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Penalty algorithm adapted for the spectral element discretization of the Darcy equations

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Abstract

Any spectral element discretization of the Darcy problem can be efficiently solved by applying the penalty method. This method leads to a system of equations with uncoupled unknowns. We prove a posteriori error estimates for a spectral element discretization of the Darcy problem. The proposed algorithm permits the optimization of the penalty parameter as a function of the error indicators.

Keywords: Penalty method; Darcy equations; Spectral element discretization

1 Introduction

The Darcy problem introduced in [1] is used to model the flow (water, petrol, ...) of an incompressible and isothermal fluid in homogeneous porous media. The unknowns are the velocity and the pressure. Any discretization by the Galerkin method leads to a system of equations where the velocity and the pressure are coupled. Many algorithms are proposed in the literature to uncouple the velocity and the pressure such as the Uzawa method [2] and the penalty method [2, 3]

The penalty method has been used extensively in finite element discretization to solve different problems (Stokes, Darcy, Navier–Stokes, ...) [4–8]. However, in spectral element discretization [9, 10], this method has been only considered for the Stokes problem [11]. In this work, We are interested in the application of the penalty method to solve the Darcy problem using spectral element discretization for its high accuracy [3, 12].

The advantage of using the penalty method is twofold: first, it permits to decouple the two unknowns (velocity and pressure), and second, it guarantees the stabilization of the discrete problem [13]. Moreover, the optimization of the penalty parameter, using error indicators, reduces considerably the computation cost for solving the discrete problem [14].

In this paper, we perform a posteriori analysis of the penalized spectral element discretization of the Darcy equations. We propose an algorithm, based on the developed error indicator, to optimize the value of the penalty parameter.

An outline of the paper is as follows:

- In Sect. 2 we present the penalized continuous problem and some regularity results.
- Section 3 is about the analysis of the penalized discrete problem.

- The a posteriori error analysis of the penalized discrete problem and a penalty adaptation algorithm are developed in Sect. 4.

2 The penalized continuous problem

Let Ω a connected domain of $\mathbb{R}^d$ ($d = 2, 3$), and $\partial\Omega$ its Lipschitz continuous boundary. We consider the following Darcy problem:

$$\begin{aligned} &\mathbf{u} + \mu\, \mathbf{grad}\, p = \mathbf{f} \quad \text{in } \Omega, \\ &\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\ &\mathbf{u}.\mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the unknowns are the velocity $\mathbf{u}$ and the pressure p, $\mathbf{f}$ represents the density of forces, μ is a positive constant equal to the quotient of the fluid viscosity by the medium permeability (μ^{-1} is called the porosity). We consider in the following $\mu = 1$. We denote by $\mathbf{x} = (x, y)$, respectively, $\mathbf{x} = (x, y, z)$, the generic point in $\mathbb{R}^2$, respectively, in $\mathbb{R}^3$.

Consider the Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$, $s \geq 0$ with associated norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H_0^s(\Omega)}$. Let $L_0^2(\Omega)$ the space of functions in $L^2(\Omega)$ where the integral vanishes on Ω, $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support in Ω and the domain $H(\operatorname{div}, \Omega)$ of the divergence operator,

$$H(\operatorname{div}, \Omega) = \{\boldsymbol{\varphi} \in L^2(\Omega)^d; \operatorname{div} \boldsymbol{\varphi} \in L^2(\Omega)\},$$

associated with the norm

$$\|\boldsymbol{\varphi}\|_{H(\operatorname{div},\Omega)} = \left(\|\boldsymbol{\varphi}\|^2_{L^2(\Omega)^d} + \|\operatorname{div} \boldsymbol{\varphi}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

The normal trace operator $\mathbf{v} \to \mathbf{v}.\mathbf{n}$ is defined from $H(\operatorname{div}, \Omega)$ into $H^{-1/2}(\partial\Omega)$ such that, for a vector fields $\boldsymbol{\varphi} \in H(\operatorname{div}, \Omega)$ and a scalar function $\psi \in \mathcal{D}(\Omega)$ [2],

$$\int_\Omega \operatorname{div} \varphi(\mathbf{x}) \psi(\mathbf{x})\, d\mathbf{x} = -\int_\Omega \varphi(\mathbf{x}).\, \mathbf{grad}\, \psi(\mathbf{x})\, d\mathbf{x} + \int_{\partial\Omega} (\boldsymbol{\varphi}.\mathbf{n})(\tau) \psi(\tau)\, d\tau.$$

This leads us to introduce its kernel

$$H_0(\operatorname{div}, \Omega) = \{\boldsymbol{\varphi} \in H(\operatorname{div}, \Omega); \boldsymbol{\varphi}.\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

The problem (1) has the following variational formulation: For $\mathbf{f} \in (L^2(\Omega))^d$, find $\mathbf{u} \in H(\operatorname{div}, \Omega)$, $p \in L_0^2(\Omega)$ such that $\forall \mathbf{v} \in H_0(\operatorname{div}, \Omega)$ and $\forall q \in L_0^2(\Omega)$

$$\begin{aligned} &\mathbf{a}(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \\ &b(\mathbf{u}, q) = 0, \end{aligned} \tag{2}$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ scalar product,

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}).\mathbf{v}(\mathbf{x})\, d\mathbf{x} \quad \text{and} \quad b(\mathbf{v}, p) = -\int_\Omega \operatorname{div} \mathbf{v}(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x}.$$

Let $\mathbf{V}$ be the kernel of the bilinear form b defined by

$$\begin{aligned}\mathbf{V} &= \left\{\boldsymbol{\varphi} \in H_0(\mathrm{div}, \Omega); \forall q \in L_0^2(\Omega), \int_\Omega \mathrm{div}\, \boldsymbol{\varphi}(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x} = 0\right\} \\ &= \left\{\boldsymbol{\varphi} \in H_0(\mathrm{div}, \Omega); \mathrm{div}\, \boldsymbol{\varphi} = 0 \text{ in } \Omega\right\}.\end{aligned}$$

The norms $\|\cdot\|_{H(\mathrm{div},\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ are equivalent on $\mathbf{V}$ [15]. This yields the ellipticity of the bilinear form $\mathbf{a}(\cdot,\cdot)$ on $\mathbf{V}$: There exists a positive constant $\lambda > 0$; such that

$$\forall \boldsymbol{\varphi} \in \mathbf{V}, \quad \mathbf{a}(\boldsymbol{\varphi}, \boldsymbol{\varphi}) \geq \lambda \|\boldsymbol{\varphi}\|_{H(\mathrm{div},\Omega)}.$$

Moreover, the inf-sup condition on the bilinear form $b(\cdot,\cdot)$: There exists a positive constant $\beta > 0$; such that

$$\forall q \in L_0^2(\Omega), \qquad \sup_{\mathbf{w}\in H(\mathrm{div},\Omega)} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{H(\mathrm{div},\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}, \tag{3}$$

is obtained by taking $\mathbf{w} = \mathbf{grad}\, \boldsymbol{\varphi}$; where $\boldsymbol{\varphi}$ is solution of a Laplace equation of data q and Neumann homogeneous boundary conditions ([2], Chap. 1, Corr 2.4).

Using the saddle-point theorem, we conclude that, for $\mathbf{f} \in L^2(\Omega)^d$, problem (2) has a unique solution $(\mathbf{u}, p) \in H(\mathrm{div}, \Omega) \times L_0^2(\Omega)$, verifying the following stability condition:

$$\|\mathbf{u}\|_{L^2(\Omega)^d} + \beta \|p\|_{L^2(\Omega)} \leq 2\|\mathbf{f}\|_{L^2(\Omega)^d}.$$

Let $H(\mathbf{curl}, \Omega)$ the domain of the $\mathbf{curl}$ operator

$$H(\mathbf{curl}, \Omega) = \left\{\boldsymbol{\varphi} \in L^2(\Omega)^d, \mathbf{curl}\, \boldsymbol{\varphi} \in L^2(\Omega)^{\frac{d(d-1)}{2}}\right\}.$$

We know (see [16]) that $H_0(\mathrm{div}, \Omega) \cap H(\mathbf{curl}, \Omega)$ is continuously imbedded in $H^{1/2}(\Omega)^d$ in general and in $H^1(\Omega)^d$ if Ω is convex. Further results are known (see [17, 18]); when Ω is a polygonal domain, a function $\mathbf{u} \in H_0(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega)$ can be written as

$$\mathbf{u} = \mathbf{u}_R + \mathbf{grad}\, S, \tag{4}$$

where $\mathbf{u}_R \in H^1(\Omega)^d$ and S is a linear combination of singular functions. We recall that each singularity in the neighborhood of a corner of the polygon with aperture ω has the form

$$r^{\pi/\omega} \varphi(\theta),$$

where r is the distance to the singular corner, θ is the polar angle and φ belongs to $\mathcal{C}^\infty(]0, 2\pi[, \mathbb{R})$. Then, in general, any such function $\mathbf{u}$, which has the further property

$$\mathrm{div}\, \mathbf{u} \in H^s(\Omega) \quad \text{and} \quad \mathbf{curl}\, \mathbf{u} \in H^s(\Omega)^3,$$

admits the expansion (4) with $\mathbf{u}_R \in H^{s+1}(\Omega)^d$ for $0 < s < \frac{2\pi}{\omega} - 1$.

Let $\alpha \in]0,1]$ the penalty parameter. We consider the following penalized problem: Find $(\mathbf{u}^\alpha, p^\alpha) \in H_0(\mathrm{div},\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} &\forall \boldsymbol{\varphi} \in H_0(\mathrm{div},\Omega), \quad \mathbf{a}\big(\mathbf{u}^\alpha, \boldsymbol{\varphi}\big) + b\big(\boldsymbol{\varphi}, p^\alpha\big) = (\mathbf{f}, \boldsymbol{\varphi}), \\ &\forall q \in L_0^2(\Omega), \quad b(\mathbf{u}, q) = \alpha \int_\Omega p^\alpha(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{5}$$

By adapting the result proved on Stokes problem [2], we conclude the following result.

Proposition 1 *For* $\mathbf{f} \in (L^2(\Omega))^d$, *problem* (5) *has a unique solution* $(\mathbf{u}^\alpha, p^\alpha) \in H_0(\mathrm{div},\Omega) \times L_0^2(\Omega)$ *such that if* $(\mathbf{u}, p)$ *is solution to problem* (2), *we have the following estimation*:

$$\big\|\mathbf{u} - \mathbf{u}^\alpha\big\|_{L^2(\Omega)^d} + \big\|p - p^\alpha\big\|_{L^2(\Omega)} \leq C\alpha \|\mathbf{f}\|_{L^2(\Omega)^d}, \tag{6}$$

where C *is a constant independent of* α.

3 The penalized discrete problem

We introduce a partition of the domain Ω without overlapping,

$$\overline{\Omega} = \bigcup_{i=1}^{I} \Omega_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \varnothing, \quad 1 \leq i < j \leq I,$$

where Ω_i are rectangles if $d = 2$ and parallelepiped rectangles if $d = 3$.

We suppose that the decomposition is conform in the sense that the intersection of the two sub-domains $\overline{\Omega_i} \cap \overline{\Omega_j}$ for $i \neq j$, if it is not empty, is an entire edge or an entire face of the two sub-domains $\overline{\Omega_i}$ and $\overline{\Omega_j}$. We choose without restriction that the edges or faces of each sub-domain $\overline{\Omega_i}$ is parallel to the axis of the coordinate system.

Let $\mathbb{P}_{nm}(\Omega)$ the space of the restriction on Ω of the polynomials of degree n in the x directions and m in the y directions in dimension $d = 2$. $\mathbb{P}_{nms}(\Omega)$ is the space of the restriction on Ω of the polynomials of degree n in the x directions, m in the y directions and s in the z directions in dimension $d = 3$.

Let $N \geq 2$ an integer. We introduce the space of discrete velocity,

$$\mathbb{D}_N(\Omega) = \big\{\varphi_N \in H_0(\mathrm{div},\Omega); \varphi_N/_{\Omega_i} \in \mathbb{P}_{N,N-1}(\Omega) \times \mathbb{P}_{N-1,N}(\Omega)\big\}$$

if $d = 2$ or

$$\mathbb{D}_N(\Omega) = \big\{\varphi_N \in H_0(\mathrm{div},\Omega); \varphi_N/_{\Omega_i} \in \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega)\big\}$$

if $d = 3$ and the space of discrete pressure,

$$\mathbb{M}_N(\Omega) = \mathbb{P}_{N-1}(\Omega) \cap L_0^2(\Omega).$$

For this choice, $\mathbb{M}_N(\Omega)$ does not contain a spurious mode and the inf-sup constant on the bilinear form $b(\cdot,\cdot)$ does not depend on N [19].

To define the discrete problem, we remember the Gauss–Lobatto–Legendre quadrature formula on the reference interval $]-1,1[$:

Let $\xi_0 = -1$ and $\xi_N = 1$, there exists a unique set of nodes ξ_k; $1 \leq k \leq N-1$, and a unique set of weights ρ_k; $0 \leq k \leq N$, such that

$$\forall \varphi \in \mathbb{P}_{2N-1}\big(]-1,1[\big), \quad \int_1^1 \varphi(\mathbf{x})\,d\mathbf{x} = \sum_{k=0}^{N} \varphi(\xi_k)\rho_k. \tag{7}$$

The weights ρ_k are positif and we have the following property:

$$\forall \varphi_N \in \mathbb{P}_N\big(]-1,1[\big), \quad \|\varphi_N\|^2_{L^2(]-1,1[)} \leq \sum_{k=0}^{N} \varphi_N^2(\xi_k)\rho_k \leq 3\|\varphi_N\|^2_{L^2(]-1,1[)}. \tag{8}$$

Let (ξ_k^i, ξ_l^i), respectively $(\xi_k^i, \xi_l^i, \xi_r^i)$, the nodes in the sub-domain Ω_i deduced from (ξ_k, ξ_l), respectively (ξ_k, ξ_l, ξ_r), by bijection in the reference domain $]-1,1[^2$, respectively $]-1,1[^3$. The local discrete scalar product is defined by: For φ and ψ two continuous functions on $\overline{\Omega}_i$,

$$(\varphi,\psi)_{N_i} = \begin{cases} \frac{|\Omega|}{4} \sum_{k=0}^{N} \sum_{l=0}^{N} \varphi(\xi_k^i, \xi_l^i)\psi(\xi_k^i, \xi_l^i)\rho_k\rho_l & \text{if } d = 2, \\ \frac{|\Omega|}{8} \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{r=0}^{N} \varphi(\xi_k^i, \xi_l^i, \xi_r^i)\psi(\xi_k^i, \xi_l^i, \xi_r^i)\rho_k\rho_l\rho_r & \text{if } d = 3. \end{cases}$$

Then the discrete scalar product on Ω is

$$(\varphi,\psi)_N = \sum_{i=1}^{I} (\varphi,\psi)_{N_i}.$$

The penalized discrete problem is written: Find $(\mathbf{u}_N^\alpha, p_N^\alpha) \in \mathbb{D}_N(\Omega) \times \mathbb{M}_N(\Omega)$ such that

$$\begin{aligned} &\forall \mathbf{v}_N \in \mathbb{D}_N(\Omega), \quad \mathbf{a}_N\big(\mathbf{u}_N^\alpha, \mathbf{v}_N\big) + b\big(\mathbf{v}_N, p_N^\alpha\big) = (\mathbf{f}, \mathbf{v}_N)_N, \\ &\forall q_N \in \mathbb{M}_N(\Omega), \quad b_N\big(\mathbf{u}_N^\alpha, q_N\big) = \alpha\big(p_N^\alpha, q_N\big)_N, \end{aligned} \tag{9}$$

where the two bilinear forms $\mathbf{a}_N(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ are defined by

$$\mathbf{a}_N(\mathbf{u}_N, \mathbf{v}_N) = (\mathbf{u}_N, \mathbf{v}_N)_N \quad \text{and} \quad b_N(\mathbf{v}_N, q_N) = -\big(\operatorname{div}(\mathbf{v}_N), q_N\big)_N.$$

According to the exactness of the quadrature formulas on the space $\mathbb{P}_{2N-1}(\Omega)$, the discrete bilinear form $b_N(\cdot,\cdot)$ coincides with the continuous bilinear form $b(\cdot,\cdot)$.

We consider Π_N the orthogonal projection operator from the space $L^2(\Omega)$ into the space $\mathbb{M}_N$, defined with respect the scalar product $L^2(\Omega)$. We prove that the penalized problem (9) is equivalent to the following uncoupled problem (see [2], Chap. 1, Sect. 4.3):

Find $\mathbf{u}_N^\alpha \in \mathbb{D}_N(\Omega)$ and $p_N \in \mathbb{M}_N(\Omega)$ such that, for all $\mathbf{v}_N \in \mathbb{D}_N(\Omega)$,

$$\mathbf{a}_N\big(\mathbf{u}_N^\alpha, \mathbf{v}_N\big) + \frac{1}{\alpha}\big(\Pi_N\big(\operatorname{div} \mathbf{u}_N^\alpha\big), \Pi_N(\operatorname{div} \mathbf{v}_N)\big)_N = (\mathbf{f}, \mathbf{v}_N)_N, \tag{10}$$

$$p_N^\alpha = -\frac{1}{\alpha}\Pi_N\big(\operatorname{div} \mathbf{u}_N^\alpha\big). \tag{11}$$

Remark 1 The penalty method permits us to uncouple the problem (9). The only unknown in equation (10) is the velocity and then we deduce the value of the pressure from equation (11).

Proposition 2 *For a continuous function* $\mathbf{f}$ *on* $\bar{\Omega}$, *problem* (10)–(11) *has a unique solution* $(\mathbf{u}_N^\alpha, p_N^\alpha) \in \mathbb{D}_N(\Omega) \times \mathbb{M}_N(\Omega)$.

Proof For $(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_N) \in \mathbb{D}_N(\Omega) \times \mathbb{D}_N(\Omega)$, we consider

$$\hat{\mathbf{a}}(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_N) = (\boldsymbol{\varphi}_N, \boldsymbol{\psi}_N)_N + \frac{1}{\alpha}\big(\Pi_N(\operatorname{div}\boldsymbol{\varphi}_N), \Pi_N(\operatorname{div}\boldsymbol{\psi}_N)\big)_N.$$

We deduce, by the triangular inequality, the continuity of the operator Π_N and the continuity of the operator div on the space $\mathbb{D}_N(\Omega)$, that the bilinear form $\hat{\mathbf{a}}(\cdot,\cdot)$ is continuous on $\mathbb{D}_N(\Omega) \times \mathbb{D}_N(\Omega)$.

Using that $\hat{\mathbf{a}}(\varphi_N, \varphi_N) \geq (\varphi_N, \varphi_N)_N$ and property (8), we deduce that the bilinear form $\hat{\mathbf{a}}(\cdot,\cdot)$ is elliptic.

The Lax–Milgram theorem permits one to conclude that problem (10)–(11) has a unique solution $(\mathbf{u}_N^\alpha, p_N^\alpha) \in \mathbb{D}_N(\Omega) \times \mathbb{M}_N(\Omega)$. □

We know that the discrete bilinear form $b_N(\cdot,\cdot)$ verifies the following inf-sup condition: For any $q_N \in \mathbb{M}_N(\Omega)$

$$\sup_{\mathbf{v}_N \in \mathbb{D}_N(\Omega)} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_{H(\operatorname{div},\Omega)}} \geq \gamma \|q_N\|_{L^2(\Omega)}, \tag{12}$$

where γ is a positive constant independent of N and of the penalty parameter α (see [19, 20]). We obtain the following a priori error estimation.

Proposition 3 *Suppose that the data function* $\mathbf{f}$ *belongs to the space* $H^\mu(\Omega)^d$, $\mu \geq \frac{d}{2}$ *and that the solutions* $(\mathbf{u}, p)$ *of problem* (2) *and* $(\mathbf{u}^\alpha, p^\alpha)$ *of problem* (5) *belongs to* $H^s(\Omega)^d \times H^s(\Omega)$, $s \geq 0$, *then the error between the solution* $(\mathbf{u}, p)$ *of problem* (2) *and* $(\mathbf{u}_N^\alpha, p_N^\alpha)$ *solution of problem* (9) *is*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_N^\alpha\|_{L^2(\omega)^d} + \gamma \|p - p_N^\alpha\|_{L^2(\Omega)} \\ &\quad \leq C\alpha\big(N^{-s}\big(\|\mathbf{u}\|_{H^s(\Omega)^d} + \|p\|_{H^s(\Omega)}\big) + N^{-\mu}\|\mathbf{f}\|_{H^\mu(\Omega)^d}\big), \end{aligned} \tag{13}$$

where C *is a positive constant independent of* N *and* α.

Proof Using the triangular inequality we have

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d} \leq \|\mathbf{u} - \mathbf{u}^\alpha\|_{L^2(\Omega)^d} + \|\mathbf{u}^\alpha - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d}, \\ &\|p - p_N^\alpha\|_{L^2(\Omega)} \leq \|p - p^\alpha\|_{L^2(\Omega)} + \|p^\alpha - p_N^\alpha\|_{L^2(\Omega)}. \end{aligned} \tag{14}$$

Using problems (5) and (9), we conclude that

$$\mathbf{a}\big(\mathbf{u}^\alpha - \mathbf{u}_N^\alpha, \mathbf{v}_N\big) + b\big(p^\alpha - p_N^\alpha, \mathbf{v}_N\big) = 0 \tag{15}$$

and

$$b\big(p^\alpha - p_N^\alpha, q_N\big) = \alpha \int_\Omega p_N^\alpha(\mathbf{x}) q_N(\mathbf{x})\, d\mathbf{x}. \tag{16}$$

Based on the inf-sub condition (3) and the continuity of the bilinear form $\mathbf{a}(\cdot,\cdot)$, there exists a positive constant C, independent of N and α such that

$$\beta\|p^\alpha - p_N^\alpha\|_{L^2(\Omega)} \le \sup_{\mathbf{v}_N\in\mathbb{D}_N(\Omega)} \frac{b(p^\alpha - p_N^\alpha, \mathbf{v}_N)}{\|\mathbf{v}_N\|_{L^2(\Omega)^d}} \le C\|\mathbf{u}^\alpha - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d}.$$

Thus

$$\|p^\alpha - p_N^\alpha\|_{L^2(\Omega)} \le C\beta^{-1}\|\mathbf{u}^\alpha - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d}. \tag{17}$$

If we choose $\mathbf{v}_N = \mathbf{u}^\alpha - \mathbf{u}_N^\alpha$ and $q_N = p^\alpha - p_N^\alpha$ in (15) and (16), we have

$$\mathbf{a}\big(\mathbf{u}^\alpha - \mathbf{u}_N^\alpha, \mathbf{u}^\alpha - \mathbf{u}_N^\alpha\big) \le -\alpha\int_\Omega p^\alpha(\mathbf{x})\big(p^\alpha - p_N^\alpha\big)(\mathbf{x})\,d\mathbf{x}.$$

Using (17), we conclude that

$$\mathbf{a}\big(\mathbf{u}^\alpha - \mathbf{u}_N^\alpha, \mathbf{u}^\alpha - \mathbf{u}_N^\alpha\big) \le \|p^\alpha\|_{L^2(\Omega)}\|\mathbf{u}^\alpha - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d}. \tag{18}$$

Then, by (16),

$$\operatorname{div}\big(\mathbf{u}^\alpha - \mathbf{u}_N^\alpha\big) = \alpha p_N^\alpha \quad \text{in } L_0^2(\Omega). \tag{19}$$

Using (18) and (19), we find that

$$\|\mathbf{u}^\alpha - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d} \le \alpha C\|p^\alpha\|_{L^2(\Omega)}. \tag{20}$$

By combining the inequalities (14), (20), (17) and (6) we conclude (13), using the standard results of spectral approximation [12]. □

4 A posteriori error analysis

We define an error indicator

$$i^\alpha = \alpha\|p_N^\alpha\|_{L^2(\Omega)} \tag{21}$$

which depends on the discrete pressure, so it is easily to calculate.

Theorem 1 *The error between the solutions* $(\mathbf{u}, p)$ *of problem* (2) *and* $(\mathbf{u}^\alpha, p^\alpha)$ *of problem* (9) *is*

$$\|\mathbf{u} - \mathbf{u}_N^\alpha\|_{L^2(\Omega)^d} + \|p - p_N^\alpha\|_{L^2(\Omega)} \le C\big(i^\alpha + \alpha\|p^\alpha - p_N^\alpha\|_{L^2(\Omega)}\big). \tag{22}$$

The estimation of the error indicator is

$$i^\alpha \le \big(\|\mathbf{u} - \mathbf{u}_N^\alpha\|_{H(\mathrm{div},\Omega)} + \alpha\|p - p_N^\alpha\|_{L^2(\Omega)}\big), \tag{23}$$

C is a positive constant independent of N and α.

Proof Making the difference between problems (2) and (9), we find, for all $\mathbf{v} \in H(\mathrm{div}, \Omega)$ and for all $q \in L^2(\Omega)$,

$$\begin{aligned} &\mathbf{a}(\mathbf{u} - \mathbf{u}^\alpha, \mathbf{v}) + b(\mathbf{v}, p - p^\alpha) = 0, \\ &b(\mathbf{u} - \mathbf{u}^\alpha, q) = -\alpha \int_\Omega p^\alpha(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{24}$$

Using the arguments presented in ([2], Chap. 1, Theorem 4.3) combined with the ellipticity of the bilinear form $\mathbf{a}(\cdot,\cdot)$ and the inf-sub condition (3), we obtain

$$\|\mathbf{u} - \mathbf{u}_N^\alpha\|_{H(\mathrm{div},\Omega)} + \|p - p_N^\alpha\|_{L^2(\Omega)} \le C\alpha \|p^\alpha\|_{L^2(\Omega)}. \tag{25}$$

By the triangular inequality

$$\|p^\alpha\|_{L^2(\Omega)} \le \|p^\alpha - p_N^\alpha\|_{L^2(\Omega)} + \|p_N^\alpha\|_{L^2(\Omega)}, \tag{26}$$

we conclude the estimation (22) with $i^\alpha = \alpha \|p_N^\alpha\|_{L^2(\Omega)}$.

Taking $q = p^\alpha$ in the second equation of (24) yields

$$\alpha \|p^\alpha\|_{L^2(\Omega)} \le \|\mathbf{u} - \mathbf{u}_N^\alpha\|_{H(\mathrm{div},\Omega)}.$$

Combining this relation with (26), we find the result (23). □

Let ϖ_i, $1 \le i \le I$, the family of error indicators which are related to the spectral element discretization

$$\varpi_i = N^{-1} \|I_N(\mathbf{f}) + \nu \mathbf{u} + \mathbf{grad}\, p_N^\alpha\|_{L^2(\Omega_i)^d} - \sum_{l=1}^{L(l)} N^{-\frac{1}{2}} \|[p_N^\alpha.\mathbf{n}]_{il}\|_{L^2(\Gamma_{il})} + \|\mathrm{div}(\mathbf{u}_N^\alpha)\|_{L^2(\Omega_i)}. \tag{27}$$

For each $1 \le i \le I$, Γ_{il}, $1 \le l \le L(l)$, are the edges in dimension $d = 2$ or the faces in dimension $d = 3$ of the sub-domain Ω_i that are not included on the boundary $\partial\Omega$ and $[p_N^\alpha.\mathbf{n}]_{il}$ represents the jump through each Γ_{il}. We denote by I_N the Lagrange interpolating operator on the Gauss–Lobatto nodes.

Theorem 2 *The a posteriori error estimate between the solutions* $(\mathbf{u}^\alpha, p^\alpha)$ *of problem* (5) *and* $(\mathbf{u}_N^\alpha, p_N^\alpha)$ *of the problem* (9) *is*

$$\|\mathbf{u} - \mathbf{u}_N^\alpha\|_{H(\mathrm{div},\Omega)} + \|p - p_N^\alpha\|_{L^2(\Omega)} \le C\left(i^\alpha + \mu\left(\sum_1^I \varpi_i\right) + \|\mathbf{f} - I_N(\mathbf{f})\|_{L^2(\Omega)}^d\right), \tag{28}$$

where C *is a positive constant independent of* N *and* α, μ *is equal to*

- 1 *if* $d = 2$ *or* Ω *is convex,*
- $N^{\frac{1}{2}}$ *if* $d = 3$ *and* Ω *not convex.*

Proof To find (28), we proceed as in ([21], Sect. 4), ([14], Sect. 3.3) and ([11], Sect. 3).

Let $\mathbf{U} = (\mathbf{u}, p)$ and $\mathbf{V} = (\mathbf{v}, q)$. We define the bilinear form

$$\mathcal{A}_\alpha(\mathbf{U},\mathbf{V}) = \mathbf{a}(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - \alpha \int_\Omega p(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}. \tag{29}$$

The bilinear form $\mathcal{A}_\alpha(\cdot,\cdot)$ is continuous on the space $\mathcal{K}(\Omega) \times \mathcal{K}(\Omega)$ where

$$\mathcal{K}(\Omega) = L^2(\Omega)^d \times L^2_0(\Omega).$$

This space is equipped with the norm

$$\big\|(\mathbf{u},p)\big\|_{\mathcal{K}(\Omega)} = \big(\|\mathbf{u}\|^2_{L^2(\Omega)^d} + \|p\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}.$$

Thanks to ([14], Lemma 3.5), the coercivity of the bilinear form $\mathbf{a}(\cdot,\cdot)$ and the inf-sup condition of the bilinear form $b(\cdot,\cdot)$, we prove an inf-sup condition on the bilinear form $\mathcal{A}_\alpha(\cdot,\cdot)$ such that there exists a constant δ_* positive independent of α:

$$\sup_{\mathbf{V}\in\mathcal{K}(\Omega)} \frac{\mathcal{A}_\alpha(\mathbf{U},\mathbf{V})}{\|\mathbf{V}\|_{\mathcal{K}(\Omega)}} \geq \delta_* \|\mathbf{U}\|_{\mathcal{K}(\Omega)}. \tag{30}$$

We need to evaluate the residual term $\mathcal{A}_\alpha(\mathbf{U}^\alpha - \mathbf{U}^\alpha_N, \mathbf{V})$, where $\mathbf{U}^\alpha = (\mathbf{u}^\alpha, p^\alpha)$ and $\mathbf{U}^\alpha_N = (\mathbf{u}^\alpha_N, p^\alpha_N)$.

According to the exactness of the quadrature formula (7) applied in the problem (9), we obtain, for $\mathbf{V}_{N-1} = (\mathbf{v}_{N-1}, 0)$, $\mathbf{v}_{N-1} \in \mathbb{D}_{N-1}$,

$$\mathcal{A}_\alpha\big(\mathbf{U}^\alpha_N, \mathbf{V}_{N-1}\big) = \int_\Omega I_N(\mathbf{f})(\mathbf{x}).\mathbf{v}_{N-1}(\mathbf{x})\,d\mathbf{x}. \tag{31}$$

Using problems (5) and (31), we have

$$\mathcal{A}_\alpha\big(\mathbf{U}^\alpha - \mathbf{U}^\alpha_N, \mathbf{V}\big) = \mathcal{A}_\alpha\big(\mathbf{U}^\alpha - \mathbf{U}^\alpha_N, \mathbf{V} - \mathbf{V}_{N-1}\big) + \int_\Omega \big(\mathbf{f} - I_N(\mathbf{f})\big)(\mathbf{x}).\mathbf{v}_{N-1}(\mathbf{x})\,d\mathbf{x},$$

and so

$$\begin{aligned}\mathcal{A}_\alpha\big(\mathbf{U}^\alpha - \mathbf{U}^\alpha_N, \mathbf{V}\big) &= \int_\Omega I_N(\mathbf{f})(\mathbf{x}).(\mathbf{v} - \mathbf{v}_{N-1})(\mathbf{x})\,d\mathbf{x} - \mathcal{A}_\alpha\big(\mathbf{U}^\alpha_N, \mathbf{V} - \mathbf{V}_{N-1}\big) \\ &\quad + \int_\Omega \big(\mathbf{f} - I_N(\mathbf{f})\big)(\mathbf{x}).\mathbf{v}(\mathbf{x})\,d\mathbf{x}. \end{aligned} \tag{32}$$

Applying an integration by part on each sub-domain Ω_i, we conclude that

$$\begin{aligned}&\int_\Omega I_N(\mathbf{f})(\mathbf{x}).(\mathbf{v} - \mathbf{v}_{N-1})(\mathbf{x})\,d\mathbf{x} - \mathcal{A}_\alpha\big(\mathbf{U}^\alpha_N, \mathbf{V} - \mathbf{V}_{N-1}\big) \\ &\quad = \sum_{i=1}^{I}\bigg(\int_{\Omega_i} \big(I_N(\mathbf{f}) + \nu\mathbf{u}^\alpha_n - \mathbf{grad}\,p^\alpha_N\big)(\mathbf{x}).(\mathbf{v} - \mathbf{v}_{N-1})(\mathbf{x})\,d\mathbf{x} \\ &\qquad + \int_{\partial\Omega_i} p^\alpha_N(\zeta).(\mathbf{v} - \mathbf{v}_{N-1})(\zeta)\,d\zeta \\ &\qquad + \int_{\Omega_i} \operatorname{div}\mathbf{u}^\alpha_N q(\mathbf{x})\,d\mathbf{x} + \alpha \int_{\Omega_i} p^\alpha_N(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}\bigg). \end{aligned} \tag{33}$$

We define $\mathcal{P}_N$ to be the orthogonal projection operator from the space $H_0(\mathrm{div},\Omega)$ into the space $\mathbb{D}_N$ associated to the scalar product of the space $H_0(\mathrm{div},\Omega)$. So for any $\mathbf{v}\in H_0(\mathrm{div},\Omega)$, we have

$$\|\mathbf{v}-\mathcal{P}_N(\mathbf{v})\|_{L^2(\Omega)}=\sup_{\kappa\in L^2(\Omega)}\frac{\int_\Omega(\mathbf{v}-\mathcal{P}_N(\mathbf{v}))(\mathbf{x})\kappa(\mathbf{x})\,d\mathbf{x}}{\|\kappa\|_{L^2(\Omega)}}.$$

For $\kappa\in L^2(\Omega)$, the problem

$$\begin{aligned}&-\Delta\psi=\kappa\quad\text{in }\Omega,\\&\psi=0\quad\text{on }\partial\Omega,\end{aligned}$$

has a unique solution $\psi\in H_0^1(\Omega)\subset H_0(\mathrm{div},\Omega)$, then

$$\begin{aligned}\int_\Omega\big(\mathbf{v}-\mathcal{P}_N(\mathbf{v})\big)(\mathbf{x})\kappa(\mathbf{x})\,d\mathbf{x}&=\int_\Omega\nabla\big(\mathbf{v}-\mathcal{P}_N(\mathbf{v})\big)(\mathbf{x})\nabla\kappa(\mathbf{x})\,d\mathbf{x}\\&=\int_\Omega\nabla(\mathbf{v})(\mathbf{x})\nabla\big(\kappa(\mathbf{x})-\mathcal{P}_N(\kappa)\big)\,d\mathbf{x}.\end{aligned}$$

Thus, we conclude that

$$\int_\Omega\big(\mathbf{v}-\mathcal{P}_N(\mathbf{v})\big)(\mathbf{x})\kappa(\mathbf{x})\,d\mathbf{x}\le\|\mathbf{v}\|_{H(\mathrm{div},\Omega)}\big\|\kappa(\mathbf{x})-\mathcal{P}_N(\kappa)\big\|_{H(\mathrm{div},\Omega)}.$$

We deduce the following inequality from the standard interpolation results [22]:

$$\big\|\kappa(\mathbf{x})-\mathcal{P}_N(\kappa)\big\|_{H(\mathrm{div},\Omega)}\le CN^{-s}\|\kappa\|_{H^s(\Omega)}.\tag{34}$$

We consider the following estimation (see [23]):

For any $\phi\in H_0^1(\Omega)\subset H_0(\mathrm{div},\Omega)$ and any sub-domain Ω_i, $1\le i\le I$,

$$\big\|\phi(\mathbf{x})-\mathcal{P}_N(\phi)\big\|_{L^2(\partial\Omega_i)}\le CN^{-\frac{1}{2}}\|\phi\|_{H(\mathrm{div},\Omega_i)}.\tag{35}$$

We conclude the a posteriori error estimation (28) applying (30), (32), (33), the Cauchy–Schwarz inequality and (34) combined with (35). □

Remark 2 We remark that in dimension $d=2$ and if Ω is convex, the a posteriori error estimation (28) is fully optimal and leads to an explicit upper bound for the error. However, the inverse estimation (the estimation of the error indicator in function of the error) is not optimal (see [23], Theorem 2.9) and we will not present it because we are not interested to the adaptability with respect N.

4.1 Penalty adaptation algorithm

We describe in this section the used strategy for the penalty adaptation in order to optimize the penalty parameter. We suppose that the data function $\mathbf{f}$ is regular. Let γ a be fixed real number and α^0 is an initial value of α:

- For $m=1,\ldots$
- For a value α^m of α

– Compute the solution $(\mathbf{u}_N^{\alpha^m}, p_N^{\alpha^m})$ of problem (10)–(11)
– Compute the associated error indicator i^{α^m} given in (21)
– Compute

$$\varpi_N = \left(\sum_{i=1}^{I} \varpi_i^2 \right)^{\frac{1}{2}},$$

where ϖ_i is defined by (27)
– If $\gamma i^{\alpha^m} \leq \varpi_N$, we obtain the optimal value α^m
– Otherwise, we choose

$$\alpha^{m+1} = \frac{\alpha^m \varpi_N}{i^{\alpha^m}},$$

and we reiterate.

5 Conclusion

This work concerns the use of the penalty technique to solve Darcy's equations discretized by the spectral elements method. This technique permits to uncouple the two unknowns, the velocity and the pressure. The construction of the error indicators, using an a posteriori error analysis is presented. This made it possible to find an optimal penalty parameter which will reduce the computational cost. The numerical validation of this result will be the subject of a forthcoming work.

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