [1,T]$, then IBVP (1.1) and (3.1)–(3.2) has at most one solution on $W_*$.*

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors equally contributed to this manuscript and approved the final version.

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