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New general decay results for a viscoelastic plate equation with a logarithmic nonlinearity

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Abstract

In this paper, we investigate the stability of the solutions of a viscoelastic plate equation with a logarithmic nonlinearity. We assume that the relaxation function g satisfies the minimal condition

$$g'(t) \le -\xi(t)G(g(t)),$$

where ξ and G satisfy some properties. With this very general assumption on the behavior of g, we establish explicit and general energy decay results from which we can recover the exponential and polynomial rates when $G(s) = s^p$ and p covers the full admissible range $[1, 2)$. Our new results substantially improve and generalize several earlier related results in the literature such as Gorka (Acta Phys. Pol. 40:59–66, 2009), Hiramatsu et al. (J. Cosmol. Astropart. Phys. 2010(06):008, 2010), Han and Wang (Acta Appl. Math. 110(1):195–207, 2010), Messaoudi and Al-Khulaifi (Appl. Math. Lett. 66:16–22, 2017), Mustafa (Math. Methods Appl. Sci. 41(1):192–204, 2018), and Al-Gharabli et al. (Commun. Pure Appl. Anal. 18(1):159–180, 2019).

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1 Introduction

In the present paper, we consider the following viscoelastic plate problem with logarithmic nonlinearity:

$$\begin{cases} u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = ku\ln|u|, & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$. The vector ν is the unit outer normal to $\partial\Omega$ and the constant k is a small positive real number. The function g is the kernel and satisfies some conditions to be specified later.

1.1 Problems with logarithmic nonlinearity

The logarithmic nonlinearity appears naturally in inflation cosmology and supersymmetric filed theories, quantum mechanics, and many other branches of physics such as nuclear physics, optics, and geophysics [1, 7–9] and [10]. These specific applications in physics and other fields attract a lot of mathematical scientists to work with such problems. Birula and Mycielski [8] and [11] introduced the following problem:

$$\begin{cases} u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 = 0, & \text{in } [a,b] \times (0,T), \\ u(a,t) = u(b,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } [a,b], \end{cases} \tag{2}$$

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit $p \to 1$ for the p-adic string equation [12, 13]. They showed that wave equations with the logarithmic nonlinearity have stable, localized, soliton-like solutions in any number of dimensions. In [14], Cazenave and Haraux established the existence and uniqueness of the solution for the following Cauchy problem:

$$u_{tt} - \Delta u = u \ln |u|^k, \quad \text{in } \mathbb{R}^3. \tag{3}$$

Gorka [1] considered the corresponding one-dimensional Cauchy problem for equation (3) and established the global existence of weak solutions for all $(u_0, u_1) \in H_0^1 \times L^2$ by using some compactness arguments. In [7], Bartkowski and Gorka investigated weak solutions and also proved existence results of classical solutions. Hiramatsu et al. [2] investigated a numerical study of the solution of the following problem:

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|. \tag{4}$$

However, there was no theoretical analysis for this problem. In [15], Han considered the initial boundary value problem (4) in $\Omega \subset \mathbb{R}^3$ and obtained global existence of weak solutions for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. For more recent work dealing with logarithmic nonlinearity, we refer to [16–20].

1.2 Plate problems

Regarding the plate equations, we start with the result obtained by Lagnese [21]. He considered a viscoelastic plate equation and introduced a dissipative mechanism on the boundary of the system, and then he proved that when the time goes to infinity, the energy decays to zero. In [22], Rivera et al. investigated the energy of the solutions of a viscoelastic plate equation and they proved that first and second order energy decays exponentially provided that the kernel also decays exponentially. Messaoudi [23] established an existence result of the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u, & \text{in } Q_T = \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_T = \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{5}$$

obtained global solution in case $m \geq p$, and proved blow-up when the initial energy is negative and $m < p$. These results, for the same problem in [23], were improved and extended by Chen and Zhou [24]. Santos and Junior [25] studied the following system:

$$\begin{cases} u_{tt} + \Delta^2 u = 0, & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ -u + \int_0^t g_1(t-s)\beta_1 u(s)\,ds = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\beta_2 u(s)\,ds = 0, & \text{on } \Gamma_2 \times (0,\infty), \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \Omega, \end{cases} \tag{6}$$

where

$$\beta_1 u = \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \beta_2 u = \frac{\partial \Delta u}{\partial \mu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

and

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} \quad \text{and} \quad B_2 u = (\nu_1 - \nu_2)u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx})$$

with boundary damping, and they obtained stability results. For more results in this direction, see [3, 26–29].

1.3 Viscoelastic problems

The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastic industry. Many advances in the studies of constitutive relations, failure theories, and life prediction of viscoelastic materials and structures were reported and reviewed in the last two decades [30]. Dafermos [31, 32] considered a one-dimensional viscoelastic problem of the form

$$\rho u_{tt} = c u_{xx} - \int_{-\infty}^t g(t-s)u_{xx}(s)\,ds,$$

and established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as t goes to infinity. However, no rate of decay has been specified. In [33], Cavalcanti et al. considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds + a(x)u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega \times (0,\infty), \tag{7}$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain Ω but satisfies $a(x) \geq a_0$ on $\omega \subset \Omega$ and g satisfies, for two positive constants ξ_1 and ξ_2,

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0.$$

They established an exponential decay result under some restrictions on ω. For more results, we refer to [34–37]. However, in all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. In 2008, Messaoudi [36, 37] generalized the decay rates allowing an extended class of relaxation functions and gave general decay rates from which the exponential and the polynomial decay rates were only special cases. However, the optimality in the polynomial decay case was not obtained. Precisely, he considered relaxation functions that satisfy

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0, \tag{8}$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function and showed that the rate of the decay of the energy is the same rate of decay of g, which is not necessarily of exponential or polynomial decay type. After that a series of papers using (8) have appeared; see, for instance, [38–44]. Inspired by the experience with frictional damping initiated in the work of Lasiecka and Tataru [45], another step forward was done by considering relaxation functions satisfying

$$g'(t) \le -\chi\bigl(g(t)\bigr). \tag{9}$$

This condition, with several constraints imposed on χ, was used by several authors with different approaches. We refer to previous studies [46–50], and [51], where general decay results in terms of χ were obtained. Here, it should be mentioned that in [52] it was the first time where Lasiecka and Wang established not only general but also optimal results in which the decay rates were characterized by an ODE of the same type as the one generated by inequality (9) satisfied by g. Mustafa and Messaoudi [53] established an explicit and general decay rate for relaxation function satisfying

$$g'(t) \le -H\bigl(g(t)\bigr), \tag{10}$$

where $H \in C^1(\mathbb{R})$ with $H(0) = 0$ and H is a linear or strictly increasing and strictly convex function C^2 near the origin. In [54], Cavalcanti et al. considered a nonlinear viscoelastic wave equation with a relaxation function satisfying (10) and some additional requirements on H. They characterized the decay of the energy by the solution of a corresponding ODE as in [45]. Messaoudi and Al-Khulaifi [4] treated the same problem considered in [54] with a relaxation function satisfying

$$g'(t) \le -\xi(t)g^p(t), \quad \forall t \ge 0, 1 \le p < \frac{3}{2}. \tag{11}$$

They obtained a more general stability result for which the results of [36, 37] are only special cases. Moreover, the optimal decay rate for the polynomial case is achieved without any extra work and conditions as in [49] and [45]. Very recently, Al-Gharabli et al. [6] considered (1) with a relaxation function which satisfies (11), proved the existence of the solutions locally and globally, and established a general decay result depending on the behavior of g. Now, there are two natural questions that arose in dealing with the general decay of viscoelastic problems:

QI. Can we extend the range of polynomial decay rate optimality from $p \in [1, \frac{3}{2})$ to $p \in [1, 2)$ in the case of (11)?

QII. Can we get a general decay result for a class of relaxation functions satisfying

$$g'(t) \leq -\xi(t)H\bigl(g(t)\bigr), \quad \forall t \geq 0, \tag{12}$$

where ξ is a positive nonincreasing differentiable defined function on $[0,\infty)$ and H is some increasing convex function such that (12) yields (11) as a special case? Mustafa [5] answered these questions for a viscoelastic wave equation and established an optimal decay result.

Motivated by the papers of Gorka [1], Hiramatsu et al. [2], Mustafa [5], and Al-Gharabli et al. [6], we intend to establish a two-fold objective:

(1) extend the work for the wave equation in [1, 2], and [5] to a viscoelastic plate equation with logarithmic nonlinearity. We believe this is a natural extension done for many problems such as thermoelastic plate.

(2) consider a more general damping instead of the one considered in [6]. In fact, the results of [6] are special cases of this work when $G(s) = s^p$ and our assumption allows p to cover the full admissible range $[1,2)$.

Remark 1.1 Let us note here that though the logarithmic nonlinearity is somehow weaker than the polynomial nonlinearity, both the existence and stability result are not obtained by straightforward application of the method used for polynomial nonlinearity. We need to make some extra condition on the nonlinearity coefficient k (see condition (A3)).

This paper is organized as follows. In Sect. 2, we present some notation and material needed for our work. In Sect. 3, we present the local and global existence of the solutions of the problem. The stability results are presented in Sect. 4.

2 Preliminaries

In this section, we present some notations and material needed for the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant, and we consider the following hypotheses:

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1- nonincreasing function satisfying

$$g(0) > 0 \quad \text{and} \quad 1 - \int_0^\infty g(s)\,ds = \ell > 0. \tag{13}$$

(A2) There exists a positive nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\xi(0) > 0$, and a C^1 function $G : (0,\infty) \to (0,\infty)$ satisfies

$$g'(t) \leq -\xi(t)G\bigl(g(t)\bigr), \qquad G(0) = G'(0) = 0, \quad \forall t \geq 0, \tag{14}$$

and G is a linear or strictly increasing and strictly convex C^2 function on $(0,r]$, $0 < r < 1$.

(A3) The constant k in (1) satisfies $0 < k < k_0$, where k_0 is the positive real number satisfying

$$\sqrt{\frac{2\pi\ell}{k_0 c_p}} = e^{-\frac{3}{2}-\frac{1}{k_0}} \tag{15}$$

and c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \le c_p \|\Delta u\|_2^2, \quad \forall u \in H_0^2(\Omega),$$

where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$.

Remark 2.1 If G is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is a strictly increasing and strictly convex C^2 function on $(0, +\infty)$. For instance, if $G(r) = a$, $G'(r) = b$, $G''(r) = C$, we can define $\overline{G}$ for $t > r$ by

$$\overline{G}(t) = \frac{C}{2}t^2 + (b - Cr)t + \left(a + \frac{C}{2}r^2 - br\right). \tag{16}$$

For simplicity, we will use G for both G and $\overline{G}$.

Remark 2.2 Since G is strictly convex on $(0, r]$ and $G(0) = 0$, then

$$G(\theta z) \le \theta G(z), \quad 0 \le \theta \le 1, \text{ and } z \in (0, r]. \tag{17}$$

Remark 2.3 The function $f(s) = \sqrt{\frac{2\pi\ell}{c_p s}} - e^{-\frac{3}{2}-\frac{1}{s}}$ is a continuous and decreasing function on $(0, \infty)$ with

$$\lim_{s \to 0^+} f(s) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -e^{-\frac{3}{2}}.$$

Then there exists unique $k_0 > 0$ such that $f(k_0) = 0$. Moreover,

$$e^{-\frac{3}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi\ell}{c_p s}}, \quad \forall s \in (0, k_0), \tag{18}$$

which implies that the selection of k in (A3) is possible.

The modified energy functional associated with problem (1) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\left(\|u_t\|_2^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \frac{k+2}{2}\|u\|_2^2\right) \\ & - \frac{1}{2}\int_\Omega u^2 \ln|u|^k\,dx + \frac{1}{2}(go\Delta u)(t), \end{aligned} \tag{19}$$

where

$$(go\Delta u)(t) = \int_0^t g(t-s)\big\|\Delta u(s) - \Delta u(t)\big\|_2^2\,ds.$$

Direct differentiation of (19), using (1), leads to

$$E'(t) = \frac{1}{2}\big(g' o\Delta u\big)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \le \frac{1}{2}\big(g' o\Delta u\big)(t) \le 0. \tag{20}$$

Lemma 2.1 ([55, 56] (Logarithmic Sobolev inequality)) *Let u be any function in* $H_0^1(\Omega)$ *and a be any positive real number. Then*

$$\int_\Omega u^2 \ln|u|\,dx \le \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{a^2}{2\pi}\|\nabla u\|_2^2 - (1+\ln a)\|u\|_2^2. \tag{21}$$

Corollary 2.1 *Let u be any function in* $H_0^2(\Omega)$ *and a be any positive real number. Then*

$$\int_\Omega u^2 \ln|u|\,dx \le \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{c_p a^2}{2\pi}\|\Delta u\|_2^2 - (1+\ln a)\|u\|_2^2. \tag{22}$$

3 Local and global existence

In this section, we state the existence results of [6] for problem (1).

Definition 3.1 Let $T > 0$. A function

$$u \in C\big([0,T], H_0^2(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big) \cap C^2\big([0,T], H^{-2}(\Omega)\big)$$

is called a weak solution of (1) on $[0,T]$ if, for any $w \in H_0^2(\Omega)$ and $t \in [0,T]$,

$$\begin{cases} \int_\Omega u_{tt}(x,t)w(x)\,dx + \int_\Omega \Delta u(x,t)\Delta w(x)\,dx + \int_\Omega u(x,t)w(x)\,dx \\ \quad - \int_\Omega \Delta w(x)\int_0^t g(t-s)\Delta u(s)\,ds\,dx = \int_\Omega u(x,t)w(x)\ln|u(x,t)|^k\,dx, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases} \tag{23}$$

Theorem 3.1 *Assume that* (A1) *and* (A3) *hold and let* $(u_0,u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. *Then problem* (1) *has a weak solution*

$$u \in C\big([0,T], H_0^2(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big) \cap C^2\big([0,T], H^{-2}(\Omega)\big). \tag{24}$$

For the global existence, we introduce the following functionals:

$$J(t) = \frac{1}{2}\left(\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t) - \int_\Omega u^2 \ln|u|^k\,dx\right) + \frac{k}{4}\|u\|_2^2 \tag{25}$$

and

$$I(t) = \left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t) - 3\int_\Omega u^2 \ln|u|^k\,dx. \tag{26}$$

From (25) and (26), one can easily see that

$$J(t) = \frac{1}{3}\left[\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t)\right] + \frac{k}{4}\|u\|_2^2 + \frac{1}{6}I(t). \tag{27}$$

Lemma 3.1 *The following inequalities hold*:

$$-kd_0\sqrt{|\Omega| c_*^3}\|\Delta u\|_2^{\frac{3}{2}} \le \int_\Omega u^2 \ln|u|^k\,dx \le kc_*^3\|\Delta u\|_2^3, \quad \forall u \in H_0^2(\Omega), \tag{28}$$

where $d_0 = \sup_{0<s<1} \sqrt{s}|\ln s|$, $|\Omega|$ *is the Lebesgue measure of* Ω, *and* c_* *is the smallest embedding constant*

$$\left(\int_\Omega |u|^3\,dx\right)^{\frac{1}{3}} \le c_* \|\Delta u\|_2, \quad \forall u \in H_0^2(\Omega) \tag{29}$$

(c_* *exists thanks to the embedding of* $H_0^2(\Omega)$ *in* $L^\infty(\Omega)$, $\Omega \subset \mathcal{R}^2$).

Proof Let

$$\Omega_1 = \{x \in \Omega : |u(x)| > 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |u(x)| \le 1\}.$$

So, using (29), we have

$$\begin{aligned}\int_\Omega u^2 \ln|u|^k\,dx &= \int_{\Omega_2} u^2 \ln|u|^k\,dx + \int_{\Omega_1} u^2 \ln|u|^k\,dx \\ &\le k\int_{\Omega_1} u^2 \ln|u|\,dx \le k\int_{\Omega_1} |u|^3\,dx \le k\int_\Omega |u|^3\,dx \le kc_*^3\|\Delta u\|_2^3,\end{aligned}$$

this gives the right inequality in (28). On the other hand, using Hölder's inequality and (29), we find

$$\begin{aligned}-\int_\Omega u^2 \ln|u|^k\,dx &= -\int_{\Omega_2} u^2 \ln|u|^k\,dx - \int_{\Omega_1} u^2 \ln|u|^k\,dx \\ &\le -k\int_{\Omega_2} u^2 \ln|u|\,dx = k\int_{\Omega_2} u^2 \bigl|\ln|u|\bigr|\,dx \\ &\le kd_0 \int_\Omega |u|^{\frac{3}{2}}\,dx \le kd_0\sqrt{|\Omega|}\left(\int_\Omega |u|^3\,dx\right)^{\frac{1}{2}} \\ &\le kd_0\sqrt{|\Omega|c_*^3}\|\Delta u\|_2^{\frac{3}{2}},\end{aligned}$$

which implies the left inequality in (28). □

Lemma 3.2 *Assume that* (A1)–(A3). *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *such that*

$$I(0) > 0 \quad \textit{and} \quad \sqrt{54}kc_*^3\left(\frac{E(0)}{\ell}\right)^{\frac{1}{2}} < \ell. \tag{30}$$

Then

$$I(t) > 0, \quad \forall t \in [0, T). \tag{31}$$

Proof From (26), we have

$$\int_\Omega u^2 \ln|u|^k\,dx = \frac{1}{3}\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{3}\|u\|_2^2 + \frac{1}{3}(go\Delta u)(t) - \frac{1}{3}I(t). \tag{32}$$

Substituting (32) in (25), we find

$$J(t) = \frac{1}{3}\left[\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t)\right] + \frac{k}{4}\|u\|_2^2 + \frac{1}{6}I(t). \tag{33}$$

Since $I(0) > 0$ and I is continuous on $[0, T]$, there exists $t_0 \in (0, T]$ such that $I(t) > 0$ for all $t \in [0, t_0)$. Let us denote by t_0 the largest real number in $(0, T]$ such that $I > 0$ on $[0, t_0)$. If $t_0 = T$, then (31) is satisfied. We assume by contradiction that $t_0 \in (0, T)$. Thus $I(t_0) = 0$ and

$$\|\Delta u(t)\|_2^2 \le \frac{3}{\ell}J(t) \le \frac{3}{\ell}E(t) \le \frac{3}{\ell}E(0), \quad \forall t \in [0, t_0). \tag{34}$$

If $\|\Delta u(t_0)\|_2^2 = 0$, then (28) and (29) give

$$0 = I(t_0) = (go\Delta u)(t_0) = \int_0^{t_0} g(s)\|\Delta u(s)\|_2^2\,ds. \tag{35}$$

Consequently, if $g > 0$ on $[0, t_0)$, we get

$$\|\Delta u(s)\|_2 = 0, \quad \forall s \in [0, t_0).$$

Then

$$I(t) = 0, \quad \forall t \in [0, t_0),$$

which is not true since $I > 0$ on $[0, t_0)$. If g is not positive on $[0, t_0)$, then let $t_1 \in [0, t_0)$ be the smallest real number such that $g(t_1) = 0$. Because $g(0) > 0$ and g is positive, nonincreasing, and continuous on $\mathbb{R}^+$ (condition (A1)), then $t_1 > 0$ and $g = 0$ on $[t_1, \infty)$. Therefore, from (35), we deduce that

$$0 = \int_0^{t_0} g(s)\|\Delta u(s)\|_2^2\,ds = \int_0^{t_1} g(s)\|\Delta u(s)\|_2^2\,ds,$$

then $\|\Delta u(s)\|_2 = 0$ for any $s \in [0, t_1)$, which implies that $I(t) = 0$ for any $t \in [0, t_1)$. As before, this is a contradiction to the fact that $I > 0$ on $[0, t_0)$. Then we conclude that $\|\Delta u(t_0)\|_2^2 > 0$. On the other hand, we have

$$I(t_0) \ge \ell\|\Delta u(t_0)\|_2^2 - 3\int_\Omega u(t_0)^2 \ln|u(t_0)|^k\,dx.$$

By using (34) and Lemma 3.1, we have

$$I(t_0) \ge \left[\ell - 3kc_*^3\left(\frac{6E(0)}{\ell}\right)^{\frac{1}{2}}\right]\|\Delta u(t_0)\|_2^2.$$

By recalling (30), we arrive at $I(t_0) > 0$, which contradicts the assumption $I(t_0) = 0$. Hence, $t_0 = T$ and then

$$I(t) > 0, \quad \forall t \in [0, T).$$

□

4 Stability

In this section, we state and prove our stability results. We start by establishing several lemmas needed for the proof of our main result.

Lemma 4.1 ([6]) *Assume that g satisfies* (A1). *Then, for* $u \in H_0^2(\Omega)$,

$$\int_\Omega \left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds \right)^2 dx \le c(go\Delta u)(t),$$

and

$$\int_\Omega \left(\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds \right)^2 dx \le -c\big(g'o\Delta u\big)(t).$$

Lemma 4.2 *Assume that* (A1)–(A3) *and* (30) *hold. Then the functional*

$$\psi_1(t) = \int_\Omega u u_t\,dx$$

satisfies, along the solutions of (1),

$$\psi_1'(t) \le \|u_t\|_2^2 - \frac{\ell}{2}\|\Delta u\|_2^2 - \|u\|_2^2 + \int_\Omega u^2 \ln|u|^k\,dx + c(go\Delta u)(t). \tag{36}$$

Proof By using (1), we easily see that

$$\begin{aligned}\psi_1' = {}& \|u_t\|_2^2 - \|\Delta u\|_2^2 - \|u\|_2^2 + \int_\Omega \Delta u \int_0^t g(t-s)\Delta u(s)\,ds\,dx \\ & + \int_\Omega u^2 \ln|u|^k\,dx.\end{aligned} \tag{37}$$

We now use Lemma 4.1 and Young's inequality to obtain, for any $\mu > 0$,

$$\begin{aligned}&\int_\Omega \Delta u(t)\left(\int_0^t g(t-s)\Delta u(s)\,ds\right) dx \\ &\quad \le \left(1-\ell+\frac{\mu}{2}\right)\|\Delta u\|_2^2 + \frac{1}{2\mu}(1-\ell)(go\Delta u)(t).\end{aligned} \tag{38}$$

By choosing $\mu = \ell$ and combining (37) and (38), we obtain (36). □

Lemma 4.3 *Assume that* (A1)–(A3) *and* (30) *hold. Then the functional*

$$\psi_2(t) = -\int_\Omega u_t \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx$$

satisfies, along the solutions of (1) *and for any* $\epsilon_0 \in (0,1)$ *and* $\delta > 0$,

$$\begin{aligned}\psi_2'(t) \le {}& \delta\|\Delta u\|_2^2 + \frac{c}{\delta}(go\Delta u)(t) + \frac{c}{\delta}\big(-g'o\Delta u\big)(t) + \left(\delta - \int_0^t g(s)\,ds\right)\|u_t\|_2^2 \\ & + c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).\end{aligned} \tag{39}$$

Proof Direct computations, using (1), yield

$$\begin{aligned}\psi_2'(t) &= \int_\Omega \Delta u \int_0^t g(t-s)\big(\Delta u(t)-\Delta u(s)\big)\,ds\,dx + \int_\Omega u \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &\quad + \int_\Omega \int_0^t g(t-s)\big(\Delta u(t)-\Delta u(s)\big)\,ds \int_0^t g(t-s)\Delta u(s)\,ds\,dx \\ &\quad - \int_\Omega u \ln|u|^k \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &\quad - \int_\Omega u_t \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx - \left(\int_0^t g(s)\,ds\right)\int_\Omega u_t^2\,dx. \end{aligned} \tag{40}$$

Similar to (37), we estimate the right-hand side terms of (40). So, by using Young's inequality, the first term gives, for any $\delta > 0$,

$$\int_\Omega \Delta u \int_0^t g(t-s)\big(\Delta u(t)-\Delta u(s)\big)\,ds\,dx \le \frac{\delta}{4}\|\Delta u\|_2^2 + \frac{c}{\delta}(go\Delta u)(t). \tag{41}$$

Using Lemma 4.1, Young's and Poincaré's inequalities, the second and fifth terms lead to

$$\int_\Omega u \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \le \frac{\delta}{4}\|\Delta u\|_2^2 + \frac{c}{\delta}(go\Delta u)(t) \tag{42}$$

and

$$-\int_\Omega u_t \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \le \delta\|u_t\|_2^2 - \frac{c}{\delta}\big(g'o\Delta u\big)(t). \tag{43}$$

Similarly, the third term can be estimated as follows:

$$\begin{aligned}&\int_\Omega \int_0^t g(t-s)\big(\Delta u(t)-\Delta u(s)\big)\,ds \int_0^t g(t-s)\Delta u(s)\,ds\,dx \\ &\quad \le \frac{\delta}{4}\|\Delta u\|_2^2 + c\left(1+\frac{1}{\delta}\right)(go\Delta u)(t).\end{aligned} \tag{44}$$

Let $\epsilon_0 \in (0,1)$ and $h(s) = s^{\epsilon_0}(|\ln s| - s)$. Notice that h is continuous on $(0,\infty)$ and its limit at 0 is 0, and its limit at ∞ is $-\infty$. Then h has a maximum d_{ϵ_0} on $[0,\infty)$, so the following inequality holds:

$$s|\ln s| \le s^2 + d_{\epsilon_0} s^{1-\epsilon_0}, \quad \forall s > 0. \tag{45}$$

Applying (45) to $u\ln|u|$, using the Cauchy–Schwarz inequality, the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$, and performing the same calculations as before, we get, for any $\delta_1 > 0$,

$$\begin{aligned}&\int_\Omega u\ln|u|^k \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &\quad \le k\int_\Omega \big(u^2 + d_{\epsilon_0}|u|^{1-\epsilon_0}\big)\left|\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\ &\quad \le c\int_\Omega |u|^2\left|\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right|\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \delta_1 \int_\Omega u^2\,dx + c_{\epsilon_0,\delta_1} \int_\Omega \left| \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds \right|^{\frac{2}{1+\epsilon_0}} dx \\
&\leq c\delta_1 \|\Delta u\|_2^2 + \frac{c}{\delta_1} \int_\Omega \left| \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds \right|^2 dx \\
&+ c_{\epsilon_0,\delta_1} \int_\Omega \left| \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds \right|^{\frac{2}{1+\epsilon_0}} dx.
\end{aligned}
$$

Then, putting $\frac{\delta}{4} = c\delta_1$ and using Hölder's inequality and Lemma 4.1, we find

$$
\begin{aligned}
\int_\Omega u \ln|u|^k \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx &\leq \frac{\delta}{4}\|\Delta u\|_2^2 + \frac{c}{\delta}(go\Delta u)(t) \\
&+ c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned} \tag{46}
$$

The above inequalities imply (39). □

Lemma 4.4 *Assume that* (A1)–(A3) *and* (30) *hold, and let* $\epsilon_0 \in (0,1)$. *Then, for* k *small enough, there exist positive constants* ε_1, ε_2, m, *and* t_0 *such that the functional*

$$
L(t) = E(t) + \varepsilon_1\psi_1(t) + \varepsilon_2\psi_2(t)
$$

satisfies

$$
L \sim E \tag{47}
$$

and

$$
L'(t) \leq -mE(t) + c(go\Delta u)(t) + c_{\epsilon_0}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t), \quad \forall t \geq t_0. \tag{48}
$$

Proof For the proof of (47), we see that, using similar calculations as before,

$$
\begin{aligned}
\big|L(t)-E(t)\big| &= \big|\varepsilon_1\psi_1(t) + \varepsilon_2\psi_2(t)\big| \\
&\leq c(\varepsilon_1+\varepsilon_2)\big(\|u_t\|_2^2 + \|\Delta u\|_2^2 + (go\Delta u)(t)\big),
\end{aligned}
$$

therefore, from (31) and (33), we obtain

$$
\big|L(t)-E(t)\big| \leq c(\varepsilon_1+\varepsilon_2)\left(\frac{1}{2}\|u_t\|_2^2 + J(t)\right) = c(\varepsilon_1+\varepsilon_2)E(t),
$$

then

$$
\big(1 - c(\varepsilon_1+\varepsilon_2)\big)E(t) \leq L(t) \leq \big(1 + c(\varepsilon_1+\varepsilon_2)\big)E(t).
$$

Hence, for ε_1, $\varepsilon_2 > 0$ satisfying

$$
1 - c(\varepsilon_1+\varepsilon_2) > 0, \tag{49}
$$

equivalence (47) holds.

Now, we prove inequality (48). Since g is positive and $g(0) > 0$, then, for any $t_0 > 0$, we have

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds = g_0 > 0, \quad \forall t \geq t_0.$$

By using (20), (36), (39), and the definition of $E(t)$, for $t \geq t_0$ and any $m > 0$, we have

$$\begin{aligned} L'(t) \leq{}& -mE(t) - \left(\varepsilon_2(g_0-\delta) - \varepsilon_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\ & - \left(\frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2}\right)\|\Delta u\|_2^2 - \left(\varepsilon_1 - \frac{(k+2)m}{4}\right)\|u\|_2^2 \\ & + \left(k\varepsilon_1 - k\frac{m}{2}\right)\int_\Omega u^2 \ln|u|\,dx + \left(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\right)(go\Delta u)(t) \\ & + \left(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\right)(g'o\Delta u)(t) + \varepsilon_2 c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \end{aligned} \tag{50}$$

Using the logarithmic Sobolev inequality, for $0 < m < 2\varepsilon_1$, we get

$$\begin{aligned} L'(t) \leq{}& -mE(t) - \left(\varepsilon_2(g_0-\delta) - \varepsilon_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\ & - \left(\frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2} - k\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\ & - \left(\varepsilon_1 - \frac{m(k+2)}{4} + k\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a) + k\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln\|u\|_2^2\right)\|u\|_2^2 \\ & + \left(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\right)(go\Delta u)(t) \\ & + \left(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\right)(g'o\Delta u)(t) + \varepsilon_2 c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \end{aligned} \tag{51}$$

At this point we choose δ so small that

$$g_0 - \delta > \frac{1}{2}g_0 \quad \text{and} \quad \delta < \frac{\ell g_0}{16}.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{g_0}{4}\varepsilon_2 < \varepsilon_1 < \frac{g_0}{2}\varepsilon_2 \tag{52}$$

will make

$$k_1 := \varepsilon_2(g_0-\delta) - \varepsilon_1 > 0 \quad \text{and} \quad k_2 := \frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta > 0.$$

Then we choose ε_1 and ε_2 so small that (49) and (52) remain valid and, further,

$$\frac{1}{2} - \frac{c\varepsilon_2}{\delta} > 0.$$

Consequently, we get (47) and

$$\begin{aligned} L'(t) &\le -mE(t) - \left(k_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\ &\quad - \left(k_2 - \frac{m}{2} - k\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\ &\quad - \left(\varepsilon_1 - \frac{m(k+2)}{4} + k\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a) + k\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln\|u\|_2^2\right)\|u\|_2^2 \\ &\quad + c(go\Delta u)(t) + c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \end{aligned} \tag{53}$$

Thanks to (A3), we choose

$$e^{-\frac{3}{2}-\frac{1}{k}} < a < \sqrt{\frac{2\pi\ell}{kc_p}}. \tag{54}$$

This selection will make

$$\ell - \frac{ka^2 c_p}{2\pi} > 0 \quad \text{and} \quad \frac{k+2}{2} + k(1+\ln a) > 0.$$

Then, using (54) and selecting m and k so small that

$$\alpha_1 = k_1 - \frac{m}{2} > 0, \qquad \alpha_2 = k_2 - \frac{m}{2} - k\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi} > 0$$

and

$$\alpha_3 = \varepsilon_1 - \frac{m(k+2)}{4} + k\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a) + k\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln\|u\|_2^2 > 0,$$

we arrive at the desired result (48). □

Remark 4.1 Using (13), (19), (25), (31), and (33), we have

$$E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 \ge J(t) \ge \frac{1}{3}(go\Delta u)(t).$$

Then, using (20),

$$(go\Delta u)(t) \le 3E(t) \le 3E(0). \tag{55}$$

Using (55), we obtain

$$\begin{aligned} (go\Delta u)(t) &= (go\Delta u)^{\frac{\epsilon_0}{1+\epsilon_0}}(t)(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \\ &\le c(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \end{aligned} \tag{56}$$

Remark 4.2 In the case of G *is linear* and since ξ is nonincreasing, we have

$$\begin{aligned}
\xi(t)(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) &= \bigl(\xi^{\epsilon_0}(t)\xi(t)(g\circ\Delta u)(t)\bigr)^{\frac{1}{1+\epsilon_0}}\\
&\le \bigl(\xi^{\epsilon_0}(0)\xi(t)(g\circ\Delta u)(t)\bigr)^{\frac{1}{1+\epsilon_0}}\\
&\le c\bigl(\xi(t)(g\circ\Delta u)(t)\bigr)^{\frac{1}{1+\epsilon_0}}\\
&\le c\bigl(-E'(t)\bigr)^{\frac{1}{(1+\epsilon_0)}}.
\end{aligned}\tag{57}$$

Lemma 4.5 *If* (A1)–(A2) *are satisfied, then we have the following estimate*:

$$(go\nabla u)(t)\le \frac{t}{q}G^{-1}\left(\frac{qI(t)}{t\xi(t)}\right),\quad \forall t>0,\tag{58}$$

where q is small enough and $\overline{G}$ is defined in Remark (2.1) *and the functional I is defined by*

$$I(t):=\bigl(-g'o\nabla u\bigr)(t)\le -cE'(t).\tag{59}$$

Proof To establish (58), let us define the following functional:

$$\lambda(t):=\frac{q}{t}\int_0^t\bigl\|\Delta(t)-\Delta(t-s)\bigr\|_2^2\,ds,\quad \forall t>0.\tag{60}$$

Then, using (19), (20), and the dentition of $\lambda(t)$, we have

$$\begin{aligned}
\lambda(t)&\le \frac{2q}{t}\left(\int_0^t\Bigl|\bigl|\Delta(t)\bigr|\Bigr|_2^2+\int_0^t\Bigl|\bigl|\Delta(t-s)\bigr|\Bigr|_2^2\,ds\right).\\
&\le \frac{4q}{\ell t}\left(\int_0^t\bigl(E(t)+E(t-s)\bigr)\,ds\right)\\
&\le \frac{8q}{\ell t}\int_0^t E(s)\,ds\\
&\le \frac{8q}{\ell t}\int_0^t E(0)\,ds=\frac{8q}{\ell}E(0)<+\infty.
\end{aligned}\tag{61}$$

Thus, q can be chosen so small so that, for all $t>0$,

$$\lambda(t)<1.\tag{62}$$

Without loss of the generality, for all $t>0$, we assume that $\lambda(t)>0$, otherwise we get an exponential decay from (48). The use of Jensen's inequality and using (59), (2.2), and (62) give

$$\begin{aligned}
I(t)&=\frac{1}{q\lambda(t)}\int_0^t\lambda(t)\bigl(-g'(s)\bigr)\int_\Omega q\bigl|\Delta(t)-\Delta(t-s)\bigr|^2\,dx\,ds\\
&\ge \frac{1}{q\lambda(t)}\int_0^t\lambda(t)\xi(s)G\bigl(g(s)\bigr)\int_\Omega q\bigl|\Delta(t)-\Delta(t-s)\bigr|^2\,dx\,ds\\
&\ge \frac{\xi(t)}{q\lambda(t)}\int_0^t G\bigl(\lambda(t)g(s)\bigr)\int_\Omega q\bigl|\Delta(t)-\Delta(t-s)\bigr|^2\,dx\,ds
\end{aligned}$$

$$\geq \frac{t\xi(t)}{q} G\left(\frac{q}{t}\int_0^t g(s)\int_\Omega \left|\Delta(t)-\Delta(t-s)\right|^2 dx\,ds\right)$$
$$= \frac{t\xi(t)}{q} \overline{G}\left(\frac{q}{t}\int_0^t g(s)\int_\Omega \left|\Delta u(t)-\Delta u(t-s)\right|^2 dx\,ds\right), \tag{63}$$

hence (58) is established. □

Theorem 4.1 *Let $(u_0,u_1)\in H_0^2(\Omega)\times L^2(\Omega)$. Assume that* (A1)–(A3) *and* (30) *hold. Then there exist positive constants C_1, C_2, t_0, and t_1 such that the solution of* (1) *satisfies, for all $t\geq t_1$,*

$$E(t)\leq C_1\left(1+\int_{t_0}^t \xi^{1+\epsilon_0}(s)\,ds\right)^{\frac{-1}{\epsilon_0}}, \quad \text{if } G \text{ is linear}, \tag{64}$$
$$E(t)\leq C_2 t^{\frac{1}{1+\epsilon_0}} K_2^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\epsilon_0}}\int_{t_1}^t \xi(s)\,ds}\right), \quad \text{if } G \text{ is nonlinear}, \tag{65}$$

where $K_2(s)=sK'(\varepsilon_1 s)$ and $K(t)=([G^{-1}]^{\frac{1}{1+\epsilon_0}})^{-1}(t)$.

Proof *Case 1*: G is linear.

We multiply (48) by $\xi(t)$ and use (56) and (57) to get

$$\xi(t)L'(t)\leq -m\xi(t)E(t)+c\big(-E'(t)\big)^{\frac{1}{1+\epsilon_0}}, \quad \forall t\geq t_0. \tag{66}$$

Multiply (66) by $\xi^{\epsilon_0}(t)E^{\epsilon_0}(t)$ and recall that $\xi'\leq 0$ to obtain

$$\xi^{\epsilon_0+1}(t)E^{\epsilon_0}(t)L'(t)\leq -m\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t)+c(\xi E)^{\epsilon_0}(t)\big(-E'(t)\big)^{\frac{1}{\epsilon_0+1}}, \quad \forall t\geq t_0.$$

Use of Young's inequality, with $q=\epsilon_0+1$ and $q^*=\frac{\epsilon_0+1}{\epsilon_0}$, gives, for any $\varepsilon'>0$,

$$\begin{aligned}\xi^{\epsilon_0+1}(t)E^{\epsilon_0}(t)L'(t)&\leq -m\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t)+c\big(\varepsilon'\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}-c_{\varepsilon'}E'(t)\big)\\&=-\big(m-\varepsilon' c\big)\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}-cE'(t), \quad \forall t\geq t_0.\end{aligned}$$

We then choose $0<\varepsilon'<\frac{m}{c}$ and use that $\xi'\leq 0$ and $E'\leq 0$ to get, for $c_1=m-\varepsilon' c$,

$$\big(\xi^{\epsilon_0+1}E^{\epsilon_0}L\big)'(t)\leq \xi^{\epsilon_0+1}(t)E^{\epsilon_0}(t)L_1'(t)\leq -c_1\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t)-cE'(t), \quad \forall t\geq t_0,$$

which implies

$$\big(\xi^{\epsilon_0+1}E^{\epsilon_0}L+cE\big)'(t)\leq -c_1\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t), \quad \forall t\geq t_0.$$

Let $L_1=\xi^{\epsilon_0+1}E^{\epsilon_0}L+cE$. Then $L_1\sim E$ (thanks to (47)) and

$$L_1'(t)\leq -c\xi^{\epsilon_0+1}(t)L_1^{\epsilon_0+1}(t), \quad \forall t\geq t_0.$$

Integrating over (t_0,t) and using the fact that $L_1\sim E$, we obtain (64).

Case 2: G is nonlinear.

Using (48), (56), and (58), we obtain, $\forall t \geq t_0$,

$$L'(t) \leq -mE(t) + ct^{\frac{1}{1+\epsilon_0}} \left[G^{-1}\left(\frac{qI(t)}{t\xi(t)} \right) \right]^{\frac{1}{1+\epsilon_0}}. \tag{67}$$

Combining the strictly increasing property of $\overline{G}$ and the fact that $\frac{1}{t} < 1$ whenever $t > 1$, we obtain

$$G^{-1}\left(\frac{qI(t)}{t\xi(t)} \right) \leq G^{-1}\left(\frac{qI(t)}{t^{\frac{1}{1+\epsilon_0}}\xi(t)} \right), \tag{68}$$

and then (67) becomes, for $\forall t \geq t_1 = \max\{t_0, 1\}$,

$$L'(t) \leq -mE(t) + ct^{\frac{1}{1+\epsilon_0}} \left[G^{-1}\left(\frac{qI(t)}{t^{\frac{1}{1+\epsilon_0}}\xi(t)} \right) \right]^{\frac{1}{1+\epsilon_0}}. \tag{69}$$

Set

$$K(t) = \left(\left[G^{-1} \right]^{\frac{1}{1+\epsilon_0}} \right)^{-1}(t), \qquad \chi(t) = \frac{qI(t)}{t^{\frac{1}{1+\epsilon_0}}\xi(t)}. \tag{70}$$

In fact,

$$K' = (1+\epsilon_0)G'\left(G^{-1}\right)^{\frac{\epsilon_0}{1+\epsilon}} > 0 \quad \text{on } (0,r],$$
$$K'' = \frac{\epsilon_0}{(G^{-1})^{1+\epsilon_0}} + (1+\epsilon_0)\left(G^{-1}\right)^{\frac{\epsilon_0}{1+\epsilon}} G'' > 0 \quad \text{on } (0,r].$$

So, (69) reduces to

$$L_1'(t) \leq -mE(t) + ct^{\frac{1}{1+\epsilon_0}} K^{-1}\left(\chi(t)\right), \quad \forall t \geq t_1. \tag{71}$$

Now, for $\varepsilon_1 < r$ and using (71) and the fact that $E' \leq 0$, $K' > 0$, $K'' > 0$ on $(0,r]$, we find that the functional L_2, defined by

$$L_2(t) := K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)} \right) L_1(t),$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 L_2(t) \leq E(t) \leq \alpha_2 L_2(t), \tag{72}$$

and, for all $t \geq t_1$,

$$L_2'(t) \leq -mE(t)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)} \right) + ct^{\frac{1}{1+\epsilon_0}} K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)} \right) K^{-1}\left(\chi(t)\right). \tag{73}$$

Let K^* be the convex conjugate of K in the sense of Young (see [57]), then

$$K^*(s) = s\left(K'\right)^{-1}(s) - K\left[\left(K'\right)^{-1}(s)\right], \quad \text{if } s \in \left(0, K'(r)\right], \tag{74}$$

and K^* satisfies the following generalized Young inequality:

$$AB \le K^*(A) + K(B), \quad \text{if } A \in \big(0, K'(r)\big], B \in (0, r]. \tag{75}$$

So, with $A = K'(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)})$ and $B = K^{-1}(\chi(t))$, we arrive at

$$\begin{aligned} L_2'(t) &\le -mE(t)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\epsilon_0}} K^*\left(K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)\right) \\ &\quad + ct^{\frac{1}{1+\epsilon_0}} \chi(t) \\ &\le -mE(t)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + c\frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \\ &\quad + ct^{\frac{1}{1+\epsilon_0}} \chi(t). \end{aligned} \tag{76}$$

Then, multiplying (76) by $\xi(t)$ and using (59), (70), we get

$$\begin{aligned} \xi(t)L_2'(t) &\le -m\xi(t)E(t)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1\xi(t) \cdot \frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \\ &\quad - cE'(t), \quad \forall t \ge t_1. \end{aligned}$$

Using the nonincreasing property of ξ, we obtain, for all $t \ge t_1$,

$$\begin{aligned} (\xi L_2 + cE)'(t) &\le -m\xi(t)E(t)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \\ &\quad + c\varepsilon_1\xi(t)\frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right). \end{aligned}$$

Therefore, by setting $L_3 := \xi L_2 + cE \sim E$, we conclude that

$$F_3'(t) \le -m\xi(t)E(t)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1\xi(t) \cdot \frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right).$$

This gives, for a suitable choice of ε_1,

$$L_3'(t) \le -k\xi(t)\left(\frac{E(t)}{E(0)}\right)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right), \quad \forall t \ge t_1,$$

or

$$k\left(\frac{E(t)}{E(0)}\right)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)\xi(t) \le -L_3'(t), \quad \forall t \ge t_1. \tag{77}$$

An integration of (77) yields

$$\int_{t_1}^{t} k\left(\frac{E(s)}{E(0)}\right)K'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(s)}{E(0)}\right)\xi(s)\,ds \le -\int_{t_1}^{t} L_3'(s)\,ds \le L_3(t_1). \tag{78}$$

Using the facts that $K', K'' > 0$ and the nonincreasing property of E, we deduce that the map $t \mapsto E(t)K'(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)})$ is nonincreasing; consequently, we have

$$\begin{aligned} &k\left(\frac{E(t)}{E(0)}\right)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)\int_{t_1}^{t} \xi(s)\,ds \\ &\quad\le \int_{t_1}^{t} k\left(\frac{E(s)}{E(0)}\right)K'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(s)}{E(0)}\right)\xi(s)\,ds \le L_3(t_1), \quad \forall t \ge t_1. \end{aligned} \tag{79}$$

Multiplying each side of (79) by $\frac{1}{t^{\frac{1}{1+\epsilon_0}}}$, we have

$$k\left(\frac{1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)K'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)\int_{t_1}^{t} \xi(s)\,ds \le \frac{k_2}{t^{\frac{1}{1+\epsilon_0}}}, \quad \forall t \ge t_1. \tag{80}$$

Next, we set $K_2(s) = sK'(\varepsilon_1 s)$, which is strictly increasing, and consequently we obtain

$$kK_2\left(\frac{1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)\int_{t_1}^{t} \xi(s)\,ds \le \frac{k_2}{t^{\frac{1}{1+\epsilon_0}}}, \quad \forall t \ge t_1. \tag{81}$$

Finally, for two positive constants k_2 and k_3, we infer

$$E(t) \le k_3 t^{\frac{1}{1+\epsilon_0}} K_2^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\epsilon_0}} \int_{t_1}^{t} \xi(s)\,ds}\right). \tag{82}$$

This finishes the proof. □

The following examples illustrate our results.

Example 4.2 Let $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that (A1) holds. Then $g'(t) = -\xi(t)G(g(t))$, where $G(t) = t$ and $\xi(t) = b$. Therefore, we can use (64) to deduce

$$E(t) \le \frac{c_1}{(1+t)^{\frac{1}{\epsilon_0}}}. \tag{83}$$

Example 4.3 Let $g(t) = \frac{a}{(1+t)^q}$, where $q > 1 + \epsilon_0$ and a is chosen so that hypothesis (A1) is satisfied. Then

$$g'(t) = -bG\bigl(g(t)\bigr), \quad \text{with } G(s) = s^{\frac{q+1}{q}},$$

where b is a fixed constant. Since $\Phi(s) = s^{\frac{(\epsilon_0+1)(q+1)}{q}}$, then (65) gives

$$E(t) \le \frac{c}{t^{\frac{q-1-\epsilon_0}{(1+\epsilon_0)^2(q+1)}}}. \tag{84}$$

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