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Existence of solutions for some two-point fractional boundary value problems under barrier strip conditions

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Abstract

In this paper, we are dedicated to researching the boundary value problems (BVPs) for equation $D^\alpha x(t) = f(t,x(t),D^{\alpha-1}x(t))$, with the boundary value conditions to be either: $x(0) = A$, $D^{\alpha-1}x(1) = B$ or $D^{\alpha-1}x(0) = A$, $x(1) = B$. Let the nonlinear term f satisfy some sign conditions, then by making use of the Leray–Schauder nonlinear alternative, some existence results are obtained. In the end, an example is given to verify the main results.

Keywords: Barrier strips; Conformable fractional derivative; Boundary value problems

1 Introduction

In the last several years, because the fractional calculus theory has been extensively used in non-Newtonian fluid mechanics, diffusion and transportation theory, engineering, biology, image processing, and other fields [9, 12, 15–19, 21, 24, 25, 35, 36, 38, 41–47], the fractional differential equations (FDEs) have been researched with different methods by many scholars. Many interesting results have been obtained [1–8, 10, 11, 13, 14, 17, 23, 26, 30–32, 35, 36, 40, 48–51].

In 1994, Kelevedjiev [27] investigated the nonlinear second order two-point BVP as follows by the use of the barrier strips argument and the topological transversality theorem [22]:

$$x''(t) = f\big(t,x(t),x'(t)\big), \quad t \in [0,1], \tag{1}$$

$$x(0) = A, \qquad x'(1) = B, \tag{2}$$

and got the existence results of solutions.

After that, the barrier strips technique was used by many researchers to study integer-order BVPs and IVPs (initial value problems). For instance, by making use of the barrier strips technique, the existence results for integer p-Laplacian BVP and first order IVP have been obtained by Kelevedjiev and Tersian, see [29] and [28]. Gao and Ma et al. generalized the idea to research the solvability for other integer BVPs such as second order three-point

BVP [33], two-point BVP on time scales [34], difference equations BVP with p-Laplacian [20]. But as far as we know, the idea was hardly used to solve fractional BVPs at that time.

Recently, Khalil et al. [30] gave the definitions of conformable fractional derivative, which have many of the basic properties of integer derivatives. These good properties are conducive for scholars to study the BVPs with conformable fractional derivative. Motivated and inspired by the above papers. In 2017, He et al. [23] generalized the idea to research the fractional BVP as follows:

$$D^{\alpha}x(t) = f\bigl(t, x(t), D^{\alpha-1}x(t)\bigr), \tag{3}$$

with the boundary value conditions to be either

$$x(0) = A, \qquad D^{\alpha-1}x(1) = B, \tag{4}$$

or

$$D^{\alpha-1}x(0) = A, \qquad x(1) = B, \tag{5}$$

where D^{α} is the standard conformable fractional derivative, $\alpha \in (1, 2]$ is a real number, and $f \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$.

Almost at the same time, Song et al. [39] considered the BVP for fractional equation (3) with inhomogeneous Dirichlet boundary conditions. By making use of the barrier strips technique and the fixed-point index theory, they acquired the existence results for the fractional Dirichlet BVP.

In this paper, we are dedicated to researching BVPs (3), (4) and (3), (5). Let the nonlinear term f satisfy certain sign conditions at the origin. Then, by making use of the Leray–Schauder nonlinear alternative [34] together with the barrier strips technique, not only can we get the existence results for BVPs (3), (4) and (3), (5), but also weaken the restrictions imposed on the nonlinear term f in Theorem 11 and Theorem 12 of [23].

The paper is laid out as follows. In Section 2, we present some necessary notions and preliminaries, which play an essential role in our proofs. In Section 3, by applying the technique of barrier strips and the Leray–Schauder nonlinear alternative, our main results are given and proved. Finally, an example is given to verify the main results obtained.

2 Preliminaries and lemmas

We recall some notions and lemmas in this section.

Definition 2.1 ([26]) Let u be n-order differentiable at $t > 0$. The fractional derivative of order $\alpha \in (n, n+1]$ of a function $u : [0, \infty) \to \mathbb{R}$ is defined as

$$D^{\alpha}u(t) = \lim_{\epsilon \to 0} \frac{u^{(n)}(t + \epsilon t^{n+1-\alpha}) - u^{(n)}(t)}{\epsilon},$$

provided the limits of the right-hand side exist.

Lemma 2.1 ([26]) *Let $t > 0$. Function $u(t)$ is α-order differentiable if and only if u is $(n+1)$-order differentiable. Furthermore, the following relation holds:*

$$D^{\alpha}u(t) = t^{n+1-\alpha}u^{(n+1)}(t).$$

Lemma 2.2 ([30]) *Suppose that* $a \geq 0$ *and* $f : [a, b] \to \mathbb{R}$ *satisfies the following conditions*:

(i) f *is continuous on* $[a, b]$,

(ii) f *is* α*-order differentiable on* (a, b).

Then there exists $e \in (a, b)$ *such that the following relation holds*:

$$f(b) - f(a) = D^\alpha f(e) \frac{b^\alpha - a^\alpha}{\alpha}.$$

Let

$$\begin{aligned} C^\alpha[0,1] &= \bigl\{ u \mid u(t) = J^\alpha_{0+} x(t) + C_n t^n + \cdots + C_1 t + C_0, \\ &\qquad C_i \in \mathbb{R}, i = 0, 1, \ldots, n, x \in C[0,1] \bigr\}, \\ \|u\|_\alpha &= \bigl\| D^\alpha u \bigr\|_0 + \bigl\| D^{\alpha-1} u \bigr\|_0 + \cdots + \bigl\| D^{\alpha-n} u \bigr\|_0 + \|u\|_0, \end{aligned}$$

where $\|u\|_0 = \max_{t \in [0,1]} |u(t)|$.

Lemma 2.3 ([23]) $(C^\alpha[0,1], \| \cdot \|_\alpha)$ *is a Banach space*.

The next theorem is Leray–Schauder nonlinear alternative, which is crucial in our proofs.

Theorem 2.1 ([37]) *Suppose that* U *is a relatively open subset of a convex set* K *in Banach space* E. *Assume that* $N : \overline{U} \to K$ *is a compact map and* $p \in U$. *Then either*

(i) N *has a fixed point in* $\overline{U}$; *or*

(ii) *there are* $\lambda \in (0, 1)$ *and* $u \in \partial U$ *such that* $u = \lambda N u + (1 - \lambda) p$.

Let $C^\alpha_{B_0}[0,1]$ be the subspace of $C^\alpha[0,1]$ such that boundary condition (4) is satisfied. Consider the BVPs:

$$D^\alpha x(t) = \lambda f\bigl(t, x(t), D^{\alpha-1} x(t)\bigr), \quad t \in (0,1), \tag{6}$$

$$x(0) = A, \qquad D^{\alpha-1} x(1) = B, \tag{7}$$

where $\lambda \in (0, 1)$ is a real number. Define $L : C^\alpha_{B_0}[0,1] \to C[0,1]$ by $Lx = D^\alpha x$. Obviously, L is one-one mapping. So, the following theorem can be easily obtained by using the nonlinear alternative theorem.

Theorem 2.2 *Suppose that* U *is an open and bounded neighborhood of* $0 \in C^{\alpha-1}[0,1]$ *and problem* (6), (7) *has no solutions in* ∂U *for* $0 < \lambda < 1$. *Then the problem*

$$D^\alpha x(t) = f\bigl(t, x(t), D^{\alpha-1} x(t)\bigr), \quad t \in (0,1), \tag{8}$$

$$x(0) = A, \qquad D^{\alpha-1} x(1) = B \tag{9}$$

has at least one solution in $\overline{U}$.

Therefore, our analysis is simplified to constructing a set U that is open and bounded such that BVP (6), (7) has no solutions in ∂U.

3 Existence results

Theorem 3.1 *Suppose that* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Let* G_1, G_2 *be two constants such that* $G_2 < B < G_1$ *and the following conditions are fulfilled*:

(H1) $f(t,x,G_1) \geq 0$ *for* $(t,x) \in [0,1] \times [-G,G]$;

(H2) $f(t,x,G_2) \leq 0$ *for* $(t,x) \in [0,1] \times [-G,G]$,

where $G > |A| + |\frac{1}{\alpha-1}| \cdot \max\{|G_2|,|G_1|\}$. *Then BVP* (3), (4) *has at least one solution*.

Proof According to the Tietze–Urysohn lemma, we can find a continuous function $\triangle$: $R^2 \to [-1,1]$ such that

$$\triangle(x,G_1) = 1, \quad x \in [-G,G], \tag{10}$$

and

$$\triangle(x,G_2) = -1, \quad x \in [-G,G]. \tag{11}$$

For $n \geq 1$, set $f_n(t,x,y) = f(t,x,y) + (1/n)\triangle(x,y)$, then

$$f_n(t,x,G_1) > 0, \quad x \in [-G,G], \tag{12}$$

$$f_n(t,x,G_2) < 0, \quad x \in [-G,G]. \tag{13}$$

Consider the BVPs

$$D^\alpha x(t) = f_n\bigl(t,x(t),D^{\alpha-1}x(t)\bigr), \tag{14}$$

$$x(0) = A, \qquad D^{\alpha-1}x(1) = B. \tag{15}$$

If we can prove that (14), (15) has a solution x_n such that

$$-G \leq x_n \leq G \quad \text{and} \quad G_2 \leq D^{\alpha-1}x_n \leq G_1 \tag{16}$$

hold for all $n \in N$, then by combining (14), (15), (16) and Arzela–Ascoli theorem, the sequence $\{x_n\}$ has a subsequence which converges in $C^\alpha[0,1]$-topology to a solution x_0 for BVP (3), (4).

The set U is defined by

$$U = \bigl\{v \in C^{\alpha-1}[0,1] \mid -G < v < G, G_2 < D^{\alpha-1}v < G_1\bigr\}. \tag{17}$$

In order to prove that (14), (15) has a solution x_n such that (16) holds, we only need to demonstrate, according to Theorem 2.1, that if $x \in C_{B_0}^{\alpha-1}[0,1]$ satisfies

$$-G \leq x \leq G \quad \text{and} \quad G_2 \leq D^{\alpha-1}x \leq G_1, \tag{18}$$

and

$$D^\alpha x(t) = \lambda f_n\bigl(t,x(t),D^{\alpha-1}x(t)\bigr), \tag{19}$$

for some $\lambda \in (0,1)$, then $x \in U$, i.e.,

$$-G < x < G \quad \text{and} \quad G_2 < D^{\alpha-1}x < G_1. \tag{20}$$

Let $x \in C_{B_0}^{\alpha-1}[0,1]$ satisfy (18), (19) for some $\lambda \in [0,1]$. By Lemma 2.1, there exists $d \in (0,t)$ such that the following relation holds:

$$x(t) - x(0) = D^{\alpha-1}x(d) \cdot \frac{t^{\alpha-1}}{\alpha - 1}, \quad d \in (0,t). \tag{21}$$

Then, by the inequality $G_2 \leq D^{\alpha-1}x \leq G_1$, there holds

$$\begin{aligned} |x(t)| &\leq \left|\frac{t^{\alpha-1}}{\alpha-1}\right| \cdot \left|D^{\alpha-1}x(d)\right| + |A| \\ &\leq \left|\frac{t^{\alpha-1}}{\alpha-1}\right| \cdot \max\{|G_2|, |G_1|\} + |A| \\ &\leq \left|\frac{1}{\alpha-1}\right| \cdot \max\{|G_2|, |G_1|\} + |A| \\ &< G. \end{aligned} \tag{22}$$

Relation (22) together with (12) and (13) implies that

$$f_n(t, x(t), G_1) > 0, \quad t \in [0,1]; \tag{23}$$

$$f_n(t, x(t), G_2) < 0, \quad t \in [0,1]. \tag{24}$$

Suppose that $D^{\alpha-1}(t_0) = G_1$ for some $t_0 \in [0,1]$. We have $t_0 < 1$ since $D^{\alpha-1}x(1) = B$. Hence $D^{\alpha}x(t_0) \leq 0$ because $D^{\alpha-1}x(t)$ attains its maximum at t_0. However, by (23) and (19), there is

$$\begin{aligned} D^{\alpha}x(t_0) &= \lambda f_n(t_0, x(t_0), D^{\alpha-1}x(t_0)) \\ &= \lambda f_n(t_0, x(t_0), G_1) \\ &> 0. \end{aligned} \tag{25}$$

This contradiction shows that $D^{\alpha-1}x(t) < G_1$. Analogously $D^{\alpha-1}x(t) > G_2$. Thus

$$G_2 < D^{\alpha-1}x < G_1, \quad t \in [0,1]. \tag{26}$$

The theorem is proven. □

Remark 3.1 Theorem 3.1 is a generalization of Theorem 11 in literature [22]. In [22], the conditions imposed on $f(t,x,p)$ are local to the variables t and p and global on x; however, in our Theorem 3.1, the variable x is also localized.

The following theorem can be obtained in a similar way.

Theorem 3.2 *Suppose that* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Let* G_3, G_4 *be two constants such that* $G_4 < A < G_3$ *and the following conditions are fulfilled*:

(H3) $f(t,x,G_3) \leq 0$ *for* $(t,x) \in [0,1] \times [-G,G]$;

(H4) $f(t,x,G_4) \geq 0$ *for* $(t,x) \in [0,1] \times [-G,G]$,

where $G > |B| + |\frac{1}{\alpha-1}| \cdot \max\{|G_4|,|G_3|\}$. *Then BVP* (3), (5) *has at least one solution*.

Remark 3.2 By comparison with the above Theorem 3.1 and Theorem 12 of [22], we can know that Theorem 3.2 is not only the "dual" of Theorem 3.1. At the same time, Theorem 3.2 is also a generalization of Theorem 12 of [22].

4 Example

Example 4.1 Consider the fractional BVP

$$D^{\frac{3}{2}}x(t) = x(t) + D^{\frac{1}{2}}x(t) + \frac{1}{2}\left[D^{\frac{1}{2}}x(t)\right]^2 + \left[D^{\frac{1}{2}}x(t)\right]^3 - 1, \tag{27}$$

$$x(0) = -1, \qquad D^{\frac{1}{2}}x(1) = 0. \tag{28}$$

Choose $G = 6$, $G_1 = 2$, $G_2 = -2$, then

$$f(t,x,G_1) \geq 0, \quad \text{for } (t,x) \in [0,1] \times [-6,6];$$

$$f(t,x,G_2) \leq 0, \quad \text{for } (t,x) \in [0,1] \times [-6,6].$$

By the use of Theorem 3.1, fractional BVP (27), (28) has at least one solution in $C^{\frac{3}{2}}[0,1]$.

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