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Infinitely many high energy solutions for fractional Schrödinger equations with magnetic field

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Abstract

In this paper we investigate the existence of infinitely many solutions for nonlocal Schrödinger equation involving a magnetic potential

$$(-\Delta)^s_A u + V(x)u = f(x, |u|)u, \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$ is fixed, $N > 2s$, $V : \mathbb{R}^N \to \mathbb{R}^+$ is an electric potential, the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function, and $(-\Delta)^s_A$ is the fractional magnetic operator. Under suitable assumptions for the potential function V and nonlinearity f, we obtain the existence of infinitely many nontrivial high energy solutions by using the variant fountain theorem.

MSC: 35R11; 35A15; 35J60; 47G20

Keywords: Schrödinger equation; Fractional magnetic operator; Variant fountain theorem

1 Introduction

The aim of the present paper is to investigate the multiplicity of solutions for the following fractional Schrödinger equation with magnetic field:

$$(-\Delta)^s_A u + V(x)u = f(x, |u|)u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1$, $N > 2s$, $V : \mathbb{R}^N \to \mathbb{R}^+$ is an electric potential, and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential. The fractional magnetic operator $(-\Delta)^s_A$ is defined along all functions $u \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$ as

$$(-\Delta)^s_A u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y)}{|x-y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N,$$

for more details about this operator see [1]. Meanwhile, in [1], based on concentration compactness arguments, ground state solutions are obtained for problem (1.1) in threedimensional space. In [2], the authors set up a bridge between the classical magnetic operator and the fractional one. Reference [3] considers the problem (1.1) with Kirchhoff function and obtains the existence of least energy solutions and infinitely many solutions under suitable conditions. In [4], the authors obtain the existence and the multiplicity of solutions for a nonlinear fractional magnetic Schrödinger equation by using variational methods and Ljusternick–Schnirelmann theory. In [5], the authors obtain the existence and the multiplicity of solutions for a nonlinear fractional magnetic Schrödinger equation with exponential critical growth. And in [6], the author obtains the existence of nontrivial solutions for a class of fractional magnetic Schrödinger equations via penalization techniques.

In previous years, the nonlinear magnetic Schrödinger equations

$$-(\nabla - iA)^2 u + V(x)u = f\big(x, |u|\big)u, \quad x \in \mathbb{R}^N,$$

have been extensively studied, we refer the interested reader to [7–13] and the references therein. $-(\nabla - iA)^2$ is a magnetic Schrödinger operator, in a suitable sense, $(-\triangle)^s_A u$ converges to $-(\nabla - iA)^2 u$ in the limit $s \uparrow 1$. In this sense, nonlocal case can be regarded as an approximation of local case.

When potential function $A \equiv 0$, the operator $(-\triangle)^s_A$ is consistence with the usual notion of fractional Laplacian $(-\triangle)^s$, and Eq. (1.1) becomes the fractional Schrödinger equation

$$(-\triangle)^s u + V(x)u = f\big(x, |u|\big)u, \quad x \in \mathbb{R}^N,$$

a great deal of research work has been done for this type of equations in recent years; for further details see for instance [14–24] and the references therein.

Inspired by the above work, we consider the existence of infinitely many solutions of problem (1.1). Firstly, we assume that the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function and the potential function $V : \mathbb{R}^N \to \mathbb{R}^+$ satisfies:

(V_1) $V \in C(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^n} V(x) \geq V_0$, where $V_0 > 0$ is a constant;

(V_2) For any $c > 0$, there exists $h > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\big(\{x \in \mathbb{R}^N : |x - y| \leq h, V(x) \leq c\}\big) = 0,$$

where meas denotes the Lebesgue measure.

The nonlinearity $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is a Carathéodory function; we require:

(f_1) There exist $C > 0$ and $p \in (2, 2^*_s)$ such that $|f(x,t)| \leq C(1 + |t|^{p-2})$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}^+$, where $2^*_s = \frac{2N}{N-2s}$ is the fractional Sobolev exponent;

(f_2) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$ uniformly for $x \in \mathbb{R}^N$;

(f_3) there exists $\mu > 2$ such that

$$0 < \mu F(x,t) = \mu \int_0^t f(x,\tau)\tau \, d\tau \leq f(x,t)t^2,$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$;

(f_4) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = \infty$ uniformly for $x \in \mathbb{R}^N$.

Now we state our main result as follows. The fractional solution space $H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ and the energy functional $J(u)$ are introduced in Sect. 2.

Theorem 1.1 *Let* (V_1)–(V_2) *and* (f_1)–(f_4) *hold. Then problem* (1.1) *possesses infinitely many high energy solutions* $u^k \in H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ *for any* $k \geq k_0$ $(k_0 \in N)$, *in the sense that* $J(u^k) \to \infty$ *as* $k \to \infty$.

Remark 1 To the best of our knowledge, Theorem 1.1 is the first result for the existence of infinitely many high energy solutions of the fractional Schrödinger equations with an external magnetic field by using the variant fountain theorem.

This paper is organized as follows. In Sect. 2 we introduce some preliminary knowledge and set up the functional. In Sect. 3, we prove Theorem 1.1 by using the variant fountain theorem.

2 Functional setting

In this section, we state some notations and preliminary knowledge which will be used in the next section.

Let $H^s_V(\mathbb{R}^N)$ is a fractional Sobolev space, defined by

$$H^s_V(\mathbb{R}^N) = \{u \in L^2_V(\mathbb{R}^N) : [u]_s < \infty\},$$

where

$$[u]_s = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy\right)^{\frac{1}{2}}$$

is the Gagliardo semi-norm and $L^2_V(\mathbb{R}^N)$ is the real valued Lebesgue space, with $V(x)|u|^2$ in $L^1(\mathbb{R}^N)$, and $H^s_V(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{H^s_V} = \left([u]_s^2 + \|u\|^2_{2,V}\right)^{\frac{1}{2}}, \qquad \|u\|^2_{2,V} = \int_{\mathbb{R}^N} V(x)|u|^2\, dx.$$

Lemma 2.1 (Theorem 2.1 of [25]) *Let* (V_1) *and* (V_2) *hold. Then, the embedding* $H^s_V(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *is continuous for any* $q \in [2, 2^*_s]$, *and the embedding* $H^s_V(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ *is compact for any* $q \in [2, 2^*_s)$.

Let $L^2(\mathbb{R}^N, \mathbb{C})$ denotes the Lebesgue space of complex functions $u : \mathbb{R}^N \to \mathbb{C}$ with $V(x)|u|^2 \in L^1(\mathbb{R}^N)$, the real scalar product of $L^2(\mathbb{R}^N, \mathbb{C})$ is endowed with

$$\langle u, v\rangle_{L^2} = \Re \int_{\mathbb{R}^N} V(x) u \bar{v}\, dx,$$

for all $u, v \in L^2(\mathbb{R}^N, \mathbb{C})$, where $\bar{v}$ denotes complex conjugation of $v \in \mathbb{C}$.

Define $H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ as the closure of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ with the norm

$$\|u\|_{H^s_{A,V}} = \left(\|u\|^2_{L^2} + [u]^2_{s,A}\right)^{\frac{1}{2}}, \tag{2.1}$$

where $[u]_{s,A}$ is the magnetic Gagliardo semi-norm

$$[u]_{s,A} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy\right)^{\frac{1}{2}}.$$

According to [1], we know that the space $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ is a real Hilbert space with the scalar product

$$\langle u,v\rangle_{H^s_{A,V}} = \langle u,v\rangle_{L^2} + \Re\int_{\mathbb{R}^{2N}} \frac{[u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)]\cdot\overline{[v(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)]}}{|x-y|^{N+2s}}\,dx\,dy. \tag{2.2}$$

Lemma 2.2 *For each $u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, then $|u|\in H^s_V(\mathbb{R}^N)$.*

Proof According to the definition of the space $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $H^s_V(\mathbb{R}^N)$, the result clearly holds. So we will not repeat it. □

Lemma 2.3 *For all $p\in[2,2^*_s]$, the embedding*

$$H^s_{A,V}\big(\mathbb{R}^N,\mathbb{C}\big)\hookrightarrow L^p\big(\mathbb{R}^N,\mathbb{C}\big)$$

is continuous.

Proof By using the pointwise diamagnetic inequality

$$\big||u(x)|-|u(y)|\big|\le\big|u(x)-e^{i(x-y)A(\frac{x+y}{2})u(y)}\big|\quad\text{for a.e. } x,y\in\mathbb{R}^N,$$

and Lemma 2.1 the continuous injection $H^s_V(\mathbb{R}^N)\hookrightarrow L^{2^*_s}(\mathbb{R}^N)$, we find

$$\|u\|_{L^{2^*_s}(\mathbb{R}^N)}\le C\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy$$

for all $u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, where $C>0$ is a real constant. Hence, by interpolation the assertion holds. □

Combining Lemma 2.1 with Lemma 2.2, we also obtain the following results.

Lemma 2.4 *Let (V_1) and (V_2) hold. Then, for any bounded sequence $(u_n)_n$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, the sequence $(|u_n|)_n$ has a subsequence converging strongly to some u in $L^p(\mathbb{R}^N)$ for every $p\in[2,2^*_s)$.*

Definition 2.5 We say that $u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ is a weak solution of problem (1.1), if for all $\phi\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, one has

$$\Re\biggl[\int_{\mathbb{R}^{2N}}\frac{[u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)]\cdot\overline{[\phi(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}\phi(y)]}}{|x-y|^{N+2s}}\,dx\,dy\biggr] + \Re\int_{\mathbb{R}^N}V(x)u\bar{\phi}\,dx = \Re\int_{\mathbb{R}^N}f\big(x,|u|\big)u\bar{\phi}\,dx. \tag{2.3}$$

For any $u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, we define the energy functional $J: H^s_{A,V}(\mathbb{R}^N,\mathbb{C})\to\mathbb{R}$ associated with the problem (1.1) as

$$J(u)=\frac{1}{2}\biggl(\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^N}V(x)|u|^2\,dx\biggr)-\int_{\mathbb{R}^N}F\big(x,|u|\big)\,dx$$

$$= \frac{1}{2}\left([u]_{s,A}^2 + \|u\|_{L^2}^2\right) - \int_{\mathbb{R}^N} F\big(x,|u|\big)\,dx$$
$$= \frac{1}{2}\|u\|_{H^s_{A,V}}^2 - \int_{\mathbb{R}^N} F\big(x,|u|\big)\,dx. \tag{2.4}$$

By direct computation, we find that J is of $C^1(H^s_{A,V}(\mathbb{R}^N,\mathbb{C}),\mathbb{R})$ and

$$\langle J'(u),v\rangle = \Re\left[\int_{\mathbb{R}^{2N}} \frac{[u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)]\cdot\overline{[v(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)]}}{|x-y|^{N+2s}}\,dx\,dy\right] + \Re\int_{\mathbb{R}^N} V(x)u\bar{v}\,dx - \Re\int_{\mathbb{R}^N} f\big(x,|u|\big)u\bar{v}\,dx,$$

for all $u,v\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$.

Since $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space, we let $\{X_j\}$ be a sequence of subspace of $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ with $\dim X_j<\infty$ for each $j\in\mathbb{N}$. Furthermore, $H^s_{A,V}(\mathbb{R}^N,\mathbb{C}) = \bigoplus_{j\in\mathbb{N}} X_j$, the closure of the direct sum of all X_j.

Set

$$W_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$$

and

$$B_k = \big\{u\in W_k : \|u\|_{H^s_{A,V}} \le \tau_k\big\}, \qquad S_k = \big\{u\in Z_k : \|u\|_{H^s_{A,V}} = \delta_k\big\}$$

for $\tau_k > \delta_k > 0$. Consider the C^1 functional $J_\lambda : H^s_{A,V}(\mathbb{R}^N,\mathbb{C}) \to \mathbb{R}$ defined by

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \lambda\in[1,2],$$

where

$$A(u) = \frac{1}{2}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u|^2\,dx\right),$$
$$B(u) = \int_{\mathbb{R}^N} F\big(x,|u|\big)\,dx.$$

Hence

$$J_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u\|_{H^s_{A,V}}^2 - \lambda\int_{\mathbb{R}^N} F\big(x,|u|\big)\,dx,$$

for all $u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\lambda\in[1,2]$.

The following variant fountain theorem was established in [26].

Theorem 2.6 (Variant fountain theorem) *Assume that* $J_\lambda \in (H^s_{A,V}(\mathbb{R}^N,\mathbb{C}),\mathbb{R})$ *defined above satisfies*:

(A_1) J_λ *maps bounded sets into bounded sets uniformly for* $\lambda\in[1,2]$, *and* $J_\lambda(-u) = J_\lambda(u)$ *for all* $(\lambda,u)\in[1,2]\times H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$;

(A_2) *$B(u) \geq 0$ for any $u \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\|_{H^s_{A,V}} \to \infty$;*
(A_3) *there exist $\tau_k > \delta_k > 0$ such that*

$$b_k(\lambda) = \inf_{u \in S_k} J_\lambda(u) > a_k(\lambda) = \max_{u \in W_k, \|u\|_{H^s_{A,V}} = \tau_k} J_\lambda(u), \quad \text{for all } \lambda \in [1,2].$$

Then

$$b_k(\lambda) \leq c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_\lambda\big(\gamma(u)\big), \quad \text{for all } \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in C(B_k, H^s_{A,V}(\mathbb{R}^N,\mathbb{C})) : \gamma$ is odd, $\gamma|_{\partial B_k} = \mathrm{id}\}$ $(k \geq 2)$, moreover, for almost every $\lambda \in [1,2]$, there exists a sequence $u^k_n(\lambda)$ such that

$$\sup_n \big\|u^k_n(\lambda)\big\|_{H^s_{A,V}} < \infty, \qquad J'_\lambda\big(u^k_n(\lambda)\big) \to 0 \quad \text{and}$$
$$J_\lambda\big(u^k_n(\lambda)\big) \to c_k(\lambda) \quad \text{as } n \to \infty.$$

3 Proofs of the main result

In order to prove Theorem 1.1, we need the following results.

Lemma 3.1 *Let $2 \leq p < 2^*_s$. For any $k \in N$, define*

$$\zeta_k := \sup\big\{\|u\|_{L^p(\mathbb{R}^N)} : u \in Z_k, \|u\|_{H^s_{A,V}} = 1\big\}.$$

Then $\zeta_k \to 0$ as $k \to \infty$.

Proof Since $Z_{k+1} \subset Z_k$, we have $0 < \zeta_{k+1} \leq \zeta_k$ for any $k \in \mathbb{N}$. Suppose $\zeta_k \to \zeta$ as $k \to \infty$ for $\zeta \geq 0$. By the definition of ζ_k, there exists $u_k \in Z_k$ such that $\|u_k\|_{L^p(\mathbb{R}^n)} < \frac{1}{2}\zeta_k$ for any $k \in \mathbb{N}$.

We know that $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ is a real Hilbert space, so a reflexive Banach space, there exist $v \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and a subsequence of u_k, without loss of generality still denoted by u_k, such that $u_k \rightharpoonup v$ in $H^s_{A,V}(\mathbb{R}^N, C)$. That is,

$$\langle u_k, \phi\rangle_{H^s_{A,V}} \to \langle v, \phi\rangle_{H^s_{A,V}} \quad \text{as } k \to \infty,$$

for all $\phi \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. Since each Z_k is convex and closed, it is closed for the weak topology. Consequently, $v \in \bigcap_{k=1}^\infty Z_k = 0$.

Hence, $|u_k| \to 0$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$. We know that ζ is nonnegative, we get $\zeta_k \to 0$ as $k \to \infty$. The proof is completed. □

Lemma 3.2 *Let (f_1)–(f_2) hold. Then there exist two sequences $\tau_k > \delta_k > 0$ such that*

$$b_k(\lambda) = \inf_{u \in S_k} J_\lambda(u) > a_k(\lambda) = \max_{u \in W_k, \|u\|_{H^s_{A,V}} = \tau_k} J_\lambda(u).$$

Proof According to assumptions (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\big|f(x,t)t\big| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1},$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$.

Therefore, for $u \in Z_k$, and ε small enough, by Lemma 2.3 and Lemma 3.1, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2_{H^s_{A,V}} - \lambda \int_{\mathbb{R}^N} F\bigl(x,|u|\bigr)\,dx \\ &\geq \frac{1}{2}\|u\|^2_{H^s_{A,V}} - \frac{\lambda\varepsilon}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{\lambda C_\varepsilon}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\ &\geq \frac{1}{4}\|u\|^2_{H^s_{A,V}} - C_1\zeta_k^p\|u\|^p_{s,A} \\ &= \left(\frac{1}{4} - C_1\zeta_k^p\|u\|^{p-2}_{H^s_{A,V}}\right)\|u\|^2_{H^s_{A,V}}. \end{aligned}$$

If we choose $\delta_k = (8C_1\zeta_k^p)^{\frac{1}{2-p}}$, we get

$$J_\lambda(u) \geq \frac{1}{8}\|u\|^2_{H^s_{A,V}} = \frac{1}{8}\bigl(8C_1\zeta_k^p\bigr)^{\frac{2}{2-p}} > 0.$$

Then, for any $u \in Z_k$, with $\|u\|_{H^s_{A,V}} = \delta_k$, we have

$$b_k(\lambda) = \inf_{u\in S_k} J_\lambda(u) \geq \frac{1}{8}\bigl(8C_1\zeta_k^p\bigr)^{\frac{2}{2-p}} > 0, \quad \text{for all } \lambda \in [1,2]. \tag{3.1}$$

Next, we prove $J_\lambda(u) \to -\infty$ as $\|u\|_{H^s_{A,V}} \to \infty$ for all $u \in W_k$. Suppose that this is not the case, then there exist a positive constant M and $\{u_n\} \subset H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that $J_\lambda(u_n) \geq -M$ as $\|u_n\|_{H^s_{A,V}} \to \infty$ (while $n \to \infty$).

Let $v_n = \frac{u_n}{\|u_n\|_{H^s_{A,V}}}$, then up to a subsequence, we get $v_n \to v$ in W_k.

We have

$$\frac{1}{2} - \frac{J_\lambda(u_n)}{\|u_n\|^2_{H^s_{A,V}}} = \lambda \int_{\mathbb{R}^N} \frac{F(x,|u_n|)}{\|u_n\|^2_{H^s_{A,V}}}\,dx = \lambda \int_{v_n(x)\neq 0} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2\,dx.$$

By (f_4) and Fatou's lemma, we deduce the contradiction that

$$\frac{1}{2} = \liminf_{n\to\infty} \lambda \int_{v_n(x)\neq 0} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2\,dx \to \infty.$$

Thus, $J_\lambda(u) \to -\infty$ as $\|u\|_{H^s_{A,V}} \to \infty$ for all $u \in W_k$. Choose $\tau_k > \delta_k > 0$ large enough and let $\|u\|_{H^s_{A,V}} = \tau_k$, we obtain

$$a_k(\lambda) = \max_{u\in W_k, \|u\|_{H^s_{A,V}} = \tau_k} J_\lambda(u) < 0. \tag{3.2}$$

The proof is completed. □

Lemma 3.3 *Let* (V_1)*–*(V_2) *and* (f_1) *hold. Then any bounded sequence* $\{u_n\}_n \subset H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ *such that* $J'_\lambda(u_n) \to 0$ *as* $n \to \infty$ *has a strongly convergent subsequence in* $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$.

Proof Assume that $\{u_n\}_n$ is bounded in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. Going if necessary to a subsequence, by Lemma 2.4, we have

$$u_n \rightharpoonup u \quad \text{in } H^s_{A,V}\bigl(\mathbb{R}^N,\mathbb{C}\bigr),$$

$$
\begin{aligned}
&|u_n| \to u \quad \text{in } L^p(\mathbb{R}^N), \\
&u_n \to u \quad \text{a.e. in } \mathbb{R}^N.
\end{aligned}
\tag{3.3}
$$

In order to prove that $\{u_n\}_n$ converges strongly to u in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, we first give a simple notation. Let $\omega \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ be fixed and denote by $\Psi(\omega)$ the linear functional on $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ defined by

$$
\langle \Psi(\omega), \varphi \rangle = \Re \int_{\mathbb{R}^{2N}} \frac{(\omega(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\omega(y))}{|x-y|^{N+2s}} \overline{(\varphi(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\varphi(y))}\,dx\,dy \tag{3.4}
$$

for any $\varphi \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$.

By (f_1), for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}^+$,

$$
|f(x,t)t| \le C(|t| + |t|^{p-1}).
$$

Using the Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \bigl|\bigl(f(x,|u_n|)u_n - f(x,|u|)u\bigr)\overline{(u_n-u)}\bigr|\,dx \\
&\quad\le \int_{\mathbb{R}^N} C\bigl[|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1}\bigr]|u_n-u|\,dx \\
&\quad\le C\bigl(\|u_n\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}\bigr)\|u_n-u\|_{L^2(\mathbb{R}^N)} \\
&\qquad + C\bigl(\|u_n\|^{p-1}_{L^p(\mathbb{R}^N)} + \|u\|^{p-1}_{L^p(\mathbb{R}^N)}\bigr)\|u_n-u\|_{L^p(\mathbb{R}^N)} \\
&\quad\le C_2\bigl(\|u_n-u\|_{L^2(\mathbb{R}^N)} + \|u_n-u\|_{L^p(\mathbb{R}^N)}\bigr).
\end{aligned}
\tag{3.5}
$$

By Lemma 2.4, we have $|u_n| \to |u|$ in $L^p(\mathbb{R}^N)$ and $|u_n| \to |u|$ in $L^2(\mathbb{R}^N)$. Hence, $u_n \to u$ in $L^p(\mathbb{R}^N,\mathbb{C})$ and $L^2(\mathbb{R}^N,\mathbb{C})$. According to the Brézis–Lieb lemma [27], we get

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} \bigl(f(x,|u_n|)u_n - f(x,|u|)u\bigr)\overline{(u_n-u)}\,dx = 0. \tag{3.6}
$$

Obviously, $\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \to 0$ as $n \to \infty$, since $u_n \rightharpoonup u$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $J'_\lambda(u_n) \to 0$ in the dual space of $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. Thus

$$
\begin{aligned}
o(1) &= \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\
&= \langle \Psi(u_n) - \Psi(u), u_n - u\rangle + \|u_n - u\|^2_{L^2} \\
&\quad - \Re \int_{\mathbb{R}^N} \bigl(f(x,|u_n|)u_n - f(x,|u|)u\bigr)\overline{(u_n-u)}\,dx.
\end{aligned}
\tag{3.7}
$$

Combining with (3.6) and (3.7), we get

$$
\lim_{n\to\infty} \bigl(\langle \Psi(u_n) - \Psi(u), u_n - u\rangle + \|u_n - u\|^2_{L^2}\bigr) = 0, \tag{3.8}
$$

which yields $u_n \to u$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. □

Proof of Theorem 1.1 By (f_3) and (2.4), we know that $B(u) \geq 0$ for all $u \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $A(u) \to \infty$ as $\|u\|_{H^s_{A,V}} \to \infty$. Moreover, $J_\lambda(-u) = J_\lambda(u)$ for all $u \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\lambda \in [1,2]$. It follows from the conditions (f_3)–(f_4) and Lemma 2.4 that J_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. Together with Lemma 2.2, the conditions (A_1)–(A_3) of Theorem 2.6 are verified. Thus, by Theorem 2.6, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u^k_n(\lambda)\}_{n=1}^\infty$ such that

$$\begin{aligned} &\sup_n \|u^k_n(\lambda)\|_{H^s_{A,V}} < \infty, \qquad J'_\lambda\big(u^k_n(\lambda)\big) \to 0 \quad \text{and} \\ &J_\lambda\big(u^k_n(\lambda)\big) \to c_k(\lambda) \quad \text{as } n \to \infty. \end{aligned} \tag{3.9}$$

We have

$$c_k(\lambda) = \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} J_\lambda\big(\gamma(u)\big) \leq \max_{u\in B_k} J_1(u) = \bar{c}_k. \tag{3.10}$$

Combining with (3.1), we have

$$c_k(\lambda) \geq b_k(\lambda) \geq \frac{1}{8}\big(8C_1\zeta_k^p\big)^{\frac{2}{2-p}} = \bar{b}_k \to \infty \quad \text{as } n \to \infty. \tag{3.11}$$

Hence

$$\bar{b}_k \leq c_k(\lambda) \leq \bar{c}_k \quad \text{for } k \geq k_0. \tag{3.12}$$

By (3.9), we know that if we choose a sequence $\lambda_m \to 1$, then the sequence $\{u^k_n(\lambda_m)\}$ is bounded. Combining with Lemma 3.3, we see that $\{u^k_n(\lambda_m)\}$ has a strong convergent subsequence as $n \to \infty$. We may assume that $u^k_n(\lambda_m) \to u^k(\lambda_m)$ as $n \to \infty$ for every $m \in \mathbb{N}$ and $k \geq k_0$. By (3.9) and (3.12), we get

$$J'_{\lambda_m}\big(u^k(\lambda_m)\big) = 0 \quad \text{and} \quad J_{\lambda_m}\big(u^k(\lambda_m)\big) \in [\bar{b}_k, \bar{c}_k] \quad \text{for } k \geq k_0. \tag{3.13}$$

Next, we show that $\{u^k(\lambda_m)\}_{m=1}^\infty$ is bounded in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. If not, we consider $v_m := \frac{u^k(\lambda_m)}{\|u^k(\lambda_m)\|_{H^s_{A,V}}}$. Then, up to a sequence, we get $v_m \rightharpoonup v$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, $|v_m| \to |v|$ in $L^p(\mathbb{R}^N)$ for $2 \leq p < 2^*_s$ and $v_m(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$.

Case 1: If $v(x) \neq 0$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, we have

$$\begin{aligned} \frac{1}{2} - \frac{J_{\lambda_m}(u^k(\lambda_m))}{\|u^k(\lambda_m)\|^2_{H^s_{A,V}}} &= \lambda_m \int_{\mathbb{R}^N} \frac{F(x,|u^k(\lambda_m)|)}{\|u^k(\lambda_m)\|^2_{H^s_{A,V}}}\,dx \\ &= \lambda_m \int_{\{v_m(x)\neq 0\}} \frac{F(x,|u^k(\lambda_m)|)}{|u^k(\lambda_m)|^2}|v_m|^2\,dx. \end{aligned} \tag{3.14}$$

Thus,

$$\lim_{m\to\infty} \lambda_m \int_{\{v_m(x)\neq 0\}} \frac{F(x,|u^k(\lambda_m)|)}{|u^k(\lambda_m)|^2}|v_m|^2\,dx = \frac{1}{2}. \tag{3.15}$$

On the other hand, by (f_4)

$$\liminf_{m\to\infty} \lambda_m \int_{\{v_m(x)\neq 0\}} \frac{F(x,|u^k(\lambda_m)|)}{|u^k(\lambda_m)|^2}|v_m|^2\,dx \to \infty. \tag{3.16}$$

That is a contradiction.

Case 2: If $v(x)=0$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, we have

$$\begin{aligned}
&\mu J_{\lambda_m}\bigl(u^k(\lambda_m)\bigr) - \bigl\langle J'_{\lambda_m}\bigl(u^k(\lambda_m)\bigr), u^k(\lambda_m)\bigr\rangle \\
&\quad= \left(\frac{\mu}{2}-1\right)\bigl\|u^k(\lambda_m)\bigr\|^2_{H^s_{A,V}} - \mu\lambda_m \int_{\mathbb{R}^N} F\bigl(x,\bigl|u^k(\lambda_m)\bigr|\bigr)\,dx \\
&\qquad+ \Re\lambda_m \int_{\mathbb{R}^N} f\bigl(x,\bigl|u^k(\lambda_m)\bigr|\bigr)u^k(\lambda_m)\overline{u^k(\lambda_m)}\,dx.
\end{aligned} \tag{3.17}$$

We have

$$\begin{aligned}
\frac{\mu}{2}-1 &= \frac{\mu J_{\lambda_m}(u^k(\lambda_m)) - \langle J'_{\lambda_m}(u^k(\lambda_m)), u^k(\lambda_m)\rangle}{\|u^k(\lambda_m)\|^2_{s,A}} \\
&\quad - \lambda_m \int_{\mathbb{R}^N} \frac{\mu F(x,|u^k(\lambda_m)|) - f(x,|u^k(\lambda_m)|)|u^k(\lambda_m)|^2}{|u^k(\lambda_m)|^2}|v_m|^2\,dx.
\end{aligned} \tag{3.18}$$

By (3.13),

$$\begin{aligned}
&\lambda_m \int_{\mathbb{R}^N} \frac{\mu F(x,|u^k(\lambda_m)|) - f(x,|u^k(\lambda_m)|)|u^k(\lambda_m)|^2}{|u^k(\lambda_m)|^2}\bigl|v_m(x)\bigr|^2\,dx \to \frac{\mu}{2}-1 \\
&\quad \text{as } m\to\infty.
\end{aligned} \tag{3.19}$$

But by the assumption (f_3),

$$\limsup_{m\to\infty} \lambda_m \int_{\mathbb{R}^N} \frac{\mu F(x,|u^k(\lambda_m)|) - f(x,|u^k(\lambda_m)|)|u^k(\lambda_m)|^2}{|u^k(\lambda_m)|^2}\bigl|v_m(x)\bigr|^2\,dx \le 0. \tag{3.20}$$

Combining with (3.19) and (3.20), we get $\frac{\mu}{2}-1\le 0$, i.e. $\mu\le 2$, which is in contradiction with the assumption. Therefore $\{u^k(\lambda_m)\}_{m=1}^\infty$ is bounded in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. By Lemma 3.3, we find that $\{u^k(\lambda_m)\}_{m=1}^\infty$ possesses a strong convergent subsequence with limit $u^k\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ for all $k\ge k_0$. Hence, u^k is a critical point of $J=J_1$ with $J(u^k)\in[\bar{b_k},\bar{c_k}]$. Since $\bar{b_k}\to\infty$ as $k\to\infty$, we have infinitely many nontrivial critical points of J. Namely, problem (1.1) has infinitely many nontrivial solutions with high energy. The proof is completed. □

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