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# Basic theory of differential equations with linear perturbations of second type on time scales

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## Abstract

In this paper, we develop the theory of differential equations with linear perturbations of second type on time scales. An existence theorem for differential equations with linear perturbations of second type on time scales is given under  $\mathcal{D}$ -Lipschitz conditions. Some fundamental differential inequalities on time scales, which are utilized to investigate the existence of extremal solutions, are also presented. The comparison principle on differential equations with linear perturbations of second type on time scales is established. Our results in this paper extend and improve some well-known results.

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**Keywords:** Linear perturbations; Existence; Differential inequalities; Comparison principle; Time scales

## 1 Introduction

In this paper, we discuss the following differential equations with linear perturbations of second type on time scales (in short DETS):

$$\begin{cases} [u(t) - f(t, u(t))]^\Delta = g(t, u(t)), & t \in J, \\ u(t_0) = u_0, \end{cases} \quad (1)$$

where  $f, g \in C_{rd}(J \times \mathbb{R}, \mathbb{R})$ .

Let  $\mathbb{T}$  be a time scale and  $J = [t_0, t_0 + a]_{\mathbb{T}} = [t_0, t_0 + a] \cap \mathbb{T}$  be a bounded interval in  $\mathbb{T}$  for some  $t_0, a \in \mathbb{R}$  with  $a > 0$ . Let  $C_{rd}(J \times \mathbb{R}, \mathbb{R})$  denote the class of rd-continuous functions  $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ . For the basic definitions and useful lemmas from the theory of calculus on time scales, one can be referred to [1].

By a solution of DETS (1), we mean a  $\Delta$ -differentiable function  $u$  such that

- (i) the function  $t \mapsto u(t) - f(t, u(t))$  is  $\Delta$ -differentiable for each  $u \in \mathbb{R}$ , and
- (ii)  $u$  satisfies the equations in (1).

The theory of time scales has drawn a lot of attention since 1988 (see [1–9]). In recent years, the theory of nonlinear differential equations with linear perturbations has been a hot research topic, see [9–15]. Dhage and Jadhav [12] discussed the following first order

hybrid differential equation with linear perturbations of second type:

$$\begin{cases} \frac{d}{dt}[x(t) - f(t, x(t))] = g(t, x(t)), & t \in [t_0, t_0 + a], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where  $[t_0, t_0 + a]$  is a bounded interval in  $\mathbb{R}$  for some  $t_0, a \in \mathbb{R}$  with  $a > 0$ , and  $f, g \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R})$ . They developed the theory of hybrid differential equations with linear perturbations of second type and gave some original and interesting results.

As far as we know, there are no results for DETS (1). From the above works, we consider the theory of DETS (1). An existence theorem for DETS (1) is given under  $\mathcal{D}$ -Lipschitz conditions. Some fundamental differential inequalities on time scales (in short DITS), which are utilized to investigate the existence of extremal solutions, are also presented. The comparison principle on DETS (1) is established. Our results in this paper extend and improve some well-known results.

The paper is organized as follows: Sect. 2 gives an existence theorem for DETS (1) under  $\mathcal{D}$ -Lipschitz conditions by the fixed point theorem in Banach algebra due to Dhage. Section 3 establishes some fundamental DITS to strict inequalities for DETS (1). Section 4 presents existence results of maximal and minimal solutions for HDTs. Section 5 proves the comparison principle for DETS (1), which is followed by the conclusion in Sect. 6.

## 2 Existence result

In this section, we discuss the existence results for DETS (1). We place DETS (1) in the space  $C_{rd}(J, \mathbb{R})$  of rd-continuous functions defined on  $J$ .  $\|\cdot\|$  denotes a supremum norm in  $C_{rd}(J, \mathbb{R})$  by

$$\|u\| = \sup_{t \in J} |u(t)|.$$

Clearly  $C_{rd}(J, \mathbb{R})$  is a Banach algebra with respect to the above norm.  $L^1(J, \mathbb{R})$  denotes the space of Lebesgue  $\Delta$ -integrable functions on  $J$  equipped with the norm  $\|\cdot\|_{L^1}$  defined by

$$\|u\|_{L^1} = \int_{t_0}^{t_0+a} |u(s)| \Delta s.$$

Some definitions and lemmas that will be used in our main results are given in what follows.

**Definition 2.1** ([12]) A mapping  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called a dominating function or, in short,  $\mathcal{D}$ -function if it is an upper semi-continuous and nondecreasing function satisfying  $\varphi(0) = 0$ . A mapping  $T : P \rightarrow P$  is called  $\mathcal{D}$ -Lipschitz if there is a  $\mathcal{D}$ -function  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying

$$\|Tu - Tv\| \leq \varphi(\|u - v\|)$$

for all  $u, v \in P$ . The function  $\varphi$  is called a  $\mathcal{D}$ -function of  $Q$  on  $P$ . If  $\varphi(t) = lt$ ,  $l > 0$ , then  $T$  is called Lipschitz with the Lipschitz constant  $l$ . In particular, if  $l < 1$ , then  $T$  is called a contraction on  $X$  with the contraction constant  $l$ . Furthermore, if  $\varphi(t) < t$  for  $t > 0$ , then  $T$  is called nonlinear  $\mathcal{D}$ -contraction and the function  $\varphi$  is called  $\mathcal{D}$ -function of  $T$  on  $X$ .

The details of different types of contractions appear in the monographs of Dhage [16] and Granas and Dugundji [17]. There do exist  $\mathcal{D}$ -functions, and the commonly used  $\mathcal{D}$ -functions are  $\varphi(t) = lt$  and  $\varphi(t) = \frac{t}{1+t}$ . These  $\mathcal{D}$ -functions have been widely used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods.

Another notion that we need in the sequel is the following definition.

**Definition 2.2** ([12]) An operator  $T$  on a Banach space  $P$  mapping into itself is called compact if  $T(P)$  is a relatively compact subset of  $P$ .  $T$  is called totally bounded if, for any bounded subset  $Q$  of  $P$ ,  $T(Q)$  is a relatively compact subset of  $P$ . If  $T$  is continuous and totally bounded, then it is called completely continuous on  $P$ .

The following fixed point theorem in Banach algebra due to Dhage [16] is useful in the proofs of our main results.

**Lemma 2.1** ([16]) *Let  $Q$  be a closed convex and bounded subset of the Banach space  $P$ , and let  $A : P \rightarrow P$  and  $B : Q \rightarrow P$  be two operators such that*

- (a)  *$A$  is a nonlinear  $\mathcal{D}$ -contraction,*
- (b)  *$B$  is compact and continuous, and*
- (c)  *$u = Au + Bv$  for all  $v \in Q \Rightarrow u \in Q$ .*

*Then the operator equation  $Au + Bu = u$  has a solution in  $Q$ .*

We present the following hypotheses:

- (A<sub>0</sub>) The function  $u \mapsto u - f(t, u)$  is increasing in  $\mathbb{R}$  for all  $t \in J$ .
- (A<sub>1</sub>) There exists a constant  $L > 0$  such that

$$|f(t, u) - f(t, v)| \leq \frac{L|u - v|}{M + |u - v|}$$

for all  $t \in J$  and  $u, v \in \mathbb{R}$ . Moreover,  $L \leq M$ .

- (A<sub>2</sub>) There exists a function  $h \in L^1(J, \mathbb{R})$  such that

$$|g(t, u)| \leq h(t), \quad t \in J$$

for all  $u \in \mathbb{R}$ .

**Lemma 2.2** *Suppose that (A<sub>0</sub>) holds. Then, for any  $v \in L^1(J, \mathbb{R})$ , the  $\Delta$ -differentiable function  $u$  is a solution of the DETS*

$$[u(t) - f(t, u(t))]^\Delta = v(t), \quad t \in J, \tag{2}$$

and

$$u(t_0) = u_0 \in \mathbb{R}, \tag{3}$$

if and only if  $u$  satisfies the integral equation

$$u(t) = u_0 - f(t_0, u_0) + f(t, u(t)) + \int_{t_0}^t v(s) \Delta s, \quad t \in J. \tag{4}$$

*Proof* Let  $u$  be a solution of problem (2) and (3). Applying  $\Delta$ -integral to (2) from  $t_0$  to  $t$ , we obtain

$$[u(t) - f(t, u(t))] - [u_0 - f(t_0, u_0)] = \int_{t_0}^t v(s) \Delta s,$$

i.e.,

$$u(t) = u_0 - f(t_0, u_0) + f(t, u(t)) + \int_{t_0}^t v(s) \Delta s, \quad t \in J.$$

Thus, (4) holds.

Conversely, suppose that  $u$  satisfies equation (4). By direct differentiation and applying  $\Delta$ -derivative on both sides of (4), then (2) is satisfied. Thus, substituting  $t = t_0$  in (4) implies

$$u(t_0) - f(t_0, u(t_0)) = u_0 - f(t_0, u_0).$$

Since the map  $u \mapsto u - f(t, u)$  is increasing in  $\mathbb{R}$  for  $t \in J$ , the map  $u \mapsto u - f(t_0, u)$  is injective in  $\mathbb{R}$  and  $u(t_0) = u_0$ . Hence, (3) also holds. □

Now we will give the following existence theorem for DETS (1).

**Theorem 2.1** *Suppose that  $(A_0)$ – $(A_2)$  hold. Then DETS (1) has a solution defined on  $J$ .*

*Proof* Set  $U = C_{rd}(J, \mathbb{R})$  and define a subset  $S$  of  $U$  by

$$S = \{u \in U \mid \|u\| \leq N\},$$

where  $N = |u_0 - f(t_0, u_0)| + L + F_0 + \|h\|_{L^1}$ , and  $F_0 = \sup_{t \in J} |f(t, 0)|$ .

Clearly,  $S$  is a closed, convex, and bounded subset of the Banach space  $U$ . By Lemma 2.2, DETS (1) is equivalent to the nonlinear integral equation

$$u(t) = u_0 - f(t_0, u_0) + f(t, u(t)) + \int_{t_0}^t g(s, u(s)) \Delta s, \quad t \in J. \tag{5}$$

Define two operators  $A : U \rightarrow U$  and  $B : S \rightarrow U$  by

$$Au(t) = f(t, u(t)), \quad t \in J, \tag{6}$$

and

$$Bu(t) = u_0 - f(t_0, u_0) + \int_{t_0}^t g(s, u(s)) \Delta s, \quad t \in J. \tag{7}$$

Then equation (5) is transformed into the operator equation as follows:

$$Au(t) + Bu(t) = u(t), \quad t \in J.$$

Next, we prove that the operators  $A$  and  $B$  satisfy all the conditions of Lemma 2.1.

Firstly, we prove that  $A$  is a nonlinear  $\mathcal{D}$ -contraction on  $U$  with a  $\mathcal{D}$ -function  $\varphi$ . Let  $u, v \in U$ . Then, by  $(A_1)$ ,

$$|Au(t) - Av(t)| = |f(t, u(t)) - f(t, v(t))| \leq \frac{L|u(t) - v(t)|}{M + |u(t) - v(t)|} \leq \frac{L\|u - v\|}{M + \|u - v\|}$$

for all  $t \in J$ . Taking supremum over  $t$ , we have

$$\|Au - Av\| \leq \frac{L\|u - v\|}{M + \|u - v\|}$$

for all  $u, v \in U$ . This shows that  $A$  is a nonlinear  $\mathcal{D}$ -contraction on  $U$  with a  $\mathcal{D}$ -function  $\varphi$  defined by  $\varphi(t) = \frac{Lt}{M+t}$ .

Next, we prove that  $B$  is a compact and continuous operator on  $S$  into  $U$ . Firstly, we prove that  $B$  is continuous on  $S$ . Let  $\{u_n\}$  be a sequence in  $S$  converging to a point  $u \in S$ . Then, by Lebesgue dominated convergence theorem adapted to time scales, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Bu_n(t) &= \lim_{n \rightarrow \infty} \left( u_0 - f(t_0, u_0) + \int_{t_0}^t g(s, u_n(s)) \Delta s \right) \\ &= u_0 - f(t_0, u_0) + \lim_{n \rightarrow \infty} \int_{t_0}^t g(s, u_n(s)) \Delta s \\ &= u_0 - f(t_0, u_0) + \int_{t_0}^t \left[ \lim_{n \rightarrow \infty} g(s, u_n(s)) \right] \Delta s \\ &= u_0 - f(t_0, u_0) + \int_{t_0}^t g(s, u(s)) \Delta s \\ &= Bu(t) \end{aligned}$$

for all  $t \in J$ . This shows that  $B$  is a continuous operator on  $S$ .

Next we prove that  $B$  is a compact operator on  $S$ . It is enough to show that  $B(S)$  is a uniformly bounded and equicontinuous set in  $U$ . On the one hand, let  $u \in S$  be arbitrary. Then, by  $(A_2)$ ,

$$\begin{aligned} |Bu(t)| &= \left| u_0 - f(t_0, u_0) + \int_{t_0}^t g(s, u(s)) \Delta s \right| \\ &\leq |u_0 - f(t_0, u_0)| + \int_{t_0}^t |g(s, u(s))| \Delta s \\ &\leq |u_0 - f(t_0, u_0)| + \int_{t_0}^t h(s) \Delta s \\ &\leq |u_0 - f(t_0, u_0)| + \|h\|_{L^1} \end{aligned}$$

for all  $t \in J$ . Taking supremum over  $t$ ,

$$\|Bu\| \leq |u_0 - f(t_0, u_0)| + \|h\|_{L^1}$$

for all  $u \in S$ . This shows that  $B$  is uniformly bounded on  $S$ .

On the other hand, let  $t_1, t_2 \in J$ . Then, for any  $u \in S$ , we get

$$\begin{aligned} |Bu(t_1) - Bu(t_2)| &= \left| \int_{t_0}^{t_1} g(s, u(s)) \Delta s - \int_{t_0}^{t_2} g(s, u(s)) \Delta s \right| \\ &\leq \left| \int_{t_2}^{t_1} |g(s, u(s))| \Delta s \right| \\ &\leq \left| \int_{t_2}^{t_1} h(s) \Delta s \right| \\ &= |p(t_1) - p(t_2)|, \end{aligned}$$

where  $p(t) = \int_{t_0}^t h(s) \Delta s$ . Since the function  $p$  is continuous on compact  $J$ , it is uniformly continuous. Hence, for  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|t_1 - t_2| < \delta \implies |Bu(t_1) - Bu(t_2)| < \varepsilon$$

for all  $t_1, t_2 \in J$  and  $u \in S$ . This shows that  $B(S)$  is an equicontinuous set in  $U$ . Now the set  $B(S)$  is a uniformly bounded and equicontinuous set in  $U$ , so it is compact by the Arzela–Ascoli theorem. As a result,  $B$  is a complete continuous operator on  $S$ .

Next, we show that (c) of Lemma 2.1 is satisfied. Let  $u \in U$  and  $v \in S$  be arbitrary such that  $u = Au + Bv$ . Then, by assumption  $(A_1)$ , we have

$$\begin{aligned} |u(t)| &\leq |Au(t)| + |Bv(t)| \\ &= |f(t, u(t))| + |u_0 - f(t_0, u_0)| + \int_{t_0}^t |g(s, v(s))| \Delta s \\ &\leq [|f(t, u(t)) - f(t, 0)| + |f(t, 0)|] + \int_{t_0}^t |g(s, v(s))| \Delta s \\ &\leq |u_0 - f(t_0, u_0)| + L + F_0 + \int_{t_0}^t h(s) \Delta s \\ &\leq |u_0 - f(t_0, u_0)| + L + F_0 + \|h\|_{L^1}. \end{aligned}$$

Taking supremum over  $t$ , we get

$$\|u\| \leq |u_0 - f(t_0, u_0)| + L + F_0 + \|h\|_{L^1} = N.$$

This shows that (c) of Lemma 2.1 is satisfied.

Thus, all the conditions of Lemma 2.1 are satisfied, and hence the operator equation  $Au + Bu = u$  has a solution in  $S$ . Therefore, DETS (1) has a solution defined on  $J$ .  $\square$

### 3 Differential inequalities on time scales

In this section, we establish DITS for DETS (1).

**Theorem 3.1** *Suppose that  $(A_0)$  holds. Assume that there exist  $\Delta$ -differentiable functions  $v, w$  such that*

$$[v(t) - f(t, v(t))]^\Delta \leq g(t, v(t)), \quad t \in J, \tag{8}$$

and

$$[w(t) - f(t, w(t))]^\Delta \geq g(t, w(t)), \quad t \in J, \tag{9}$$

one of the inequalities being strict. Then  $v(t_0) < w(t_0)$  implies

$$v(t) < w(t) \tag{10}$$

for all  $t \in J$ .

*Proof* Assume that inequality (9) is strict. Suppose that the claim is false. Then there exists  $t_1 \in J, t_1 > t_0$  such that  $v(t_1) = w(t_1)$  and  $v(t) < w(t)$  for  $t_0 \leq t < t_1$ .

Define

$$V(t) = v(t) - f(t, v(t)) \quad \text{and} \quad W(t) = w(t) - f(t, w(t))$$

for all  $t \in J$ . Then we obtain  $V(t_1) = W(t_1)$  and, by  $(A_0)$ , we have  $V(t) < W(t)$  for all  $t < t_1$ .

By  $V(t_1) = W(t_1)$ , we get

$$\frac{V(t_1 + h) - V(t_1)}{h} > \frac{W(t_1 + h) - W(t_1)}{h}$$

for sufficiently small  $h < 0$ . The above inequality implies that

$$V^\Delta(t_1) \geq W^\Delta(t_1)$$

because of  $(A_0)$ . Then we obtain

$$g(t_1, v(t_1)) \geq V^\Delta(t_1) \geq W^\Delta(t_1) > g(t_1, w(t_1)).$$

This is a contradiction with  $v(t_1) = w(t_1)$ . Hence conclusion (10) is valid. □

The next result is concerned with nonstrict DITS which needs a Lipschitz condition.

**Theorem 3.2** *Suppose that the conditions of Theorem 3.1 hold with inequalities (8) and (9). Suppose that there exists a real number  $K > 0$  such that*

$$g(t, u_1) - g(t, u_2) \leq K \sup_{t_0 \leq s \leq t} (u_1(s) - f(s, u_1(s)) - (u_2(s) - f(s, u_2(s))))), \quad t \in J \tag{11}$$

for all  $u_1, u_2 \in \mathbb{R}$  with  $u_1 \geq u_2$ . Then  $v(t_0) \leq w(t_0)$  implies  $v(t) \leq w(t)$  for all  $t \in J$ .

*Proof* Let  $\varepsilon > 0$  and a real number  $K > 0$  be given. Define

$$w_\varepsilon(t) - f(t, w_\varepsilon(t)) = w(t) - f(t, w(t)) + \varepsilon e^{2L(t-t_0)},$$

so that we get

$$w_\varepsilon(t) - f(t, w_\varepsilon(t)) > w(t) - f(t, w(t)) \quad \Rightarrow \quad w_\varepsilon(t) > w(t).$$

Let  $W_\varepsilon(t) = w_\varepsilon(t) - f(t, w_\varepsilon(t))$  so that  $W(t) = w(t) - f(t, w(t))$  for  $t \in J$ . By (9), we obtain

$$W_\varepsilon^\Delta(t) = W^\Delta(t) + 2K\varepsilon e^{2L(t-t_0)} \geq g(t, w(t)) + 2L\varepsilon e^{2L(t-t_0)}.$$

From (11), we have

$$g(t, w_\varepsilon(t)) - g(t, w(t)) \leq K \sup_{t_0 \leq s \leq t} (W_\varepsilon(s) - W(s)) = K\varepsilon e^{2L(t-t_0)}$$

for all  $t \in J$ , then we obtain

$$W_\varepsilon^\Delta(t) \geq g(t, w_\varepsilon(t)) - K\varepsilon e^{2L(t-t_0)} + 2K\varepsilon e^{2L(t-t_0)} > g(t, w_\varepsilon(t)),$$

i.e.,

$$[w_\varepsilon(t) - f(t, w_\varepsilon(t))]^\Delta > g(t, w_\varepsilon(t))$$

for all  $t \in J$ . Also, we get  $w_\varepsilon(t_0) > w(t_0) > v(t_0)$ . Hence, an application of Theorem 3.1 with  $w = w_\varepsilon$  implies that  $v(t) < w_\varepsilon(t)$  for all  $t \in J$ . By the arbitrariness of  $\varepsilon > 0$ , taking the limits as  $\varepsilon \rightarrow 0$ , we have  $v(t) \leq w(t)$  for all  $t \in J$ . □

#### 4 Existence of maximal and minimal solutions

In this section, we give the existence of maximal and minimal solutions for DETS (1) on  $J = [t_0, t_0 + a]_{\mathbb{T}}$ .

**Definition 4.1** A solution  $r$  of DETS (1) is said to be maximal if, for any other solution  $u$  to DETS (1), one has  $u(t) \leq r(t)$  for all  $t \in J$ . Similarly, a solution  $\rho$  of DETS (1) is said to be minimal if  $\rho(t) \leq u(t)$  for all  $t \in J$ , where  $u$  is any solution of DETS (1) on  $J$ .

We discuss the case of maximal solution only. Similarly, the case of minimal solution can be obtained with the same arguments with appropriate modifications. Given an arbitrarily small real number  $\varepsilon > 0$ , discuss the following initial value problem of DETS:

$$\begin{cases} [u(t) - f(t, u(t))]^\Delta = g(t, u(t)) + \varepsilon, & t \in J, \\ u(t_0) = u_0 + \varepsilon, \end{cases} \tag{12}$$

where  $f, g \in C_{rd}(J \times \mathbb{R}, \mathbb{R})$ .

An existence theorem for DETS (12) can be stated as follows.

**Theorem 4.1** *Suppose that  $(A_0)$ – $(A_2)$  hold. Then, for every small number  $\varepsilon > 0$ , DETS (12) has a solution defined on  $J$ .*

*Proof* The proof is similar to that of Theorem 2.1, and we omit the proofs. □

Our main existence theorem for maximal solution for DETS (1) is as follows.

**Theorem 4.2** *Suppose that  $(A_0)$ – $(A_2)$  hold. Then DETS (1) has a maximal solution defined on  $J$ .*



*Proof* Let  $\{\varepsilon_n\}_0^\infty$  be a decreasing sequence of positive real numbers such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . Then, for any solution  $x$  of DETS (1), by Theorem 4.1, we get

$$x(t) < r(t, \varepsilon_n)$$

for all  $t \in J$  and  $n \in \mathbb{N} \cup \{0\}$ , where  $r(t, \varepsilon_n)$  defined on  $J$  is a solution of the DETS

$$\begin{cases} [u(t) - f(t, u(t))]^\Delta = g(t, u(t)) + \varepsilon_n, & t \in J, \\ u(t_0) = u_0 + \varepsilon_n. \end{cases} \tag{13}$$

By Theorem 3.2,  $\{r(t, \varepsilon_n)\}$  is a decreasing sequence of positive real numbers, the limit

$$r(t) = \lim_{n \rightarrow \infty} r(t, \varepsilon_n) \tag{14}$$

exists. We prove that the convergence in (14) is uniform on  $J$ . Next, we show that the sequence  $\{r(t, \varepsilon_n)\}$  is equicontinuous in  $C_{rd}(J, \mathbb{R})$ . Let  $t_1, t_2 \in J$  with  $t_1 < t_2$  be arbitrary. Then we have

$$\begin{aligned} & |r(t_1, \varepsilon_n) - r(t_2, \varepsilon_n)| \\ &= \left| u_0 + \varepsilon_n - f(t_0, u_0 + \varepsilon_n) + f(t_1, r(t_1, \varepsilon_n)) + \int_{t_0}^{t_1} g(s, r(s, \varepsilon_n)) \Delta s \right. \\ &\quad \left. + \int_{t_0}^{t_1} \varepsilon_n \Delta s - \left( u_0 + \varepsilon_n - f(t_0, u_0 + \varepsilon_n) + f(t_2, r(t_2, \varepsilon_n)) \right. \right. \\ &\quad \left. \left. + \int_{t_0}^{t_2} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_2} \varepsilon_n \Delta s \right) \right| \\ &\leq |f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| + \left| \int_{t_0}^{t_1} g(s, r(s, \varepsilon_n)) \Delta s \right. \\ &\quad \left. - \int_{t_0}^{t_2} g(s, r(s, \varepsilon_n)) \Delta s \right| + \left| \int_{t_0}^{t_1} \varepsilon_n \Delta s - \int_{t_0}^{t_2} \varepsilon_n \Delta s \right| \\ &= |f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| + \left| \int_{t_1}^{t_2} g(s, r(s, \varepsilon_n)) \Delta s \right| + \left| \int_{t_1}^{t_2} \varepsilon_n \Delta s \right| \\ &\leq |f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| + \left| \int_{t_1}^{t_2} h(s) \Delta s \right| + |t_1 - t_2| \varepsilon_n \\ &\leq |f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| + |p(t_1) - p(t_2)| + |t_1 - t_2| \varepsilon_n, \end{aligned}$$

where  $p(t) = \int_{t_0}^t h(s) \Delta s$ .

Since  $f$  is continuous on a compact set  $J \times [-N, N]$ , it is uniformly continuous there. Hence,

$$|f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2$$

uniformly for all  $n \in \mathbb{N}$ . Similarly, since the function  $p$  is continuous on a compact set  $J$ , it is uniformly continuous and hence

$$|p(t_1) - p(t_2)| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2.$$

Therefore, we obtain

$$|r(t_1, \varepsilon_n) - r(t_2, \varepsilon_n)| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2$$

uniformly for all  $n \in \mathbb{N}$ . Therefore,

$$r(t, \varepsilon_n) \rightarrow r(t) \quad \text{as } n \rightarrow \infty$$

for all  $t \in J$ .

Next, we prove that the function  $r(t)$  is a solution of DETS (1) defined on  $J$ . Since  $r(t, \varepsilon_n)$  is a solution of DETS (13), we get

$$r(t, \varepsilon_n) = u_0 + \varepsilon_n - f(t_0, u_0 + \varepsilon_n) + f(t, r(t, \varepsilon_n)) + \int_{t_0}^t g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^t \varepsilon_n \Delta s \quad (15)$$

for all  $t \in J$ . Taking the limit as  $n \rightarrow \infty$  in equation (15) implies

$$r(t) = u_0 - f(t_0, u_0) + f(t, r(t)) + \int_{t_0}^t g(s, r(s)) \Delta s$$

for all  $t \in J$ . Thus, the function  $r$  is a solution of DETS (1) on  $J$ . Finally, from inequality (13), it follows that  $x(t) \leq r(t)$  for all  $t \in J$ . Hence, DETS (1) has a maximal solution on  $J$ .  $\square$

### 5 Comparison theorems on time scales

The main problem of the DITS is to estimate a bound for the solution set for the DITS related to DETS (1). In this section, we present the maximal and minimal solutions serve as bounds for the solutions of the related DITS to DETS (1) on  $J = [t_0, t_0 + a]_{\mathbb{T}}$ .

**Theorem 5.1** *Suppose that  $(A_0)$ – $(A_2)$  hold. Assume that there exists a  $\Delta$ -differentiable function  $u$  such that*

$$\begin{cases} [x(t) - f(t, x(t))]^\Delta \leq g(t, x(t)), & t \in J, \\ x(t_0) \leq u_0. \end{cases} \quad (16)$$

Then

$$x(t) \leq r(t) \quad (17)$$

for all  $t \in J$ , where  $r$  is a maximal solution of DETS (1) on  $J$ .

*Proof* Let  $\varepsilon > 0$  be arbitrarily small. From Theorem 4.2,  $r(t, \varepsilon)$  is a maximal solution of DETS (12) and the limit

$$r(t) = \lim_{\varepsilon \rightarrow 0} r(t, \varepsilon) \quad (18)$$

is uniform on  $J$ , and the function  $r$  is a maximal solution of DETS (1) on  $J$ . Hence, we have

$$\begin{cases} [r(t, \varepsilon) - f(t, r(t, \varepsilon))]^\Delta = g(t, r(t, \varepsilon)) + \varepsilon, & t \in J, \\ r(t_0, \varepsilon) = u_0 + \varepsilon. \end{cases}$$

The above inequality implies that

$$\begin{cases} [r(t, \varepsilon) - f(t, r(t, \varepsilon))]^\Delta > g(t, r(t, \varepsilon)), t \in J, \\ r(t_0, \varepsilon) > u_0. \end{cases} \tag{19}$$

Now, we apply Theorem 3.2 to inequalities (16) and (19) and conclude that  $x(t) < r(t, \varepsilon)$  for all  $t \in J$ . Thus, (18) implies that inequality (17) holds on  $J$ .  $\square$

**Theorem 5.2** *Suppose that  $(A_0)$ – $(A_2)$  hold. Assume that there exists a  $\Delta$ -differentiable function  $u$  such that*

$$\begin{cases} [y(t) - f(t, y(t))]^\Delta \geq g(t, y(t)), \quad t \in J, \\ y(t_0) \geq u_0. \end{cases}$$

Then

$$\rho(t) \leq y(t)$$

for all  $t \in J$ , where  $\rho$  is a minimal solution of DETS (1) on  $J$ .

Note that Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for DETS (1) on  $J$ . We have the following result.

**Theorem 5.3** *Suppose that  $(A_0)$ – $(A_2)$  hold. Assume that there exists a function  $G : J \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that*

$$g(t, u_1) - g(t, u_2) \leq G(t, (u_1 - f(t, u_1)) - (u_2 - f(t, u_2))), \quad t \in J$$

for all  $u_1, u_2 \in \mathbb{R}$  with  $u_1 \geq u_2$ . If an identically zero function is the only solution of the differential equation

$$m^\Delta(t) = G(t, m(t)), \quad t \in J, \quad m(t_0) = 0, \tag{20}$$

then DETS (1) has a unique solution on  $J$ .

*Proof* From Theorem 2.1, DETS (1) has a solution defined on  $J$ . Suppose that there are two solutions  $x_1$  and  $x_2$  of DETS (1) existing on  $J$  with  $x_1 > x_2$ . Define  $m : J \rightarrow \mathbb{R}^+$  by

$$m(t) = (x_1(t) - f(t, x_1(t))) - (x_2(t) - f(t, x_2(t))).$$

By  $(A_0)$ , we conclude that  $m(t) > 0$ . Then we obtain

$$\begin{aligned} m^\Delta(t) &= [x_1(t) - f(t, x_1(t))]^\Delta - [x_2(t) - f(t, x_2(t))]^\Delta \\ &= g(t, x_1) - g(t, x_2) \\ &\leq G\left(t, \frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)}\right) = G(t, m(t)) \end{aligned}$$

for  $t \in J$ , and that  $m(t_0) = 0$ .

Now, we apply Theorem 5.1 with  $f(t, u) \equiv 0$  to get that  $m(t) \leq 0$  for all  $t \in J$ , where an identically zero function is the only solution of DETS (20).  $m(t) \leq 0$  is a contradiction with  $m(t) > 0$ . Then we have  $x_1 = x_2$ .  $\square$

**Remark 5.1** When  $f \equiv 0$  and  $\mathbb{T} = \mathbb{R}$  in our results of this paper, we obtain the differential inequalities and other related results given in Lakshmikantham and Leela [18] for the IVP of ordinary nonlinear differential equation

$$u'(t) = g(t, u(t)), \quad t \in [t_0, t_0 + a], \quad u(t_0) = u_0.$$

**Remark 5.2** The main results in this paper extend and improve some well-known results in [12].

## 6 Conclusion

In this paper, we have developed the theory of DETS (1). By the fixed point theorem in Banach algebra due to Dhage, we have presented an existence theorem for DETS (1) under  $\mathcal{D}$ -Lipschitz conditions. We have also established some DITS for DETS (1) which are used to investigate the existence of extremal solutions. The comparison principle on DETS (1) has been given. Our results in this paper extend and improve some well-known results.

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### Availability of data and materials

Not applicable.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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